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ON FUNCTIONAL DIFFERENTIAL INCLUSIONS IN HILBERT SPACES

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Abstract

We prove the existence of monotone solutions, of the functional differential inclusion $\dot{x}(t) \in f(t, T(t)x) + F(T(t)x)$ in a Hilbert space, where f is a Carathéodory single-valued mapping and F is an upper semicontinuous set-valued mapping with compact values contained in the Clarke subdifferential $\partial_c V(x)$ of a uniformly regular function V.

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1. Introduction

Let H be a separable Hilbert space with the norm $\|.\|$ and the scalar product $\langle .,. \rangle$. For any segment I in $\mathbb{R}$, we denote by $\mathcal{C}(I, H)$ the Banach space of continuous functions from I to H equipped with the norm $\| x(.) \|_\infty := \sup\{\| x(t) \|; t \in I\}$. For all positive reals a, we put $\mathcal{C}_a := \mathcal{C}([-a, 0], H)$ and for any $t \in [0, T]$, $T > 0$, we define the operator $T(t)$ from $\mathcal{C}([-a, T], H)$ to $\mathcal{C}_a$ by $(T(t)x)(s) = x(t+s)$. For a given nonempty subset K of H, we introduce the set $\mathcal{K}_0 := \{\varphi \in \mathcal{C}_a; \varphi(0) \in K\}$. The aim of this paper is to prove the existence of solutions to the following functional differential inclusion:

$$(1.1) \qquad \begin{cases} \dot{x}(t) \in f(t, T(t)x) + F(T(t)x) & \text{a.e. on } [0, \tau]; \\ x(s) = \varphi(s), \ \forall s \in [-a, 0]; \\ x(s) \in P(x(t)), \ \forall t \in [0, \tau], \ \forall s \in [t, \tau]; \end{cases}$$

where F is an upper semicontinuous multifunction with compact values, f is a Carathéodory function and P is a lower semicontinuous multifunction.

Functional differential inclusions have deserved the attention of many authors. For review of results on functional differential inclusions, we refer the reader to the papers by Haddad [10, 11], Gavioli and Malaguti [9], Syam [13] and the references therein.

Recently, Cernea and Lupulescu [5] proved the existence of solutions to the problem (1.1) in the case where H is finite-dimensional space and F is cyclically monotone. They used the following tangential condition: for all $(t, \varphi) \in \mathbb{R} \times \mathcal{K}_0$ and for all $v \in F(\varphi)$

$$\liminf_{h \mapsto 0^+} \frac{1}{h} d_K \left(\varphi(0) + hv + \int_t^{t+h} f(s, \varphi) ds \right) = 0. \tag{1.2}$$

This work extends result which is presented in [5]. Indeed, we assume that $F(\varphi)$ is contained in the Clarke subdifferential $\partial_c V(\varphi(0))$, where V belongs to the class of uniformly regular functions which contains strictly the class of convex functions and the class of lower-C^2 functions. For the problem (1.1), we shall use a tangency condition which is weaker than (1.2). We suppose that for all $(t, \varphi) \in \mathbb{R} \times \mathcal{K}_0$ there exists $v \in F(\varphi)$ such that

$$\liminf_{h \mapsto 0^+} \frac{1}{h} d_{P(\varphi(0))} \left(\varphi(0) + hv + \int_t^{t+h} f(s, \varphi) ds \right) = 0.$$

The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, in Section 3, we give some preliminary results, while in Section 4, we prove the existence of solutions for (1.1).

2. Preliminaries and statement of the main result

For $x \in H$ and $r > 0$ let $B(x, r) := \{y \in H; \| y - x \| < r\}$ be the open ball centered at x with radius r, $\bar{B}(x, r)$ be its closure and let $B = B(0, 1)$. For $\varphi \in \mathcal{C}_a$ let $B_a(\varphi, r) := \{\psi \in \mathcal{C}_a; \| \varphi - \psi \|_\infty < r\}$ and $\bar{B}_a(\varphi, r)$ be its closure. For $x \in H$ and for a set $A \subset H$ we denote by $d_A(x)$ the distance from x to A given by $d_A(x) := \inf\{\| y - x \|; y \in A\}$.

We shortly review the definitions of the various extensions of derivatives used in this paper (see [6, 7, 12] as general references).

Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and x be any point where V is finite. The generalized Rockafellar directional derivative $V^\uparrow(x, .)$ is

$$V^\uparrow(x, v) := \limsup_{x' \to x, V(x') \to V(x), t \to 0^+} \inf_{v' \to v} \frac{V(x' + tv') - V(x')}{t}.$$

The upper generalized Clarke directional derivative $V^o(x,.)$ is

$$V^o(x,v) := \limsup_{h\to 0^+\ y\to x} \frac{V(y+hv)-V(y)}{h}.$$

Analogously the lower generalized Clarke directional derivative $V_o(x,.)$ is

$$V_o(x,v) := \liminf_{h\to 0^+\ y\to x} \frac{V(y+hv)-V(y)}{h}.$$

If V is Lipschitz around x, then $V^{\uparrow}(x,v)$ coincides with $V^o(x,v)$ for all $v \in H$. We also recall that the Clarke subdifferential of V at x is defined by

$$\partial_c V(x) := \Big\{ y \in H \,:\, \langle y, v\rangle \,\le\, V^{\uparrow}(x,v), \text{ for all } v \in H \Big\},$$

and that the proximal subdifferential $\partial_p V(x)$ of V at x is the set of all $y \in H$ for which there exist δ, $\sigma > 0$ such that for all $x' \in x + \delta \bar{B}$

$$\langle y, x'-x\rangle \,\le V(x') - V(x) + \sigma \|x'-x\|^2.$$

Note that $\partial_c V(x)$ is convex and closed and $\partial_p V(x)$ is convex, but not necessarily closed. On the other hand, one always has $\partial_p V(x) \subset \partial_c V(x)$.

In the following proposition we summarize some useful properties of Clarke generalized directional derivatives.

Proposition 2.1. *Let $V : H \to \mathbb{R}\cup\{+\infty\}$ be locally Lipschitz. Then the following conditions hold:*

(i) $\partial_c V(x) = \Big\{ p \in H : V^o(x,v) \ge \langle p, v\rangle, \forall v \in H \Big\}$
$= \Big\{ p \in H : V_o(x,v) \le \langle p, v\rangle, \forall v \in H \Big\};$

(ii) $V^o(x,v) = \max \Big\{ \langle p, v\rangle, p \in \partial_c V(x) \Big\}$ *and*
$V_o(x,v) = \min \Big\{ \langle p, v\rangle, p \in \partial_c V(x) \Big\} = -V^o(x,-v).$

Let us recall the definition of the concept of regularity that will be used in the sequel.

Definition 2.2 [3]**.** Let $V : H \to \mathbb{R}\cup\{+\infty\}$ be a lower semi-continuous function and let $U \subset DomV$ be a nonempty open subset. We will say that V is uniformly regular over U if there exists a positive number β such that for all $x \in U$ and for all $\xi \in \partial_p V(x)$ one has

$$\langle \xi, x'-x\rangle \,\le V(x') - V(x) + \beta \|x'-x\|^2 \quad \text{for all } x' \in U.$$

We say that V is uniformly regular over a closed set S if there exists an open set U containing S such that V is uniformly regular over U.

The class of functions that are uniformly regular over sets is quite large. Any l.s.c. proper convex function V is uniformly regular over any nonempty subset of its domain with $\beta = 0$. For more details on the concept of regularity, we refer the reader to [3].

The following proposition summarizes some important properties for uniformly regular locally Lipschitz functions over sets.

Proposition 2.3 [3]**.** *Let $V : H \to \mathbb{R}$ be a locally Lipschitz function and S be a nonempty closed set. If V is uniformly regular over S, then the following conditions hold:*

(a) *The proximal subdifferential of V is closed as a multifunction over S, that is, for every $x_n \to x \in S$ with $x_n \in S$ and every $\xi_n \to \xi$ weakly with $\xi_n \in \partial_p V(x_n)$ one has $\xi \in \partial_p V(x)$.*

(b) *The proximal subdifferential of V coincides with the Clarke subdifferential of V for any point x.*

(c) *The proximal subdifferential of V is upper hemicontinuous over S, that is, the support function $x \mapsto \sigma\big(v, \partial_p V(x)\big)$ is u.s.c. over S for every $v \in H$.*

In the following lemma we recall some useful properties for the measure of noncompactness β. For instance see Proposition 9.1 in [8].

Lemma 2.4. *Let X be an infinite dimensional real Banach space and D_1, D_2 be two bounded subsets of X.*

(i) $\beta(D_1) = 0 \Leftrightarrow D_1$ *is relatively compact.*

(ii) $\beta(\lambda D_1) = |\lambda|\beta(D_1)$; $\lambda \in \mathbb{R}$.

(iii) $D_1 \subseteq D_2 \Rightarrow \beta(D_1) \leq \beta(D_2)$.

(iv) $\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$.

(v) *If $x_0 \in X$ and r is a positive real number then $\beta(B(x_0, r)) = 2r$.*

Now let us state the main result. Assume that

(H1) (a) K is a nonempty locally compact subset in H and $V : H \to \mathbb{R}$ is a locally Lipschitz function which is uniformly regular over all closed subset of K,

(b) $P : H \to 2^K$ is a lower semicontinuous set-valued map satisfying

(i) For all $x \in K$, $x \in P(x)$,

(ii) For all $x \in K$ and all $y \in P(x)$ we have $P(y) \subseteq P(x)$;

(H2) $F : \mathcal{K}_0 \to 2^H$ is an upper semicontinuous set-valued map with compact values satisfying $F(\varphi) \subset \partial_c V(\varphi(0))$ for all $\varphi \in \mathcal{K}_0$;

(H3) $f : \mathbb{R} \times \mathcal{K}_0 \to H$ is a function with the following properties:

(1) For all $\varphi \in \mathcal{K}_0$, $t \mapsto f(t,\varphi)$ is measurable,

(2) For all $t \in \mathbb{R}$, $\varphi \mapsto f(t,\varphi)$ is continuous,

(3) For all bounded subset S of $\mathcal{K}_0$, there exists $m(.) \in L^2(\mathbb{R},\mathbb{R}^+)$ such that

$$\|f(t,\varphi)\| \leq m(t), \quad \forall (t,\varphi) \in \mathbb{R} \times S.$$

(H4) (**Tangential condition**) For all $(t,\varphi) \in \mathbb{R} \times \mathcal{K}_0$, there exists $v \in F(\varphi)$ such that

$$\liminf_{h \mapsto 0^+} \frac{1}{h} d_{P(\varphi(0))} \left(\varphi(0) + hv + \int_t^{t+h} f(s,\varphi)ds \right) = 0.$$

The aim of this paper is to prove the following theorem.

Theorem 2.5. *If assumptions* (H1)–(H4) *are satisfied, then for all* $\varphi \in \mathcal{K}_0$, *there exist* $T > 0$ *and a continuous function* $x(.) : [-a,T] \to H$, *which is absolutely continuous on* $[0,T]$ *and such that* $x(.)$ *is a solution of* (1.1).

In the paper, we suppose that the assumptions (H1)–(H4) are satisfied, we fix $\varphi \in \mathcal{K}_0$ and we choose $r > 0$ such that $K_0 = K \cap \overline{B}(\varphi(0),r)$ is compact and V is Lipschitz continuous on $B(\varphi(0),2r)$ with Lipschitz constant $\lambda > 0$. Then $\partial_c V(x) \subset \lambda \overline{B}$ for every $x \in B(\varphi(0),2r)$. Let $m(.) \in L^2(\mathbb{R},\mathbb{R}^+)$ be such that

$$\|f(t,\varphi)\| \leq m(t), \quad \forall (t,\varphi) \in \mathbb{R} \times \overline{B}_a(\varphi,2r).$$

For $\varepsilon > 0$ set

$$\eta(\varepsilon) := \sup \Bigg\{ \rho \in]0,\varepsilon] : \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon \text{ and}$$

$$\left| \int_{t_1}^{t_2} (\lambda + a + m(s))ds \right| < \varepsilon \text{ if } |t_1 - t_2| \leq \rho \Bigg\}. \tag{2.1}$$

Remark 2.6. If $K \cap \overline{B}(\varphi(0), r)$ is closed in H, then $\mathcal{K}_0 \cap \overline{B}_a(\varphi, r)$ is closed in $\mathcal{C}_a$. Indeed, let $(\psi_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{K}_0 \cap \overline{B}_a(\varphi, r)$ which converges to ψ. We have $\psi_n \in \overline{B}_a(\varphi, r)$ for all $n \in \mathbb{N}$, so $\psi \in \overline{B}_a(\varphi, r)$. On the other hand since $\psi_n \in \mathcal{K}_0 \cap \overline{B}_a(\varphi, r)$ for all $n \in \mathbb{N}$, one has $\psi_n(0) \in K \cap \overline{B}(\varphi(0), r)$ for all $n \in \mathbb{N}$, so by the closedness of $K \cap \overline{B}(\varphi(0), r)$ we get $\psi(0) \in K$. Hence $\psi \in \mathcal{K}_0 \cap \overline{B}_a(\varphi, r)$.

3. Preliminary results

In this section, we shall prove some auxiliary results needed in the next section. Consider first the following hypotheses which we shall use throughout this section.

(A1) $G : K \to 2^H$ is an upper semicontinuous multifunction with compact values satisfying $G(x) \subset \partial_c V(x)$ for all $x \in K$;

(A2) $g : \mathbb{R} \times H \to H$ is a function with the following properties:

- (i) For all $x \in H$, $t \mapsto g(t, x)$ is measurable,
- (ii) For all $t \in \mathbb{R}$, $x \mapsto g(t, x)$ is continuous,
- (iii) For all $(t, x) \in \mathbb{R} \times \overline{B}(\varphi(0), 2r)$

$$\|g(t, x)\| \leq m(t);$$

(A3) (**Tangential condition**) $\forall (t, x) \in [0, 1] \times K$, $\exists v \in G(x)$ such that

$$\liminf_{h \mapsto 0^+} \frac{1}{h} d_{P(x)}\left(x + hv + \int_t^{t+h} g(s, x)ds \right) = 0.$$

In the sequel, we will use the following important Lemma. It will play a crucial role in the proof of Proposition 3.3.

Lemma 3.1. *If assumptions* (A1)–(A3) *are satisfied, then for all* $\varepsilon > 0$, *there exists* $\eta > 0$ $(\eta < \varepsilon)$ *such that* $\forall (t, x) \in [0, 1] \times K_0$, *there exist* $h_{t,x} \in [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$ *and* $u \in G(x) + \varepsilon B$ *such that*

$$\left(x + h_{t,x} u + \int_t^{t+h_{t,x}} g(s, x)ds \right) \in P(x).$$

Proof. Let $\varepsilon > 0$ and $(t, x) \in [0, 1] \times K_0$ be fixed. Since G is upper semicontinuous on x, there exists $\delta_x > 0$ such that $G(\bar{x}) \subset G(x) + \frac{\varepsilon}{2} B$ for all $\bar{x} \in B(x, \delta_x)$.

Let $(s,\bar{x}) \in [0,1] \times K_0$. By the tangential condition, there exist $v \in G(\bar{x})$ and $h_{s,\bar{x}} \in]0, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$ such that

$$d_{P(\bar{x})}\left(\bar{x} + h_{s,\bar{x}}v + \int_s^{s+h_{s,\bar{x}}} g(\tau,\bar{x})d\tau\right) < \frac{h_{s,\bar{x}}\varepsilon}{8}.$$

Consider the subset $N(s,\bar{x})$ of all $(\tilde{s},\tilde{x})$ in $\mathbb{R} \times B(\varphi(0),2r)$ such that

$$d_{P(\tilde{x})}\left(\tilde{x} + h_{s,\bar{x}}v + \int_{\tilde{s}}^{\tilde{s}+h_{s,\bar{x}}} g(\tau,\tilde{x})d\tau\right) < \frac{h_{s,\bar{x}}\varepsilon}{8}.$$

Moreover, by hypothesis (A2), the dominated convergence theorem shows that the function

$$(\tilde{s},\tilde{x}) \mapsto \tilde{x} + h_{s,\bar{x}}v + \int_{\tilde{s}}^{\tilde{s}+h_{s,\bar{x}}} g(\tau,\tilde{x})d\tau$$

is continuous. Since P is lower semicontinuous, by Corollary 1.2.1 in [2], the function

$$(\tilde{s},\tilde{x}) \mapsto d_{P(\tilde{x})}\left(\tilde{x} + h_{s,\bar{x}}v + \int_{\tilde{s}}^{\tilde{s}+h_{s,\bar{x}}} g(\tau,\tilde{x})d\tau\right)$$

is upper semicontinuous. So $N(s,\bar{x})$ is open. Furthermore, since $(s,\bar{x})$ belongs to $N(s,\bar{x})$, there exists $0 < \eta_{s,\bar{x}} < \delta_x$ such that $B((s,\bar{x}),\eta_{s,\bar{x}})$ is contained in $N(s,\bar{x})$, therefore, the compact subset $[0,1] \times K_0$ can be covered by q such balls $B((s_i,\bar{x}_i),\eta_{s_i,\bar{x}_i})$. For simplicity, set $h_i := h_{s_i,\bar{x}_i}$ and $\eta_i := \eta_{s_i,\bar{x}_i}$, $i = 1,\ldots,q$. Put $\eta = \min\{h_i/1 \leq i \leq q\}$. There exists $i \in \{1,\ldots,q\}$ such that $(t,x) \in B((s_i,\bar{x}_i),\eta_{s_i,\bar{x}_i})$, hence $(t,x) \in N(s_i,\bar{x}_i)$. Then there exists $v_i \in G(\bar{x}_i)$ such that

$$d_{P(x)}\left(x + h_i v_i + \int_t^{t+h_i} g(\tau,x)d\tau\right) < \frac{h_i\varepsilon}{8}.$$

Let $x_i \in P(x)$ be such that

$$\frac{1}{h_i}\left\|x_i - \left(x + h_i v_i + \int_t^{t+h_i} g(\tau,x)d\tau\right)\right\|$$
$$\leq \frac{1}{h_i} d_{P(x)}\left(x + h_i v_i + \int_t^{t+h_i} g(\tau,x)d\tau\right) + \frac{\varepsilon}{4},$$

hence

$$\left\|\frac{1}{h_i}\left(x_i - x - \int_t^{t+h_i} g(\tau,x)d\tau\right) - v_i\right\| < \frac{\varepsilon}{2}.$$

Set

$$u = \frac{1}{h_i}\left(x_i - x - \int_t^{t+h_i} g(\tau, x)d\tau\right),$$

then $u \in G(\bar{x}_i) + \frac{\varepsilon}{2}B$ and

$$x_i = \left(x + h_i u + \int_t^{t+h_i} g(\tau, x)d\tau\right) \in P(x).$$

Since $\|x - \bar{x}_i\| \leq \delta_x$ we have $u \in G(x) + \varepsilon B$. ∎

In the sequel, we need the following Lemma.

Lemma 3.2. *For all* $0 < \varepsilon < a$ *there exists* $0 < \alpha < \varepsilon$ *such that for all* $t_0 \in [0,1]$ *and* $x \in B(\varphi(0), r)$, *there exist* $\rho \in]0,1]$ *and* $b \in [\alpha, \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$ *satisfying* $B(x, \rho) \subset B(\varphi(0), r)$ *and* $\int_{t_0}^{t_0+b}(a + \lambda + m(s))ds \leq \frac{\rho}{2}$.

Proof. Let $0 < \varepsilon < a$, $t_0 \in [0,1]$ and $x \in B(\varphi(0), r)$ be fixed. Consider $0 < \rho \leq 1$ such that $B(x, \rho) \subset B(\varphi(0), r)$. Let $(\bar{\rho}, t) \in [0,1]^2$. There exists $b_{\bar{\rho},t} \in]0, \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$ such that $\int_t^{t+b_{\bar{\rho},t}}(a+\lambda+m(s))ds < \frac{\bar{\rho}}{4}$. Consider the open subset

$$N(\bar{\rho}, t) = \left\{(\mu, \nu) \in \mathbb{R}^2 \ / \ \int_\nu^{\nu+b_{\bar{\rho},t}} (a + \lambda + m(s))ds < \frac{\mu}{4}\right\}.$$

Since $(\bar{\rho}, t) \in N(\bar{\rho}, t)$ then there exists $\tau > 0$ such that $B((\bar{\rho}, t), \tau) \subset N(\bar{\rho}, t)$. The compact subset $[0,1]^2$ can be covered by q such balls $B((\bar{\rho}_i, t_i), \tau_i)$. Set $b_i = b_{\bar{\rho}_i, t_i}$ and $\alpha = \inf\{b_i, 0 \leq i \leq q\}$. Let $i \in \{1, \ldots, q\}$ be such that $(\rho, t_0) \in B((\bar{\rho}_i, t_i), \tau_i)$. Hence $(\rho, t_0) \in N(\bar{\rho}_i, t_i)$. So $\int_{t_0}^{t_0+b_i}(a + \lambda + m(s))ds < \frac{\rho}{4}$, where $b_i \in [\alpha, \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$. ∎

In the paper, for $\varepsilon > 0$ we denote by $\alpha(\varepsilon)$ the constant α given by Lemma 3.2.

In the next section, we need the following Proposition.

Proposition 3.3. *If assumptions* (A1)–(A3) *are satisfied, then for all* $\varepsilon \in]0, a[$, $t_0 \in [0,1]$ *and* $x_0 \in K \cap \overline{B}(\varphi(0), r)$ *there exist* $b_0 \in [\alpha(\varepsilon), \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$, *a continuous function* $x(.) : [t_0, +\infty[\to H$, *a function* $u(.) : [t_0, +\infty[\to H$ *and step functions* $\theta(.), \bar{\theta}(.) : [t_0, +\infty[\to [t_0, +\infty[$ *such that*

(i) $x(t_0) = x_0$, $x(\bar{\theta}(t)) \in P(x(\theta(t)))$ *and* $x(\theta(t)) \in K \cap \overline{B}(\varphi(0), r)$ *for all* $t \in [t_0, t_0 + b_0]$;

(ii) $\dot{x}(t) - g(t, x(\theta(t))) \in G(x(\theta(t))) + \varepsilon B$ *a.e. on* $[t_0, t_0 + b_0]$;

(iii) $\|\dot{x}(t)\| \leq \lambda + a + m(t)$ *for almost all* $t \in [t_0, t_0 + b_0]$;

(iv) $0 \leq t - \theta(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leq \bar{\theta}(t) - t \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ *and* $u(t) \in G(x(\theta(t))) + \varepsilon B$ *for all* $t \in [t_0, t_0 + b_0]$.

Proof. Let $0 < \varepsilon < a$, $t_0 \in [0, 1]$ and $x_0 \in K \cap \overline{B}(\varphi(0), r)$. By Lemma 3.2 there exist $\rho \in]0, 1]$ and $b_0 \in [\alpha(\varepsilon), \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$ such that $B(x_0, \rho) \subset B(\varphi(0), r)$ and

$$\int_{t_0}^{t_0+b_0} (a + \lambda + m(s))ds < \frac{\rho}{4}. \tag{3.1}$$

By Lemma 3.1, there exist $\eta > 0$, $h_0 \in [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$ and $u_0 \in G(x_0) + \varepsilon B$ such that

$$x_1 = \left(x_0 + h_0 u_0 + \int_{t_0}^{t_0+h_0} g(s, x_0)ds \right) \in P(x_0).$$

Set $t_1 = t_0 + h_0$. If $h_0 \leq b_0$, by (A1) and (A2), we have

$$\begin{aligned} \|x_1 - x_0\| &= \left\| h_0 u_0 + \int_{t_0}^{t_0+h_0} g(s, x_0)ds \right\| \\ &\leq h_0 \|u_0\| + \int_{t_0}^{t_0+h_0} \|g(s, x_0)\| ds \\ &\leq \int_{t_0}^{t_0+h_0} (\lambda + a + m(s))ds \leq \rho. \end{aligned}$$

Thus $x_1 \in K \cap \overline{B}(\varphi(0), r)$. Set $h_{-1} = 0$. We reiterate this process for constructing sequences $(h_p)_{p\geq 0} \subset [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$, $(t_p)_{p\geq 0}$, $(x_p)_{p\geq 0}$, $(u_p)_{p\geq 0}$ such that

(a) $t_p = t_0 + \sum\limits_{i=0}^{p-1} h_i$ and $x_p \in P(x_{p-1})$;

(b) $x_p = x_{p-1} + h_{p-1}u_{p-1} + \int\limits_{t_{p-1}}^{t_p} g(s, x_{p-1})ds$;

(c) $x_p \in K \cap \overline{B}(\varphi(0), r)$ if $\sum\limits_{i=0}^{p-1} h_i \leq b_0$;

(d) $u_{p-1} \in G(x_{p-1}) + \varepsilon B$.

It is easy to see that for $p = 1$ the assertions (a)–(d) are fulfilled. Let now $p \geq 1$. Assume that (a)–(d) are satisfied for any $p = 1, \dots, q$. If $t_0 + b_0 \leq t_q$, then we stop this process of iterations and we get (a)–(d) satisfied with $t_{q-1} < t_0 + b_0 \leq t_q$. On the other case, we can apply for (t_q, x_q) the same technique applied for (t_0, x_0)

at the beginning of this proof, and we get (a), (b) and (d) satisfied for $p = q+1$. It remains to prove (c). By induction, we have

$$x_{q+1} = x_0 + \sum_{i=0}^{q} h_i u_i + \sum_{j=0}^{q} \int_{t_0+\sum_{i=0}^{j} h_{i-1}}^{t_0+\sum_{i=0}^{j} h_i} g(s, x_j) ds.$$

Then, if $t_{q+1} \leq t_0 + b_0$, by (A1) and (A2), we have

$$\begin{aligned} \|x_{q+1} - x_0\| &\leq \sum_{i=0}^{q} h_i(\lambda + a) + \sum_{j=0}^{q} \int_{t_0+\sum_{i=0}^{j} h_{i-1}}^{t_0+\sum_{i=0}^{j} h_i} m(s) ds \\ &\leq \int_{t_0}^{t_{q+1}} (\lambda + a + m(s)) ds \leq \rho. \end{aligned}$$

Hence $x_{q+1} \in K \cap \overline{B}(\varphi(0), r)$. Since $h_p \geq \eta > 0$ then there exists an integer s such that

$$t_s < t_0 + b_0 \leq t_{s+1}.$$

Define on $[t_0, +\infty[$ the functions $x(.)$, $u(.)$, $\theta(.)$ and $\bar{\theta}(.)$ as follows:

$$x(t) = x_{q-1} + (t - t_{q-1}) u_{q-1} + \int_{t_{q-1}}^{t} g(s, x_{q-1}) ds \text{ for all } t \in [t_{q-1}, t_q];$$

$\theta(t) = t_{q-1}$, $u(t) = u_{q-1}$ and $\bar{\theta}(t) = t_q$ for all $t \in [t_{q-1}, t_q[$. Finally, the above definitions will enable us to derive the assertions (i)–(v). ■

4. Proof of the main result

Set $\varphi(0) = x_0$ and let

$$T = \inf\left\{1, \frac{1}{4}\eta\left(\frac{r}{8}\right)\right\}.$$

We shall show the following Proposition. It will be used in order to obtain a sequence of approximated solutions.

Proposition 4.1. *For all $0 < \varepsilon < a$ there exist continuous maps $x(.) : [-a, +\infty[\to H$, $\Gamma(.) : [0, +\infty[\to \mathcal{C}_a$ and step functions $\theta(.), \bar{\theta}(.), \tilde{\theta}(.) : [0, +\infty[\to [0, +\infty[$ such that*

(i) *$x(\theta(t)) \in K \cap \overline{B}(\varphi(0), r)$ and $x(\bar{\theta}(t)) \in P(x(\theta(t)))$, for all $t \in [0, T]$ and $x \equiv \varphi$ on $[-a, 0]$;*

(ii) $\dot{x}(t) - f(t, \Gamma(t)) \in F(\Gamma(t)) + \varepsilon B$ *for almost all* $t \in [0, T]$;

(iii) $0 \leq t - \theta(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leq t - \tilde{\theta}(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ *and* $0 \leq \bar{\theta}(t) - t \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$ *for all* $t \in [0, T]$;

(iv) $\|\dot{x}(t)\| \leq \lambda + a + m(t)$ *for almost all* $t \in [0, T]$;

(v) *For all* $t \in [0, T]$

$$\Gamma(t)(s) = \begin{cases} x\big(\tilde{\theta}(t) + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s\big) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})} x(\tilde{\theta}(t)) + \left(1 + \frac{4s}{\eta(\frac{\varepsilon}{4})}\right) x(\theta(t)) & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0. \end{cases}$$

Proof. Let $0 < \varepsilon < a$ be fixed. Set $t_0 = 0$ and put $x(t) = \varphi(t)$ for all $t \in [-a, 0]$. Consider the function $\Gamma_0 : H \to \mathcal{C}_a$ defined as follows: for all $z \in H$

$$\Gamma_0(z)(s) = \begin{cases} x\left(t_0 + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s\right) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})} x(t_0) + \left(1 + \frac{4s}{\eta(\frac{\varepsilon}{4})}\right) z & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0. \end{cases}$$

The mappings $G_0 : K \to 2^H$ and $g_0 : \mathbb{R} \times H \to H$ defined by $G_0(x) = F(\Gamma_0(x))$ and $g_0(t, x) = f(t, \Gamma_0(x))$ satisfy all assumptions (A1)–(A3). By Proposition 3.3, there exist $b_0 \in [\alpha(\varepsilon), \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$, a continuous map $x_0(.) : [t_0, +\infty[\to H$ and step functions $\theta_0(.), \bar{\theta}_0(.) : [t_0, +\infty[\to [t_0, +\infty[$ such that

(i) $x_0(t_0) = x_0$, $x_0(\bar{\theta}_0(t)) \in P(x_0(\theta_0(t)))$ and $x_0(\theta_0(t)) \in K \cap \overline{B}(\varphi(0), r)$ for all t in $[t_0, t_0 + b_0]$;

(ii) $\dot{x}_0(t) - f(t, \Gamma_0(x_0(\theta_0(t)))) \in F(\Gamma_0(x_0(\theta_0(t)))) + \varepsilon B$ for almost all $t \in [t_0, t_0 + b_0]$;

(iii) $\|\dot{x}_0(t)\| \leq \lambda + a + m(t)$ for almost all $t \in [t_0, t_0 + b_0]$;

(iv) For all $t \in [t_0, t_0 + b_0]$, $0 \leq t - \theta_0(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leq \bar{\theta}_0(t) - t \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$.

Set $t_1 = t_0 + b_0$ and $x(t) = x_0(t)$ for all $t \in [t_0, t_1]$.

We reiterate this process for constructing sequences $x_i(.) : [t_i, +\infty[\to H$, $\theta_i(.), \bar{\theta}_i(.) : [t_i, +\infty[\to [t_i, +\infty[$, $\Gamma_i : H \to \mathcal{C}_a$ and continuous function $x(.) : [-a, t_{i+1}] \to H$ satisfying the following assertions for $i \geq 0$:

(a) $t_{i+1} = t_i + b_i$, $x(t) = x_i(t)$, $x_i(\bar{\theta}_i(t)) \in P(x_i(\theta_i(t)))$ and $x_i(\theta_i(t)) \in K \cap \overline{B}(\varphi(0), r)$ for all $t \in [t_i, t_{i+1}]$;

(b) $\dot{x}_i(t) - f(t, \Gamma_i(x_i(\theta_i(t)))) \in F(\Gamma_i(x_i(\theta_i(t)))) + \varepsilon B$ for almost all $t \in [t_i, t_{i+1}]$;

(c) $\|\dot{x}_i(t)\| \leq \lambda + a + m(t)$ for almost all $t \in [t_i, t_{i+1}]$;

(d) For all $t \in [t_i, t_{i+1}]$, $0 \leq t - \theta_i(t) \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leq \bar{\theta}_i(t) - t \leq \frac{1}{4}\eta(\frac{\varepsilon}{4})$;

(e) For all $z \in H$

$$\Gamma_i(z)(s) = \begin{cases} x\left(t_i + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s\right) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})} x(t_i) + \left(1 + \frac{4s}{\eta(\frac{\varepsilon}{4})}\right) z & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0. \end{cases}$$

The assertions (a)–(e) are fulfilled for $i = 0$. Let now $i \geq 1$. Assume that (a)–(e) are satisfied for any $i = 1, \ldots, q$. If $T \leq t_{q+1}$, then we stop this process of iterations and we get (a)–(e) satisfied with $t_q < T \leq t_{q+1}$. On the other case: $t_{q+1} < T$, consider the function $\Gamma_{q+1} : H \to \mathcal{C}_a$ defined as follows: for all $z \in H$,

$$\Gamma_{q+1}(z)(s) = \begin{cases} x\left(t_{q+1} + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s\right) & -a \leq s \leq -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})} x(t_{q+1}) + \left(1 + \frac{4s}{\eta(\frac{\varepsilon}{4})}\right) z & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leq s \leq 0. \end{cases}$$

The set-valued map $G_{q+1} : K \to 2^H$ and the map $g_{q+1} : \mathbb{R} \times H \to H$, defined by $G_{q+1}(x) = F(\Gamma_{q+1}(x))$ and $g_{q+1}(t, x) = f(t, \Gamma_{q+1}(x))$, satisfy all assumptions (A1)–(A3). In view of Proposition 3.3, there exist $b_{q+1} \in [\alpha(\varepsilon), \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$, a continuous function $x_{q+1}(.)$, and step functions $\theta_{q+1}(.)$ and $\bar{\theta}_{q+1}(.)$, defined on $[t_{q+1}, +\infty[$, satisfying (a)–(e) for $i = q+1$. Set $t_{q+2} = t_{q+1} + b_{q+1}$, $x(t) = x_{q+1}(t)$ for all $t \in [t_{q+1}, t_{q+2}]$. Thus the conditions (a)–(e) are satisfied for $i = q+1$. Since $t_{i+1} - t_i = b_i \geq \alpha(\varepsilon)$, there exists an integer s such that $t_s < T \leq t_{s+1}$. Further on, we define the functions $\theta(.), \bar{\theta}(.), \tilde{\theta}(.) : [0, +\infty[\to [0, +\infty[$ and $\Gamma : [0, +\infty[\to \mathcal{C}_a$ as follows: for all $t \in [t_q, t_{q+1}[$, $\theta(t) = \theta_q(t)$, $\bar{\theta}(t) = \bar{\theta}_q(t)$, $\tilde{\theta}(t) = t_q$ and $\Gamma(t) = \Gamma_q(x_q(\theta_q(t)))$. Hence the proof of Proposition 4.1 is complete. ■

Now we are ready to prove our Theorem 2.5. Let $k \in \mathbb{N}^*$ be such that

$$\frac{1}{k} < \inf\left\{a, \frac{r}{2}\right\}.$$

By Proposition 4.1, we can define sequences $s_k \in \mathbb{N}^*$, $(t_q^k)_{0 \leq q \leq s_k+1}$, $x_k(.) : [-a, +\infty[\to H$, $\theta_k(.), \bar{\theta}_k(.), \tilde{\theta}_k(.) : [0, +\infty[\to [0, +\infty[$ and $\Gamma_k(.) : [0, +\infty[\to \mathcal{C}_a$ such that

(1) $x_k(\theta_k(t)) \in \overline{B}(\varphi(0), r)$ and $x_k(\bar{\theta}_k(t)) \in P(x_k(\theta_k(t)))$, for all $t \in [0, T]$ and $x_k \equiv \varphi$ on $[-a, 0]$;

(2) $\dot{x}_k(t) - f(t, \Gamma_k(t)) \in F(\Gamma_k(t)) + \frac{1}{k}B$ for almost all t in $[0, T]$;

(3) $0 \leq t - \theta_k(t) \leq \frac{1}{4}\eta(\frac{1}{4k})$, $0 \leq t - \tilde{\theta}_k(t) \leq \frac{1}{4}\eta(\frac{1}{4k})$ and $0 \leq \bar{\theta}_k(t) - t \leq \frac{1}{4}\eta(\frac{1}{4k})$ for all $t \in [0, T]$;

(4) $\|\dot{x}(t)\| \leq \lambda + a + m(t)$ for almost all $t \in [0, T]$;

(5) For all $t \in [0, T]$

$$\Gamma_k(t)(s) = \begin{cases} x_k\big(\tilde{\theta}(t) + \frac{1}{4}\eta(\frac{1}{4k}) + s\big) & -a \leq s \leq -\frac{1}{4}\eta(\frac{1}{4k}) \\ -\frac{4s}{\eta(\frac{1}{4k})} x_k(\tilde{\theta}_k(t)) + \Big(1 + \frac{4s}{\eta(\frac{1}{4k})}\Big) x_k(\theta_k(t)) & -\frac{1}{4}\eta(\frac{1}{4k}) \leq s \leq 0. \end{cases}$$

Claim 4.2. $\Gamma_k(t) \in \mathcal{K}_0 \cap \overline{B}_a(\varphi, r)$ *and* $\|T(t)x_k - \Gamma_k(t)\|_{+\infty} \leq \frac{1}{k}$ *for all* $t \in [0, T]$.

Proof. First, remark that for all $t, \bar{t} \in [-a, T]$ such that $|t - \bar{t}| \leq \eta(\rho)$ we have $\|x_k(t) - x_k(\bar{t})\| \leq 2\rho$. Indeed, let $t, \bar{t} \in [-a, T]$ such that $|t - \bar{t}| \leq \eta(\rho)$. If $t, \bar{t} \in [0, T]$ and $\bar{t} \leq t$, by (4) we have

$$\|x_k(t) - x_k(\bar{t})\| \leq \int_{\bar{t}}^{t} \|\dot{x}_k(s)\| ds \leq \int_{\bar{t}}^{t} (\lambda + a + m(s)) ds \leq \rho.$$

If $t, \bar{t} \in [-a, 0]$, by construction, $\|x_k(t) - x_k(\bar{t})\| = \|\varphi(t) - \varphi(\bar{t})\| \leq \rho$. If $t \in [0, T]$ and $\bar{t} \in [-a, 0]$, one has $|t| \leq \eta(\rho)$ and $|\bar{t}| \leq \eta(\rho)$. Then

$$\|x_k(t) - x_k(\bar{t})\| \leq \|x_k(t) - x_k(0)\| + \|\varphi(\bar{t}) - \varphi(0)\| \leq \rho + \rho = 2\rho.$$

Hence we conclude that for all $t, \bar{t} \in [-a, T]$ such that $|t - \bar{t}| \leq \eta(\rho)$, we have $\|x_k(t) - x_k(\bar{t})\| \leq 2\rho$. Now, let $t \in [0, T]$, if $-a \leq s \leq -\frac{1}{4}\eta(\frac{1}{4k})$ we have

$$\begin{aligned} &\Big|\tilde{\theta}_k(t) + \frac{1}{4}\eta\Big(\frac{1}{4k}\Big) + s - s\Big| \\ &\leq \tilde{\theta}_k(t) - t + t + \frac{1}{4}\eta\Big(\frac{r}{8}\Big) \\ &\leq \frac{1}{4}\eta\Big(\frac{1}{4k}\Big) + \frac{1}{4}\eta\Big(\frac{r}{8}\Big) + \frac{1}{4}\eta\Big(\frac{r}{8}\Big) \leq \eta\Big(\frac{r}{8}\Big). \end{aligned}$$

Then

$$\|\Gamma_k(t)(s)-\varphi(s)\| = \Big\|x_k\Big(\tilde{\theta}_k(t)+\frac{1}{4k}\eta\Big(\frac{1}{4k}\Big)+s\Big)-x_k(s)\Big\| \le 2\frac{r}{8} \le r.$$

If $-\frac{1}{4}\eta(\frac{1}{4k}) \le s \le 0$ we get

$$\Big|\theta_k(t)-\tilde{\theta}_k(t)\Big| \le |\theta_k(t)-t|+|\tilde{\theta}_k(t)-t| \le \frac{1}{4}\eta\Big(\frac{r}{8}\Big)+\frac{1}{4}\eta\Big(\frac{r}{8}\Big) \le \eta\Big(\frac{r}{8}\Big)$$

and

$$|\theta_k(t)-s| \le \frac{1}{4}\eta\Big(\frac{r}{8}\Big)+\frac{1}{4}\eta\Big(\frac{1}{4k}\Big) \le \eta\Big(\frac{r}{8}\Big).$$

So

$$\begin{aligned}
&\|\Gamma_k(t)(s)-\varphi(s)\| \\
&= \Big\|x_k(s)+\frac{4s}{\eta(\frac{1}{4k})}x_k(\tilde{\theta}_k(t))-\Big(1+\frac{4s}{\eta(\frac{1}{4k})}\Big)x_k(\theta_k(t))\Big\| \\
&\le \|x_k(s)-x_k(\theta_k(t))\|+\|x_k(\tilde{\theta}_k(t))-x_k(\theta_k(t))\| \le \frac{4r}{8} \le r.
\end{aligned}$$

Thus we deduce $\Gamma_k(t) \in \overline{B}_a(\varphi,r)$. Since $\Gamma_k(t)(0)=x_k(\theta_k(t)) \in K$, we have $\Gamma_k(t) \in \mathcal{K}_0 \cap \overline{B}_a(\varphi,r)$.

For the second assertion, let $t \in [0,T]$. If $-a \le s \le -\frac{1}{4}\eta(\frac{1}{4k})$ we have

$$\Big|\tilde{\theta}_k(t)+\frac{1}{4}\eta\Big(\frac{1}{4k}\Big)+s-t-s\Big| \le |\tilde{\theta}_k(t)-t|+\frac{1}{4}\eta\Big(\frac{1}{4k}\Big) \le \frac{1}{4}\eta\Big(\frac{1}{4k}\Big)+\frac{1}{4}\eta\Big(\frac{1}{4k}\Big) \le \eta\Big(\frac{1}{4k}\Big).$$

Then

$$\|T(t)x_k(s)-\Gamma_k(t)(s)\| = \Big\|x_k(t+s)-x_k\Big(\tilde{\theta}_k(t)+\frac{1}{4}\eta\Big(\frac{1}{4k}\Big)+s\Big)\Big\| \le \frac{1}{2k}.$$

If $-\frac{1}{4}\eta(\frac{1}{4k}) \le s \le 0$ we get

$$\Big|\theta_k(t)-t-s\Big| \le |\theta_k(t)-t|+|s| \le \frac{1}{4}\eta\Big(\frac{1}{4k}\Big)+\frac{1}{4}\eta\Big(\frac{1}{4k}\Big) \le \eta\Big(\frac{1}{4k}\Big)$$

and

$$|\theta_k(t)-\tilde{\theta}_k(t)| \le |\theta_k(t)-t|+|\tilde{\theta}_k(t)-t| \le \frac{1}{4}\eta\Big(\frac{1}{4k}\Big)+\frac{1}{4}\eta\Big(\frac{1}{4k}\Big) \le \eta\Big(\frac{1}{4k}\Big).$$

So

$$\begin{aligned}
&\|T(t)x_k(s) - \Gamma_k(t)(s)\| \\
&= \left\| x_k(t+s) + \frac{4s}{\eta(\frac{1}{4k})} x_k(\tilde{\theta}_k(t)) - \Big(1 + \frac{4s}{\eta(\frac{1}{4k})}\Big) x_k(\theta_k(t)) \right\| \\
&\leq \|x_k(t+s) - x_k(\theta_k(t))\| + \|x_k(\tilde{\theta}_k(t)) - x_k(\theta_k(t))\| \\
&\leq \frac{1}{2k} + \frac{1}{2k} \leq \frac{1}{k}.
\end{aligned}$$

Thus $\|T(t)x_k - \Gamma_k(t)\|_{+\infty} \leq \frac{1}{k}$ for all $t \in [0,T]$. ∎

Now we continue the proof of Theorem 2.5. By (4) the sequence $(x_k(.))_k$ is equicontinuous. In order to apply Ascoli-Arzela theorem, we are going to show that for every $t \in [0,T]$, the set $S(t) = \{x_k(t) : k \geq k_0\}$, where $k_0 \in \mathbb{N} \setminus \{0\}$, is relatively compact in H. By (1), for all $t \in [0,T]$, $x_k(\theta_k(t)) \in K_0$. Thus for all $t \in [0,T]$, the set $\{x_k(\theta_k(t)) : k \geq k_0\}$ is relatively compact in H, hence by Lemma 2.4, $\beta\Big(\{x_k(\theta_k(t)) : k \geq k_0\}\Big) = 0$. Next, for all $t \in [0,T]$

$$\beta(S(t)) = \beta\Big(\{x_k(t) : k \geq k_0\}\Big) = \beta\Big(\{x_k(t) - x_k(\theta_k(t)) + x_k(\theta_k(t)) : k \geq k_0\}\Big).$$

Then by Lemma 2.4, we obtain

$$\begin{aligned}
\beta(S(t)) &\leq \beta\Big(\{x_k(t) - x_k(\theta_k(t)) : k \geq k_0\}\Big) + \beta\Big(\{x_k(\theta_k(t)) : k \geq k_0\}\Big) \\
&\leq \beta\Big(\{x_k(t) - x_k(\theta_k(t)) : k \geq k_0\}\Big) \\
&= \beta\left(\left\{ \int_{\theta_k(t)}^t \dot{x}_k(s)ds : k \geq k_0 \right\}\right) \\
&\leq \beta\left(B\left(0, \int_{\theta_k(t)}^t (\lambda + a + m(s))ds\right)\right) \\
&= 2\int_{\theta_k(t)}^t (\lambda + a + m(s))ds.
\end{aligned}$$

Since $\int_{\theta_k(t)}^t (\lambda + a + m(s))ds$ converges to 0 as $k \to \infty$, $\beta(S(t)) = 0$. Hence $S(t)$ is relatively compact in H. Therefore, by Arzelà-Ascoli's Theorem, we can select a subsequence, again denoted by $(x_k(.))_k$ which converges uniformly to an absolutely continuous function $x(.)$ on $[0,T]$. Moreover $\dot{x}_k(.)$ converges weakly to $\dot{x}(.)$ in $L^2([0,T],H)$. Also, since all functions $x_k(.)$ agree with $\varphi(.)$ on $[-a,0]$, we deduce that $(x_k(.))_k$ converges uniformly to $x(.)$ on $[-a,T]$ extending $x(.)$ in such a way that $x(.) \equiv \varphi(.)$ on $[-a,0]$. Now, let $t \in [0,T]$. Since

$$\lim_{k\to+\infty} \|x(t) - x_k(\theta_k(t))\| = 0,$$

$x_k(\theta_k(t)) \in K_0$ and K_0 is closed we obtain $x(t) \in K_0 \subset K$. By (H1), we conclude that $x(t) \in P(x(t))$ for all $t \in [0,T]$. It remains to prove that if $t' < t$ then $x(t) \in P(x(t'))$. Let $t', t \in [0,T]$ be such that $t' < t$. Then for k large enough we can find $p, q \in \{0,\dots,s_k\}$ such that $q = p+i$ where $0 \le i \le s_k$, $t' \in [t_p^k, t_{p+1}^k]$, $t \in [t_q^k, t_{q+1}^k]$, $\lim_{k\to+\infty} t_q^k = t$ and $\lim_{k\to+\infty} t_p^k = t'$. Note that, by construction, one has $x_k(t_{q-1}^k) \in P(x_k(t_{q-2}^k))$, which together with (H1) gives

$$P(x_k(t_{q-1}^k)) \subseteq P(x_k(t_{q-2}^k)).$$

Similarly, $P(x_k(t_{q-2}^k)) \subseteq P(x_k(t_{q-3}^k))$. If we continue for $i-1$ steps, we obtain $P(x_k(t_{q-1}^k)) \subseteq P(x_k(t_p^k)))$. By the fact that $x_k(t_q^k) \in P(x_k(t_{q-1}^k))$, we conclude that $x_k(t_q^k) \in P(x_k(t_p^k))$. By letting $k \to +\infty$, we get $x(t) \in P(x(t'))$. Remark that, from Claim 4.2, we deduce that $\Gamma_k(t)$ converges to $T(t)x$ in $\mathcal{C}_a$ and $T(t)x \in \mathcal{K}_0 \cap \overline{B}_a(\varphi, r)$.

Proposition 4.3. *For almost all $t \in [0,T]$, $\dot{x}(t) - f(t, T(t)x) \in \partial_c V(x(t))$.*

Proof. The weak convergence of $\dot{x}_k(.)$ to $\dot{x}(.)$ in $L^2([0,T],H)$ and the Mazur's Lemma entail $\dot{x}(t) \in \bigcap_k \bar{co}\big\{\dot{x}_m(t) : m \ge k\big\}$, for almost all $t \in [0,T]$. Then for all $y \in H$,

$$\langle y, \dot{x}(t)\rangle \ \le\ \inf_m \sup_{k\ge m} \langle y, \dot{x}_k(t)\rangle \text{ a.e. on } [0,T].$$

By (2) and (H2) one has

$$\dot{x}_k(t) \in \left(\partial_c V(x_k(\theta_k(t))) + f(t, \Gamma_k(t)) + \frac{1}{k}B\right) \text{ a.e. on } [0,T].$$

Thus

$$\langle y, \dot{x}(t)\rangle \ \le\ \inf_m \sup_{k\ge m} \sigma\left(y, \partial_c V(x_k(\theta_k(t))) + f(t, \Gamma_k(t)) + \frac{1}{k}B\right) \text{ a.e. on } [0,T],$$

from which we deduce that

$$\langle y, \dot{x}(t)\rangle \ \le\ \limsup_{k\to\infty} \sigma\left(y, \partial_c V(x_k(\theta_k(t))) + f(t, \Gamma_k(t)) + \frac{1}{k}B\right) \text{ a.e. on } [0,T].$$

By Proposition 2.3, the function $x \mapsto \sigma\big(y, \partial_c V(x)\big)$ is $u.s.c$ and hence we get

$$\langle y, \dot{x}(t)\rangle \ \leq \ \sigma\Big(y, \partial_c V(x(t)) + f(t, T(t)x)\Big) \text{ a.e. on } [0,T].$$

So, the convexity and the closedness of the set $\partial_c V(x(t))$ (see [6, 7]) ensure

$$\dot{x}(t) - f(t, T(t)x) \in \partial_c V(x(t)) \text{ a.e. on } [0,T].$$

■

In the sequel, we need the following proposition.

Proposition 4.4. *The set* $\big\{\langle p, \dot{x}(t)\rangle, p \in \partial_c V(x(t))\big\}$ *is reduced to the singleton* $\big\{\frac{d}{dt}V(x(t))\big\}$ *for almost every* $t \in [0,T]$.

Proof. Since $x(.)$ is absolutely continuous function and V is locally Lipschitz continuous, then the function $Vox(.)$ is absolutely continuous and for almost all t there exists $\frac{d}{dt}V(x(t))$. Let $t \in [0,T]$ be such that there exist both $\dot{x}(t)$ and $\frac{d}{dt}V(x(t))$. There exists $\delta > 0$ such that $x(t+h) - x(t) - h\dot{x}(t) = r(h)$ for every $|h| < \delta$, where $\lim_{h\to 0} \|r(h)\|/h = 0$. Since a locally Lipschitz function on a compact set is globally Lipschitz continuous, we can assume that

$$|V(x(t+h)) - V(x(t) + h\dot{x}(t))| \leq \lambda \|r(h)\|, \ \lambda \in \mathbb{R}^+$$

whenever $|h| < \delta$. Consequently, the function $h \to V(x(t) + h\dot{x}(t))$ is differentiable at $h = 0$, and its derivative is the same as the derivative of $h \to V(x(t+h))$ at $h = 0$. Hence

$$\frac{d}{dt}V(x(t)) = \lim_{h\to 0} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}. \tag{4.1}$$

Since V is uniformly regular over K_0, there exists an open set U containing K_0 and such that V is uniformly regular over U. Then there exists $\beta \geq 0$ such that for all $x \in U$ and for all $\xi \in \partial_p V(x)$ one has

$$\langle \xi, x' - x\rangle \ \leq V(x') - V(x) + \beta\|x' - x\|^2 \quad \text{for all } x' \in U. \tag{4.2}$$

By construction, $x(t) \in K_0 \subset U$, hence there exists $\rho > 0$ such that $B(x(t), \rho) \subset U$. In view of the convexity of $B(x(t), \rho)$ there exists $\nu > 0$ such that $x(t) + h\dot{x}(t) \in U$ and $x(t) - h\dot{x}(t) \in U$ whenever $0 < h < \nu$. Now, let $0 < h < \nu$. Applying the inequality (4.2) with $x' = x(t) + h\dot{x}(t)$ and $x = x(t)$, we have

$$\langle \xi, h\dot{x}(t)\rangle \ \leq V(x(t) + h\dot{x}(t)) - V(x(t)) + \beta\|h\dot{x}(t)\|^2.$$

Then

$$\langle \xi, \dot{x}(t) \rangle \leq \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h} + \beta h \|\dot{x}(t)\|^2.$$

By passing to the limit, we get

$$\langle \xi, \dot{x}(t) \rangle \leq \lim_{h \to 0^+} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}.$$

By (4.1), it follows that

$$\max \Big\{ \langle \xi, \dot{x}(t) \rangle, \xi \in \partial_p V(x) \Big\} \leq \frac{d}{dt} V(x(t)).$$

In view of Proposition (2.3) and Proposition (2.1), one has

$$V^o(x(t), \dot{x}(t)) \leq \frac{d}{dt} V(x(t)). \tag{4.3}$$

If we Apply the inequality (4.2) with $x' = x(t) + h(-\dot{x}(t))$ and $x = x(t)$, we obtain by the same argument

$$\langle \xi, -\dot{x}(t) \rangle \leq \lim_{h \to 0^+} \frac{V(x(t) + (-h)\dot{x}(t)) - V(x(t))}{h}.$$

Thus

$$\langle \xi, -\dot{x}(t) \rangle \leq - \lim_{h \to 0^-} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}.$$

Consequently

$$V^o(x(t), -\dot{x}(t)) \leq -\frac{d}{dt} V(x(t)).$$

Since $V^o(x(t), -\dot{x}(t)) = -V_o(x(t), \dot{x}(t))$, we have

$$-V_o(x(t), \dot{x}(t)) \leq -\frac{d}{dt} V(x(t)).$$

In other words

$$\frac{d}{dt} V(x(t)) \leq V_o(x(t), \dot{x}(t)). \tag{4.4}$$

By (4.3) and (4.4), we deduce that

$$V^o(x(t), \dot{x}(t)) \leq \frac{d}{dt} V(x(t)) \leq V_o(x(t), \dot{x}(t)),$$

which implies that

$$V^o(x(t),\dot{x}(t)) = \frac{d}{dt}V(x(t)) = V_o(x(t),\dot{x}(t)).$$

This means that for almost all t the set $\{\langle p,\dot{x}(t)\rangle, p\in\partial_c V(x(t))\}$ reduces to the singleton $\{\frac{d}{dt}V(x(t))\}$. ■

Proposition 4.5. $\dot{x}(t)\in f(t,T(t)x)+F(T(t)x)$ *for almost all* $t\in[0,T]$.

Proof. In view of Proposition 4.3 and Proposition 4.4, we obtain

$$\frac{d}{dt}V(x(t)) \;=\; \langle\dot{x}(t),\dot{x}(t)-f(t,T(t)x)\rangle \text{ a.e. on } [0,T].$$

Therefore,

$$V(x(T))-V(x_0)=\int_0^T \|\dot{x}(s)\|^2\,ds-\int_0^T\langle\dot{x}(s),f(s,T(s)x)\rangle ds. \tag{4.5}$$

For simplicity, in the rest of the paper, we take $t^k_{s_k+1}=T$. On the other hand, by construction, for all $q=0,\dots,s_k$ and $t\in]t^k_q,t^k_{q+1}[$ we have

$$\left(\dot{x}_k(t)-f(t,\Gamma_k(t))+\frac{1}{k}b_q\right)\in\partial_c V(x_k(t^k_q)),$$

where $b_q\in B$. Since V is uniformly regular over K_0, by Definition 2.2 there exists $\beta\geq 0$ such that

$$\begin{aligned}
&V(x_k(t_k^{q+1}))-V(x_k(t_k^q))\\
&\geq\left\langle x_k(t_k^{q+1})-x_k(t_k^q),\dot{x}_k(t)-f(t,\Gamma_k(t))+\frac{1}{k}b_q\right\rangle\\
&\quad-\beta\,\|x_k(t_k^{q+1})-x_k(t_k^q)\|^2\\
&\geq\left\langle\int_{t_k^q}^{t_k^{q+1}}\dot{x}_k(s)ds,\dot{x}_k(t)-f(t,\Gamma_k(t))+\frac{1}{k}b_q\right\rangle\\
&\quad-\beta\,\|x_k(t_k^{q+1})-x_k(t_k^q)\|^2\\
&\geq\int_{t_k^q}^{t_k^{q+1}}\langle\dot{x}_k(s),\dot{x}_k(s)\rangle ds-\int_{t_k^q}^{t_k^{q+1}}\Big\langle\dot{x}_k(s),f(s,\Gamma_k(s))\Big\rangle ds\\
&\quad+\frac{1}{k}\int_{t_k^q}^{t_k^{q+1}}\langle\dot{x}_k(s),b_q\rangle ds-\beta\,\|x_k(t_k^{q+1})-x_k(t_k^q)\|^2\,.
\end{aligned}$$

By adding, we obtain

$$
\begin{aligned}
& V(x_k(T)) - V(x_0) \\
& \geq \int_0^T \| \dot{x}_k(s) \|^2 \, ds - \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \Big\langle \dot{x}_k(s), f(s, \Gamma_k(s)) \Big\rangle ds \\
& \quad + \frac{1}{k} \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \langle \dot{x}_k(s), b_q \rangle ds - \sum_{q=0}^{s_k} \beta \| x_k(t_k^{q+1}) - x_k(t_k^q) \|^2 .
\end{aligned}
\tag{4.6}
$$

Claim 4.6.

$$
\lim_{k \to +\infty} \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \Big\langle \dot{x}_k(s), f(s, \Gamma_k(s)) \Big\rangle ds = \int_0^T \Big\langle \dot{x}(s), f(s, T(s)x) \Big\rangle ds.
$$

Proof. We have

$$
\begin{aligned}
& \left| \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \Big\langle \dot{x}_k(s), f(s, \Gamma_k(s)) \Big\rangle ds - \int_0^T \Big\langle \dot{x}(s), f(s, T(s)x) \Big\rangle ds \right| \\
& = \left| \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \Big(\Big\langle \dot{x}_k(s), f(s, \Gamma_k(s)) \Big\rangle - \Big\langle \dot{x}(s), f(s, T(s)x) \Big\rangle \Big) ds \right| \\
& \leq \left| \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \Big(\Big\langle \dot{x}_k(s), f(s, \Gamma_k(s)) \Big\rangle - \Big\langle \dot{x}_k(s), f(s, T(s)x) \Big\rangle \Big) ds \right| \\
& \quad + \left| \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \Big(\Big\langle \dot{x}_k(s), f(s, T(s)x) \Big\rangle - \Big\langle \dot{x}(s), f(s, T(s)x) \Big\rangle \Big) ds \right| \\
& \leq \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \left| \Big\langle \dot{x}_k(s), f(s, \Gamma_k(s)) \Big\rangle - \Big\langle \dot{x}_k(s), f(s, T(s)x) \Big\rangle \right| ds \\
& \quad + \left| \int_0^T \Big(\Big\langle \dot{x}_k(s), f(s, T(s)x) \Big\rangle - \Big\langle \dot{x}(s), f(s, T(s)x) \Big\rangle \Big) ds \right| \\
& \leq \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \|\dot{x}_k(s)\| \|f(s, \Gamma_k(s)) - f(s, T(s)x)\| ds \\
& \quad + \left| \int_0^T \Big(\Big\langle \dot{x}_k(s), f(s, T(s)x) \Big\rangle - \Big\langle \dot{x}(s), f(s, T(s)x) \Big\rangle \Big) ds \right|.
\end{aligned}
$$

Since

$$\| \dot{x}_k(t) \| \leq \lambda + a + m(t), \quad \lim_{k\to+\infty} f(s,\Gamma_k(s)) = f(s,T(s)x),$$

and $\dot{x}_k(.)$ converges weakly to $\dot{x}(.)$, the last term in the inequality converges to 0. This completes the proof of the Claim. □

Claim 4.7.

$$\lim_{k\to+\infty} \sum_{q=0}^{s_k} \beta \|x_k(t_k^{q+1}) - x_k(t_k^q)\|^2 = 0 \text{ and } \lim_{k\to+\infty} \frac{1}{k} \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \langle \dot{x}_k(s), b_q \rangle ds = 0.$$

Proof. By (4) and (2.1) we have

$$\begin{aligned}
\|x_k(t_k^{q+1}) - x_k(t_k^q)\|^2 &\leq \left(\int_{t_k^q}^{t_k^{q+1}} \|\dot{x}_k(t)\| dt \right)^2 \\
&\leq \left(\int_{t_k^q}^{t_k^{q+1}} \Big(\lambda + a + m(t)\Big) dt \right)^2 \\
&\leq \frac{1}{4k} \int_{t_k^q}^{t_k^{q+1}} \Big(\lambda + a + m(t)\Big) dt.
\end{aligned}$$

Then

$$\sum_{q=0}^{s_k} \|x_k(t_k^{q+1}) - x_k(t_k^q)\|^2 \leq \frac{1}{4k} \int_0^T \Big(\lambda + a + m(t)\Big) dt.$$

Hence

$$\lim_{k\to+\infty} \sum_{q=0}^{s_k} \beta \|x_k(t_k^{q+1}) - x_k(t_k^q)\|^2 = 0.$$

By the same argument, one has

$$\begin{aligned}
\left| \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \langle \dot{x}_k(s), b_q \rangle ds \right| &\leq \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} |\langle \dot{x}_k(s), b_q \rangle| ds \\
&\leq \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \|\dot{x}_k(s)\| \|b_q\| ds \\
&\leq \sum_{q=0}^{s_k} \int_{t_k^q}^{t_k^{q+1}} \Big(\lambda + a + m(t)\Big) dt \\
&= \int_0^T \Big(\lambda + a + m(t)\Big) dt.
\end{aligned}$$

Thus

$$\lim_{k\to+\infty}\frac{1}{k}\sum_{q=0}^{s_k}\int_{t_k^q}^{t_k^{q+1}}\langle\dot{x}_k(s),b_q\rangle ds=0.$$

□

By passing to the limit for $k\to\infty$ in (4.6) and using the continuity of the function V on the ball $B(\varphi(0),r)$, we obtain

$$V(x(T))-V(x_0)\ \geq\ \limsup_{k\to+\infty}\int_0^T\parallel\dot{x}_k(s)\parallel^2 ds-\int_0^T\Big\langle\dot{x}(s),f(s,T(s)x)\Big\rangle ds.$$

Moreover, by (4.5), we have $\parallel\dot{x}(.)\parallel_2^2\geq\limsup_{k\to+\infty}\parallel\dot{x}_k(.)\parallel_2^2$ and since $\parallel\dot{x}(.)\parallel_2^2\leq\liminf_{k\to+\infty}\parallel\dot{x}_k(.)\parallel_2^2$, we get $\parallel\dot{x}(.)\parallel_2^2=\lim_{k\to+\infty}\parallel\dot{x}_k(.)\parallel_2^2$. Finally, $(\dot{x}_k(.))_k$ converges strongly in $L^2([0,T],H)$ to $\dot{x}(.)$. Hence there exists a subsequence (still denoted $(\dot{x}_k(.))_k$) which converges point-wisely a.e. to $\dot{x}(.)$. Now, since

$$\Big(\dot{x}_k(t)-f(t,\Gamma_k(t))\Big)\in\Big(F(\Gamma_k(t))+\frac{1}{k}B\Big)\text{ a.e. on }[0,T],$$

one has

$$d_{grF}\Big(\Big(\Gamma_k(t),\dot{x}_k(t)-f(t,\Gamma_k(t))\Big)\Big)\ \leq\ \frac{1}{k},$$

hence

$$\lim_{k\to+\infty}d_{grF}\Big(\Big(\Gamma_k(t),\dot{x}_k(t)-f(t,\Gamma_k(t))\Big)\Big)=0,$$

from which we conclude that $d_{grF}\Big(\big(T(t)x,\dot{x}(t)-f(t,T(t)x)\big)\Big)=0$ and as F has a closed graph, we obtain for almost all $t\in[0,T]$, $\dot{x}(t)\in f(t,T(t)x)+F(T(t)x)$. The proof is complete. ■

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