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NONLINEAR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH ANTI-PERIODIC TYPE INTEGRAL BOUNDARY CONDITIONS

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Abstract

This article studies a boundary value problem of nonlinear fractional differential inclusions with anti-periodic type integral boundary conditions. Some existence results are obtained via fixed point theorems. The cases of convex-valued and nonconvex-valued right hand sides are considered. Several new results appear as a special case of the results of this paper.

Keywords: fractional differential inclusions, anti-periodic, integral boundary conditions, existence, fixed point theorems.

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1. Introduction

In this article, we discuss the existence of solutions for a boundary value problem of nonlinear fractional differential inclusions with anti-periodic type integral boundary conditions given by

$$
(1) \qquad \begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad t \in [0,T], \ T > 0, \quad 1 < q \le 2, \\ x(0) - \lambda_1 x(T) = \mu_1 \displaystyle\int_0^T g(s, x(s))ds, \\ x'(0) - \lambda_2 x'(T) = \mu_2 \displaystyle\int_0^T h(s, x(s))ds, \end{cases}
$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $g, h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ with $\lambda_1 \neq 1, \lambda_2 \neq 1$.

The boundary conditions in problem (1) reduce to anti-periodic boundary conditions by choosing $\lambda_1 = -1 = \lambda_2, \mu_1 = 0 = \mu_2$. So the boundary conditions of (1) can be thought as an extension of anti-periodic boundary conditions to integral case. For a detailed description of the integral boundary conditions and some recent work on boundary value problems of fractional order with integral boundary conditions, we refer the reader to the papers [1, 2, 3, 4, 5, 6, 8, 9, 14, 18, 19] and references therein.

The present work is motivated by a recent paper [7], in which the authors discussed the existence of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions. The aim here is to establish existence results for the problem (1), when the right hand side is convex as well as nonconvex valued. In the first result (Theorem 5) we consider the case when the right hand side has convex values, and prove an existence result via nonlinear alternative for Kakutani maps. In the second result (Theorem 7), we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result (Theorem 9), we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

The method of proof employed in this article is standard, however its exposition in the framework of problem (1) is new.

2. Preliminaries

Let us recall some basic definitions [15, 20].

Definition. For $q > 0$, the Riemann-Liouville fractional integral of order q is defined as

$$
I^q x(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{x(s)}{(t-s)^{1-q}} ds,
$$

provided the integral exists.

Definition. For an at least $(n-1)$-times absolutely continuous function $x : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q x(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} x^{(n)}(s)ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

In order to define the solution for (1), we recall the following Lemma [7].

Lemma 1. *For a given $\sigma \in C([0,T],\mathbb{R}) \cap L((0,1),\mathbb{R})$, the unique solution of the boundary value problem*

$$\begin{cases} ^cD^q x(t) = \sigma(t), \quad t \in [0,T], \ T > 0, \quad 1 < q \le 2, \\ x(0) - \lambda_1 x(T) = \mu_1 \int_0^T g(s,x(s))ds, \\ x'(0) - \lambda_2 x'(T) = \mu_2 \int_0^T h(s,x(s))ds, \end{cases} \tag{2}$$

is given by

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds, \ t \in [0,T], \end{aligned}$$

where

$$\xi_1 = \frac{1}{(\lambda_1 - 1)}, \quad \xi_2 = \frac{1}{(\lambda_2-1)(\lambda_1-1)}.$$

3. Existence results

Let us begin this section by giving a definition of the solution for the problem (1).

Definition. A function $x \in AC^1([0,T],\mathbb{R})$ is a solution of the problem (1) if $x(0) - \lambda_1 x(T) = \mu_1 \int_0^T g(s,x(s))ds$, $x'(0) - \lambda_2 x'(T) = \mu_2 \int_0^T h(s,x(s))ds$, and there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,T]$ and

$$
\begin{aligned}
x(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\
& + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\
& + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds.
\end{aligned}
$$

3.1. The Carathéodory case

In this subsection, we are concerned with the existence of solutions for the problem (1) when the right hand side has convex values. We first recall some preliminary facts.

For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

Definition. A multi-valued map $G : X \to \mathcal{P}(X)$:

(i) is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$;

(ii) is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);

(iii) is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;

(v) is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$;

(v) has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Remark 2. It is known that, if the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

Definition. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{\|y - z\| : z \in G(t)\}$$

is measurable.

Definition. A multivalued map $F : [0,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in X$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a.e. $t \in [0,T]$.

For each $y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

The consideration of this subsection is based on the following fixed point Theorem ([13]).

Theorem 3 (Nonlinear alternative for Kakutani maps) ([13])**.** *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is an upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

The following lemma will be used in the sequel.

Lemma 4 ([17])**.** *Let X be a Banach space. Let $F : [0,T] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T],X)$ to $C([0,T],X)$. Then the operator*

$$\Theta \circ S_F : C([0,T],X) \to \mathcal{P}_{cp,c}(C([0,T],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,T],X) \times C([0,T],X)$.

Theorem 5. *Assume that:*

(H_1) *$F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact convex values;*

(H_2) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,T]\times\mathbb{R};$$

(H_3) *there exists a continuous nondecreasing function* $\psi_1 : [0,\infty) \to (0,\infty)$ *and a function* $p_1 \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$|h(t,x)| \le p_1(t)\psi_1(\|x\|) \quad \text{for each } (t,x) \in [0,T] \times \mathbb{R};$$

(H_4) *there exists a continuous nondecreasing function* $\psi_2 : [0,\infty) \to (0,\infty)$ *and a function* $p_2 \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$|g(t,x)| \le p_2(t)\psi_2(\|x\|) \quad \text{for each } (t,x) \in [0,T] \times \mathbb{R};$$

(H_5) *there exists a number* $M > 0$ *such that*

$$\frac{M}{\dfrac{\psi(M)T^q}{\Gamma(q+1)}\Lambda_1\|p\| + \psi_1(M)\Lambda_2\|p_1\|_{L^1} + \psi_2(M)\Lambda_3\|p_2\|_{L^1}} > 1,$$

where

$$\begin{aligned}
\Lambda_1 &= 1 + |\xi_1\lambda_1| + q|\xi_2||\lambda_2|(1+|\lambda_1|),\\
\Lambda_2 &= |\xi_2||\mu_2|(1+|\lambda_1|)T,\\
\Lambda_3 &= |\mu_1\xi_1|.
\end{aligned}$$

Then the boundary value problem (1) *has at least one solution on* $[0,T]$.

Proof. In order to transform boundary value problem (1) into a fixed point problem, consider the multivalued operator $\Omega : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ defined by

$$\Omega(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ \displaystyle +\xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\ \displaystyle +\xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds \\ \displaystyle -\mu_1\xi_1 \int_0^T g(s,x(s))ds, \quad 0 \le t \le 1, \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$. Clearly, according to Lemma 1, the fixed points of Ω are solutions to boundary value problem (1). We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps.

As a first step, *we show that Ω is convex for each $x \in C([0,T],\mathbb{R})$*. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,T]$, we have

$$\begin{aligned} h_i(t) \;=\; & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds, \; i = 1,2. \end{aligned}$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,T]$, we have

$$\begin{aligned} & [\omega h_1 + (1-\omega)h_2](t) \\ & = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\ & \quad + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}[\omega f_1(s) + (1-\omega) f_2(s)]ds \\ & \quad + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds. \end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), then it follows that $\omega h_1 + (1-\omega)$ $h_2 \in \Omega(x)$.

Next, *we show that Ω maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$*. For a positive number ρ, let $B_\rho = \{x \in C([0,T],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) \;=\; & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds, \; t \in [0,T]. \end{aligned}$$

Then

$$
\begin{aligned}
|h(t)| \le\ & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s)| ds + \frac{|\xi_1 \lambda_1|}{\Gamma(q)} \int_0^T (T-s)^{q-1} |f(s)| ds \\
& + \frac{|\xi_2 \lambda_2 [\lambda_1 T + (1-\lambda_1) t]|}{\Gamma(q-1)} \int_0^T (T-s)^{q-2} |f(s)| ds \\
& + |\xi_2 \mu_2 [\lambda_1 T + (1-\lambda_1) t]| \int_0^T |h(s, x(s))| ds + |\mu_1 \xi_1| \int_0^T |g(s, x(s))| ds \\
\le\ & \frac{\psi(\|x\|) T^q}{\Gamma(q+1)} \left[1 + |\xi_1 \lambda_1| + q |\xi_2| |\lambda_2| (1 + |\lambda_1|)\right] \|p\| \\
& + |\xi_2| |\mu_2| (1 + |\lambda_1|) T \psi_1(\|x\|) \int_0^T p_1(s) ds + |\mu_1 \xi_1| \psi_2(\|x\|) \int_0^T p_2(s) ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|h\| \le\ & \frac{\psi(\rho) T^q}{\Gamma(q+1)} \left[1 + |\xi_1 \lambda_1| + q |\xi_2| |\lambda_2| (1 + |\lambda_1|)\right] \|p\| \\
& + |\xi_2| |\mu_2| (1 + |\lambda_1|) T \psi_1(\rho) \|p_1\|_{L^1} + |\mu_1 \xi_1| \psi_2(\rho) \|p_2\|_{L^1}.
\end{aligned}
$$

Now *we show that Ω maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.* Let $t', t'' \in [0,T]$ with $t' < t''$ and $x \in B_\rho$, where B_ρ is a bounded set of $C([0,T],\mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$
\begin{aligned}
|h(t'') - h(t')| \le\ & \left| \frac{\psi(\rho) \|p\|}{\Gamma(q)} \int_0^{t'} [(t''-s)^{q-1} - (t'-s)^{q-1}] ds \right| \\
& + \left| \frac{\psi(\rho) \|p\|}{\Gamma(q)} \int_{t'}^{t''} (t''-s)^{q-1} f(s) ds \right| \\
& + \frac{|\xi_2 \lambda_2 (1-\lambda_1)| |t'' - t'| \psi(\rho) \|p\|}{\Gamma(q-1)} \int_0^T (T-s)^{q-2} ds \\
& + |\xi_2 \mu_2 (1-\lambda_1)| |t'' - t'| \psi_1(\rho) \|p_1\|_{L^1}.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t'' - t' \to 0$. Since Ω satisfies the above three assumptions, then it follows by the Arzelá-Ascoli theorem that $\Omega : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

In our next step, *we show that Ω has a closed graph.* Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. We need to show that $h_* \in \Omega(x_*)$. There exist $f_n \in S_{F,x_n}$ associated with $h_n \in \Omega(x_n)$ and such that for each $t \in [0,T]$,

$$\begin{aligned} h_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_n(s)ds \\ &+ \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_n(s)ds \\ &+ \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x_n(s))ds - \mu_1\xi_1 \int_0^T g(s,x_n(s))ds. \end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\ &+ \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds \\ &+ \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x_*(s))ds - \mu_1\xi_1 \int_0^T g(s,x_*(s))ds. \end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &+ \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\ &+ \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds, \ t \in [0,T]. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| &= \Bigg\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(f_n(s) - f_*(s))ds \\ &- \frac{\xi_1\lambda_1}{\Gamma(q)} \int_0^T (T-s)^{q-1}(f_n(s) - f_*(s))ds \\ &+ \frac{\xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{\Gamma(q-1)} \int_0^T (T-s)^{q-1}(f_n(s) - f_*(s))ds \\ &+ \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T [h(s,x_n(s)) - h(s,x_*(s))]ds \\ &- \mu_1\xi_1 \int_0^T [g(s,x_n(s)) - g(s,x_*(s))]ds \Bigg\| \to 0, \end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 4 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, then, we have

$$\begin{aligned} h_*(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s, x_*(s))ds - \mu_1\xi_1 \int_0^T g(s, x_*(s))ds, \ t \in [0,T], \end{aligned}$$

for some $f_* \in S_{F,x_*}$.

Finally, *we discuss a priori bounds on solutions.* Let x be a solution of (1). Then there exists $f \in L^1([0,T],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,T]$, we have, using the computations on the second step

$$\begin{aligned} |x(t)| \le & \frac{\psi(\|x\|)T^q}{\Gamma(q+1)} [1 + |\xi_1\lambda_1| + q|\xi_2||\lambda_2|(1+|\lambda_1|)] \|p\| \\ & + |\xi_2||\mu_2|(1+|\lambda_1|)T\psi_1(\|x\|) \int_0^T p_1(s)ds + |\mu_1\xi_1|\psi_2(\|x\|) \int_0^T p_2(s)ds \\ \le & \frac{\psi(\|x\|)T^q}{\Gamma(q+1)} \Lambda_1\|p\| + \psi_1(\|x\|)\Lambda_2\|p_1\|_{L^1} + \psi_2(\|x\|)\Lambda_3\|p_2\|_{L^1}. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\frac{\psi(\|x\|)T^q}{\Gamma(q+1)} \Lambda_1\|p\| + \psi_1(\|x\|)\Lambda_2\|p_1\|_{L^1} + \psi_2(\|x\|)\Lambda_3\|p_2\|_{L^1}} \le 1.$$

In view of (H_5), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,T],\mathbb{R}) : \|x\| < M+1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,T],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 3), we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1). This completes the proof. ∎

3.2. The lower semi-continuous case

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [10] for lower semi-continuous maps with decomposable values.

Definition. Let X be a nonempty closed subset of a Banach space E and let $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E.

Definition. Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

Definition. A subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,T] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,T] \times \mathbb{R}) \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,T],\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,T]\},$$

which is called the Nemytskii operator associated with F.

Definition. Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 6 ([10])**.** *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,T],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 7. *Assume that* (H_2)–(H_5) *and the following condition holds:*

(H_6) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

- (a) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*
- (b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,T]$.

Then the boundary value problem (1) *has at least one solution on* $[0,T]$.

Proof. It follows from (H_6) and (H_2) that F is of l.s.c. type. Then from Lemma 6, there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,T],\mathbb{R})$.
Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad t \in [0,T], \ T > 0, \quad 1 < q \le 2, \\ x(0) - \lambda_1 x(T) = \mu_1 \displaystyle\int_0^T g(s,x(s))ds, \\ x'(0) - \lambda_2 x'(T) = \mu_2 \displaystyle\int_0^T h(s,x(s))ds, \end{cases} \tag{3}$$

in the space $C([0,T],\mathbb{R})$. It is clear that if $x \in C([0,T],\mathbb{R})$ is a solution of the problem (3), then x is a solution to the problem (1). In order to transform the problem (3) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(x(s))ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(x(s))ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds, \ t \in [0,T]. \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 5. So we omit it. This completes the proof. ■

3.3. The Lipschitz case

Now we prove the existence of solutions for the problem (1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [12].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [16]).

Definition. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called:

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y) \text{ for each } x,y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 8 (Covitz-Nadler) ([12]). *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Definition. A measurable multi-valued function $F : [0, T] \to \mathcal{P}(X)$ is said to be integrably bounded if there exists a function $h \in L^1([0, T], X)$ such that for all $v \in F(t)$, $\|v\| \leq h(t)$ for a.e. $t \in [0, T]$.

Theorem 9. *Assume that the following conditions hold:*

(H_7) *$F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0, T] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(H_8) *$H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x - \bar{x}|$ for almost all $t \in [0, T]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in C([0, T], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq m(t)$ for almost all $t \in [0, T]$.*

(H_9) *There exist constants $c_1, c_2 > 0$ such that*

$$|h(t, x) - h(t, \bar{x})| \leq c_1|x - \bar{x}|, \quad |g(t, x) - g(t, \bar{x})| \leq c_2|x - \bar{x}|$$

for all $t \in [0, T]$ and $x, \bar{x} \in \mathbb{R}$.

Then the boundary value problem (1) has at least one solution on $[0, T]$ if

$$\frac{T^q}{\Gamma(q + 1)}\Lambda_1\|m\| + T(c_1\Lambda_2 + c_2\Lambda_3) < 1,$$

where Λ_1, Λ_2 and Λ_3 are given by (H_5).

Proof. We transform the problem (1) into a fixed point problem. Consider the set-valued map $\Omega : C([0, T], \mathbb{R}) \to \mathcal{P}(C([0, T], \mathbb{R}))$ defined at the beginning of the proof of Theorem 5. It is clear that points fixed point of Ω are solutions of the problem (1).

Note that, by the assumption (H_7), since the set-valued map $F(\cdot, x)$ is measurable, it admits a measurable selection $f : [0, T] \to \mathbb{R}$ (see Theorem III.6 [11]). Moreover, from assumption (H_8)

$$|f(t)| \leq m(t) + m(t)|x(t)|,$$

i.e., $f(\cdot) \in L^1([0, T], X)$. Therefore, the set $S_{F,x}$ is nonempty. Also note that since $S_{F,x} \neq \emptyset$, then $\Omega(x) \neq \emptyset$ for any $x \in C([0, T], \mathbb{R})$.

Now we show that the operator Ω satisfies the assumptions of Lemma 8. To show that $\Omega(x) \in \mathcal{P}_{cl}((C[0, T], \mathbb{R}))$ for each $x \in C([0, T], \mathbb{R})$, let $\{u_n\}_{n \geq 0} \in \Omega(x)$

be such that $u_n \to u$ $(n \to \infty)$ in $C([0,T],\mathbb{R})$. Then $u \in C([0,T],\mathbb{R})$ and there exist $v_n \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_n(s)ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_n(s)ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds. \end{aligned}$$

As F has compact values, we may pass to a subsequence (if necessary) to obtain that v_n converge to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) \to u(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v(s)ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v(s)ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds. \end{aligned}$$

Hence, $u \in \Omega(x)$ and $\Omega(x)$ is closed.

Next we show that Ω is a contraction on $C([0,T],\mathbb{R})$, i.e. there exists $\gamma < 1$ such that

$$H_d(\Omega(x),\Omega(\bar{x})) \le \gamma\|x-\bar{x}\| \text{ for each } x,\bar{x} \in C([0,T],\mathbb{R}).$$

Let $x,\bar{x} \in C([0,T],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t,x(t))$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_1(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_1(s)ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_1(s)ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s,x(s))ds - \mu_1\xi_1 \int_0^T g(s,x(s))ds. \end{aligned}$$

By (H_8), we have

$$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w(t) \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \quad t \in [0, T].$$

Define $U : [0, T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w(t)| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [11]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0, T]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$. For each $t \in [0, T]$, let us define

$$\begin{aligned} h_2(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds - \xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_2(s)ds \\ & + \xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_2(s)ds \\ & + \xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t] \int_0^T h(s, \bar{x}(s))ds - \mu_1\xi_1 \int_0^T g(s, \bar{x}(s))ds, \ t \in [0, T]. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |v_1(s) - v_2(s)| ds \\ & + \frac{|\lambda_1||\xi_1|}{\Gamma(q)} \int_0^T (T-s)^{q-1} |v_1(s) - v_2(s)| ds \\ & + \frac{|\xi_2\lambda_2[\lambda_1 T + (1-\lambda_1)t]|}{\Gamma(q-1)} \int_0^T (T-s)^{q-2} |v_1(s) - v_2(s)| ds \\ & + |\xi_2\mu_2[\lambda_1 T + (1-\lambda_1)t]| \int_0^T |h(s, x(s)) - h(s, \bar{x}(s))| ds \\ & + |\mu_1\xi_1| \int_0^T |g(s, x(s)) - g(s, \bar{x}(s))| ds \\ \leq & \frac{\|x - \bar{x}\| T^q}{\Gamma(q+1)} \left[1 + |\xi_1\lambda_1| + q|\xi_2||\lambda_2|(1 + |\lambda_1|)\right] \|m\| \\ & + |\xi_2||\mu_2|(1 + |\lambda_1|) T^2 c_1 \|x - \bar{x}\| + |\mu_1\xi_1| T c_2 \|x - \bar{x}\|. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \left\{ \frac{T^q}{\Gamma(q+1)} \Lambda_1 \|m\| + T(c_1\Lambda_2 + c_2\Lambda_3) \right\} \|x - \bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\Omega(x),\Omega(\bar{x})) &\le \gamma\|x-\bar{x}\| \\ &\le \left\{\frac{T^q}{\Gamma(q+1)}\Lambda_1\|m\| + T(c_1\Lambda_2 + c_2\Lambda_3)\right\}\|x-\bar{x}\|. \end{aligned}$$

Since Ω is a contraction, it follows by Lemma 8 that Ω has a fixed point x which is a solution of (1). This completes the proof. ∎

Example 10. Consider the following boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t,x(t)), \quad t\in[0,1], \\ x(0) - \dfrac{1}{2}x(1) = \dfrac{1}{2}\displaystyle\int_0^1 s^4(1+|x|)ds, \\ x'(0) - 2x'(T) = \dfrac{1}{8}\displaystyle\int_0^1 s^2(1+|x|)ds. \end{cases} \tag{4}$$

Here, $q = \frac{3}{2}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 2$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{8}$, $\xi_1 = -2$, $\xi_2 = -2$ and $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{|x|^3}{8(|x|^3+3)}, \frac{|x|}{9(|x|+1)} + \frac{1}{100}\right].$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{|x|^3}{8(|x|^3+3)}, \frac{|x|}{9(|x|+1)} + \frac{1}{100}\right) \le \frac{1}{8}, \quad x\in\mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y\in F(t,x)\} \le \frac{1}{8} = p(t)\psi(\|x\|), \quad x\in\mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|) = \frac{1}{8}$. Also $\psi_1(x) = 1+x$, $\psi_2(x) = 1+x$, $\|p_1\|_{L^1} = \frac{1}{5}$, $\|p_2\|_{L^1} = \frac{1}{3}$, $\Lambda_1 = 11$, $\Lambda_2 = \frac{3}{8}$ and $\Lambda_3 = 1$.

Further, using the condition (H_5) we find that $M > 2.438334$. Clearly, all conditions of Theorem 5 are satisfied. So there exists at least one solution of the problem (4) on $[0,1]$.

Remark 11. The results for an anti-periodic boundary value problem of fractional differential inclusions of order $q\in(1,2]$ follow as a special case by taking $\lambda_1 = -1 = \lambda_2$, $\mu_1 = 0 = \mu_2$ in the results of this article. In case of $\lambda_1 = 0 = \lambda_2$, our results become the ones for the initial-integral boundary conditions

$$x(0) = \mu_1\int_0^T g(s,x(s))ds, \; x'(0) = \mu_2\int_0^T h(s,x(s))ds.$$

For $q = 2$, we obtain new results for a second order boundary value problem with anti-periodic type integral boundary conditions. Thus, several new results appear as a special case of the results obtained in this paper by fixing the parameters involved in the problem (1).

References

[1] B. Ahmad, A. Alsaedi and B. Alghamdi, *Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions*, Nonlinear Anal. Real World Appl. **9** (2008) 1727–1740. doi:10.1016/j.nonrwa.2007.05.005

[2] B. Ahmad and J.J. Nieto, *Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions*, Bound. Value Probl. 2009, Art. ID 708576, 11 pp.

[3] B. Ahmad and V. Otero-Espinar, *Existence of solutions for fractional differential inclusions with anti-periodic boundary conditions*, Bound. Value Probl. 2009 (2009), Article ID 625347, 11 pages.

[4] B. Ahmad and S. Sivasundaram, *Existence of solutions for impulsive integral boundary value problems of fractional order*, Nonlinear Anal. Hybrid Syst. **4** (2010) 134–141. doi:10.1016/j.nahs.2009.09.002

[5] B. Ahmad and J.J. Nieto, *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory*, Topological Methods in Nonlinear Analysis **35** (2010) 295–304.

[6] B. Ahmad, S.K. Ntouyas and A. Alsaedi, *New existence results for nonlinear fractional differential equations with three-point integral boundary conditions*, Adv. Difference Equ. 2011, Art. ID 107384, 11 pp.

[7] B. Ahmad, J. Nieto and A. Alsaedi, *Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions*, Acta Math. Scientia **31B** (2011) 2122–2130. doi:10.1016/S0252-9602(11) 60388-3

[8] M. Benchohra, John R. Graef and S. Hamani, *Existence results for boundary value problems with nonlinear fractional differential equations*, Appl. Anal. **87** (2008) 851–863.

[9] A. Boucherif, *Second-order boundary value problems with integral boundary conditions*, Nonlinear Anal. **70** (2009) 364–371. doi:10.1016/j.na.2007.12.007

[10] A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988) 69–86.

[11] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[12] H. Covitz and S.B. Nadler Jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970) 5–11. doi:10.1007/BF02771543

[13] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.

[14] S. Hamani, M. Benchohra and John R. Graef, *Existence results for boundary value problems with nonlinear fractional inclusions and integral conditions*, Electronic J. Differential Equations **2010** (20) (2010) 1–16.

[15] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[16] M. Kisielewicz, Differential Inclusions and Optimal Control, Kluwer, Dordrecht, The Netherlands, 1991.

[17] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. Ser. Math. **13** (1965) 781–786.

[18] X. Liu, M. Jia and B. Wu, *Existence and uniqueness of solution for fractional differential equations with integral boundary conditions*, E.J. Qualitative Theory of Diff. Equ. **69** (2009) 1–10.

[19] H.A.H. Salem, *Fractional order boundary value problem with integral boundary conditions involving Pettis integral*, Acta Math. Scientia **31** (2011) 661–672. doi:10.1016/S0252-9602(11)60266-X

[20] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.

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