MONOTONIC SOLUTIONS FOR QUADRATIC INTEGRAL EQUATIONS

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Abstract

Using the Darbo fixed point theorem associated with the measure of noncompactness, we establish the existence of monotonic integrable solution on a half-line $\mathbb{R}_+$ for a nonlinear quadratic functional integral equation.

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1. Introduction

In this paper we study the following functional integral equation

$$(1) \qquad x(t) \; = \; g(t) \; + \; f\!\left(t, \; x(t) \int_{\alpha}^{\beta} u(t, s, x(s)) \; ds\right).$$

In particular, in a special case it cover so-called quadratic integral equations. Nonlinear quadratic functional integral equations are often applicable for instance in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, in the traffic theory and in numerous branches of mathematical physics [30]. Especially, the quadratic integral equation of Chandrasekhar type

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \varphi(s)x(s)ds$$

can be very often encountered in many applications (cf. [3, 11, 14, 17, 18, 25]).

The particular cases of our equation, were investigated for existence for both continuous (cf. [1, 12, 15, 24] and integrable solutions ([6, 10, 11]). The existence of different subclasses of solutions were proved (nonnegative functions, monotone, having limit at infinity etc.). Let us note, that the problem is investigated for finite or infinite intervals. We extend the existing results dealing the monotonicity problem in a half-line for the most complicated problem of the Uryson operators. For continuous solutions such a property was recently investigated in [22, 24], for instance.

By applying Darbo fixed point theorem associated with the measure of noncompactness, we obtain the sufficient conditions for the existence of monotonic solutions of equation (1), which are integrable. The results presented in this paper are motivated by the recent works of Banaś and Chlebowicz [6], Banaś and Rzepka [12, 13] and extend these papers in many ways.

2. Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^+$ be the interval $[0, \infty)$ and $L^1(I)$ be the space of Lebesgue integrable functions on a measurable subset I of $\mathbb{R}$, with the standard norm. In the paper we will denote a finite interval $[a, b]$ by I.

One of the most important operators studied in nonlinear functional analysis is the so-called superposition operator [2]. Assume that a function $f(t, x) = f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions i.e., it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then, to every function $x(t)$ being measurable on I, we may assign the function

$$(Fx)(t) = f(t, x(t)), \quad t \in I.$$

The operator F is called the superposition operator generated by the function f. We have the following theorem (see [30]):

Theorem 2.1. *Suppose, that f satisfies Carathéodory conditions. The superposition operator F maps the space $L^1(I)$ into $L^1(I)$ if and only if*

$$(2) \qquad\qquad |f(t,x)| \leq a(t) + b|x|,$$

for all $t \in I$ and $x \in \mathbb{R}$, where $a(t) \in L^1(I)$ and $b \geq 0$. Moreover, this operator is continuous.

Let $S = S(I)$ denote the set of measurable (in Lebesgue sense) functions on I and let *meas* stand for the Lebesgue measure in $\mathbb{R}$. Identifying the functions equal almost everywhere the set S furnished with the metric

$$d(x,y) = \inf_{a>0}[a + meas\{s : |x(s) - y(s)| \geq a\}]$$

becomes a complete space. Moreover, the convergence in measure on I is equivalent to the convergence with respect to d (Proposition 2.14 in [29]). For σ-finite subsets of $\mathbb{R}$ we say that the sequence x_n is convergent in finite measure to x if it is convergent in measure on each set T of finite measure. The compactness in such spaces we will call a "compactness in measure" ("in finite measure") and such sets have very nice properties when considered as subsets of L^p-spaces of integrable functions.

Let X be a bounded subset of $L^1(I)$. Assume that there is a family of subsets $(\Omega_c)_{0 \leq c \leq b-a}$ of the interval I such that *meas* $\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \geq x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$. Such a family is equimeasurable ([4]) and the set X is compact in measure in $L^1(I)$. It is clear, that putting $\Omega_c = [0,c) \cup E$ or $\Omega_c = [0,c) \setminus E$, where E is a set of a null measure, this family contains nonincreasing functions (possibly except for a set E). We will call the functions from this family "a.e. nonincreasing" functions. This is the case, when we choose an integrable and nonincreasing *function y* and all functions equal a.e. to y satisfy the above condition. Thus we can write, that *elements* from $L^1(I)$ belong to this class of functions.

Due to the compactness criterion in the space of measurable functions (with the topology of the convergence in measure) (see Lemma 4.1 in [4]) we have a desired theorem concerning the compactness in measure of a subset X of $L^1(I)$ (cf. Corollary 4.1 in [4] or Section III.2 in [23]).

Theorem 2.2. *Let X be a bounded subset of $L^1(I)$ consisting of functions which are a.e. nonincreasing (or a.e. nondecreasing) on the interval I. Then X is compact in measure in $L^1(I)$.*

If we consider the set of indices $c \geq 0$ in the definition of the family of a.e. nonincreasing functions, we are able to extend this result for the space $L^1(\mathbb{R}^+)$. For simplicity, we will denote such a space by L^1. Due to some results of Väth we are able to extend the desired result from the interval $I = [a, b]$ into the σ-finite subsets of $\mathbb{R}$ and the topology of the convergence in finite measure.

Theorem 2.3. *Let X be a bounded subset of $L^1(\mathbb{R}^+)$ consisting of functions which are a.e. nonincreasing (or a.e. nondecreasing) on the half-line $\mathbb{R}^+$. Then X is compact in finite measure in $L^1(\mathbb{R}^+)$.*

Proof. If we consider the space $L^1(T)$ for a σ-finite measure space T, then there is some equivalent finite measure ν ($\nu(\mathbb{R}^+) < \infty$) (Proposition 2.1. in [29] or Corollary 2.20 in [29]). Then the convergence of sequences in S are the same for the metric d and for

$$d_\nu(x, y) = \inf_{a>0}[a + \nu\{s : |x(s) - y(s)| \geq a\}]$$

(Proposition 2.2 in [28]). Take an arbitrary bounded sequence $(x_n) \subset X$. As a subset of a metric space $X = (L^1(\mathbb{R}^+), d_\nu)$ the sequence is compact in this metric space (Theorem 2.2). Then there exists a subsequence (x_{n_k}) of (x_n) which is convergent in the space X to some x i.e.

$$d_\nu(x_{n_k}, x) \overset{k \to \infty}{\longrightarrow} 0.$$

As claimed above these metrics have the same convergent sequences, then

$$d(x_{n_k}, x) \overset{k \to \infty}{\longrightarrow} 0.$$

This means that X is compact in finite measure in $L^1(\mathbb{R}^+)$. ∎

We have also an important

Theorem 2.4 (Lemma 4.2 in [4]). *Suppose the function $t \to f(t, x)$ is a.e. nonincreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \to f(t, x)$ is a.e. nonincreasing on $\mathbb{R}$ for any $t \in I$. Then the superposition operator F generated by f transforms functions being a.e. nonincreasing on I into functions having the same property.*

We need to recall some basic fact about the Uryson operator:

$$(Ux)(t) = \int_\alpha^\beta u(t, s, x(s)) \ ds \quad t \in \mathbb{R}_+.$$

For the operator U we fix $\alpha, \beta \in I$. Let $u(t, s, x) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \ \to \ \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s). If the operator

$$(U_0 x)(t) = \int_\alpha^\beta V(t, s, x(s)) \ ds,$$

maps the space $L^p(\alpha, \beta)$ into $L^q(\alpha, \beta)$ $(q < \infty)$, where

$$V(t, s, x) = \max_{\|v\| \leq \|x\|} |u(t, s, v)|,$$

then the operator U maps continuously $L^p(\alpha, \beta)$ into $L^q(\alpha, \beta)$ $(q < \infty)$. Unfortunately, for the most interesting case of operators with values in $L^\infty(\alpha, \beta)$ such a characterization is not valid. Let us note, that in the main result, we will restrict ourselves to the set of a.e. monotone functions. Thus we have to assume an additional monotonicity property of u to preserve this property too (cf. [12]).

Consider the operator

$$K(x)(t) = x(t) \cdot U(x)(t).$$

For such a type of operators we need to verify that it takes the values in a (expected) space $L^1(\mathbb{R}^+)$. For continuous solutions on a finite interval this is described, for instance, in [21], but for integrable solutions the problem is not sufficiently noticed and examined (sometimes even skipped). It is necessary, that the continuity of such an operator should be verified.

As a consequence of the Riesz representation theorem we obtain the following

Lemma 2.1. *Assume, that the operator U maps continuously the space $L^1(\mathbb{R}^+)$ into $L^\infty(\mathbb{R}^+)$. Then the operator $K(x)(t) = x(t) \cdot U(x)(t)$ is a continuous operator from $L^1(\mathbb{R}^+)$ into itself.*

The last question is to describe some conditions under which the Uryson operator has the above property (cf. [30], section 10.1.7). Unfortunately,

the problem is complicated and exceed the scope and the aim of this paper. Let us formulate the main result by using most useful assumptions from the point of view of applicability of the results. Namely, even the simplest case: $|u(t, s, x)| \leq m(s)$ for sufficiently good functions m is really applicable (cf. Section 4).

The following theorem, which is a particular case of a much more general result, will be very useful in the proof of the main result:

Lemma 2.2 [30]. *Let $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions i.e., it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s). Assume, that*

$$|u(t, s, x)| \leq k(t, s),$$

where the nonnegative function k is measurable in (t, s) and such that the linear integral operator with the kernel $k(t, s)$ maps L^1 into L^∞. Then the operator U maps L^1 into L^∞. If for each non-negative $z(t) \in L^1$ this operator satisfies

$$\lim_{measD \to 0} \sup_{|x| \leq z} \left\| \int_D u(t, s, x(s)) ds \right\|_{L^\infty} = 0$$

and for arbitrary $h > 0$

$$\lim_{\delta \to 0} \left\| \int_D \max_{|x_i| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| \ ds \right\|_{L^\infty} = 0,$$

then U is a continuous operator.

We mention also that some particular conditions guaranteeing the continuity of the operator U may be found in [27, 30]. For the continuous case the situation is simpler (cf. [15, 24], for instance).

Next, we give some definitions and results which will be needed further on. Assume that $(E, \| \cdot \|)$ is an arbitrary Banach space with zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If X is a subset of E, then $\bar{X}$ and $convX$ denote the closure and convex closure of X, respectively. We denote the standard algebraic operations on sets by the symbols λX and $X + Y$. Moreover, we denote by M_E the family of all nonempty and bounded subsets of E and N_E its subfamily consisting of all relatively compact subsets.

Now we present the concept of a regular measure of noncompactness:

Definition 2.1 [9]**.** A mapping $\mu : M_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

  (i) $\mu(X) = 0 \Leftrightarrow X \in N_E$.
 (ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(ConvX) = \mu(X)$.
 (iv) $\mu(\lambda X) = |\lambda|\mu(X)$, for $\lambda \in \mathbb{R}$.
  (v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
 (vi) $\mu(X \bigcup Y) = \max\{\mu(X), \mu(Y)\}$.
(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \ldots$, and $\lim_{n \to \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

An example of such a mapping is the following:

Definition 2.2 [9]**.** Let X be a nonempty and bounded subset of E. The Hausdorff measure of noncompactness $\chi(X)$ is defined as

$$\chi(X) = \inf\{r > 0 : \text{there exists a finite subset Y of E such that } x \subset Y + B_r\}.$$

Another regular measure was defined in the space L^1 in [7] (cf. [8]). Restricted to the family of compact in finite measure subsets of this space, it forms a regular measure of noncompactness. For any $\epsilon > 0$, let c be a measure of equiintegrability of the set X:

$$c(X) = \lim_{\epsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_D |x(t)| dt, D \subset \mathbb{R}^+, \ measD \leq \epsilon \right] \right\} \right\},$$

and

$$d(X) = \lim_{T \to \infty} \left\{ \sup \left[\int_T^\infty |x(t)| \ dt : x \in X \right] \right\}.$$

Put

$$\gamma(X) = c(X) + d(X).$$

Then we have the following theorem, which clarify the connections between measures $\chi(x)$ and $\gamma(x)$ ([7]).

Theorem 2.5. *Let X be a nonempty, bounded and compact in measure subset of L^1. Then*

$$\chi(x) \leq \gamma(x) \leq 2\chi(x).$$

An importance of such a kind of functions can be clarified by using the contraction property with respect to this measure instead of compactness in the Schauder fixed point theorem. Namely, we have a theorem due to Darbo ([9, 19]):

Theorem 2.6. *Let Q be a nonempty, bounded, closed and convex subset of E and let $H : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e., there exists $k \in [0, 1)$ such that*

$$\mu(HX) \leq k\mu(X),$$

for any nonempty subset X of E. Then H has at least one fixed point in the set E.

3. Main result

Denote by H the operator associated with the right hand side of equation (1) which takes the form

$$x = Hx,$$

where

$$(Hx)(t) = g(t) + f(t, x(t) \int_\alpha^\beta u(t, s, x(s)) \ ds), \quad t \geq 0.$$

The operator H will be written as the product $Hx(t) = g(t) + FKx(t)$ of the superposition operator

$$(Fx)(t) = f(t, x(t))$$

and the Uryson integral operator of the form

$$(Ux)(t) = \int_\alpha^\beta u(t, s, x(s))ds.$$

Thus equation (1) becomes

(3) $$x = g + FKx.$$

We shall treat equation (1) under the following assumptions which are listed below.

(i) $g \in L^1(\mathbb{R}^+)$ and is a.e. nonincreasing on $\mathbb{R}^+$.

(ii) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and there are a positive function $a \in L^1$ and a constant $b \geq 0$ such that

$$|f(t, x)| \leq a(t) + b|x|,$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Moreover, $f(t, x) \geq 0$ for $x \geq 0$ and f is assumed to be nonincreasing with respect to both variable t and x separately.

(iii) $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions i.e., it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s). The function u is nonincreasing with respect to each variable, separately. Moreover, for arbitrary fixed $s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ the function $t \to u(t, s, x(s))$ is integrable.

(iv) There exists a measurable function k such that:

$$|u(t, s, x)| \leq k(t, s)$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$. A measurable nonnegative function $k : \mathbb{R}^+ \to \mathbb{R}^+$ is supposed to be nonincreasing with respect to each variable separately and such that the linear integral operator K_0 with kernel $k(t, s)$ maps L^1 into L^∞. Moreover, for each non-negative $z \in L^1$ let

$$\lim_{measD \to 0} \sup_{|x| \leq z} \| \int_D u(t, s, x(s)) ds \|_{L^\infty} = 0$$

and assume that for arbitrary $h > 0$ $(i = 1, 2)$

$$\lim_{\delta \to 0} \| \int_D \max_{|x_i| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| \, ds \|_{L^\infty} = 0.$$

(v) $b \cdot \|K_0\|_\infty < 1$.

Then we can prove the following theorem.

Theorem 3.1. *Let the assumptions* (i)–(v) *be satisfied. Then equation* (1) *has at least one solution a.e. nonincreasing on* $\mathbb{R}^+$ *which is locally integrable.*

Proof. First of all observe that by Assumption (ii) and Theorem 2.1 F is a continuous operator from L^1 into itself. Moreover, by (iv) U is a continuous operator from L^1 into L^∞ (see Lemma 2.2) and then by Lemma 2.1 the

operator K maps L^1 into itself. Finally, for a given $x \in L^1$ the function Hx belongs to L^1 (cf. Section 2) and is continuous.

Using (3) together with assumptions (iii) and (iv), we get

$$
\begin{aligned}
\|Hx\| &\leq \|g\| + \|FKx(t)\| \\
&\leq \|g\| + \int_0^\infty [a(t) + b|x(t)| \int_\alpha^\beta |u(t,s,x(s))| \ ds]dt \\
&\leq \|g\| + \|a\| + b \int_0^\infty |x(t)|[\int_\alpha^\beta k(t,s) \ ds \ ]dt \\
&\leq \|g\| + \|a\| + b \int_0^\infty [|x(t)| \cdot \|K_0(t)\|_\infty]dt \\
&= \|g\| + \|a\| + b \cdot \|K_0\|_\infty \cdot \|x\|.
\end{aligned}
$$

From the above estimate it follows, that there is a constant $r > 0$ such that H maps the ball B_r into itself. Indeed, by (v) we get

$$
\begin{aligned}
\|Hx\| &\leq \|g\| + \|a\| + b \cdot \|K_0\|_\infty \cdot \|x\| \\
&\leq \|g\| + \|a\| + b \cdot \|K_0\|_\infty \cdot r
\end{aligned}
$$

and then we obtain that $H(B_r) \subset B_r$, where

$$
r = \frac{\|g\| + \|a\|}{1 - b\|K_0\|_\infty}.
$$

Further, let Q_r stand for the subset of B_r consisting of all functions which are a.e. nonincreasing on $\mathbb{R}^+$. Similarly as claimed in [5] this set is nonempty $(x(t) = e^{\frac{t}{r}} \in B_r \cap Q_r$, for instance), bounded by r, convex (direct calculation from the definition) and closed in $L^1(\mathbb{R}^+)$. To prove the last property, let (y_n) be a sequence of elements in Q_r convergent in L^1 to y. Then the sequence is convergent in finite measure and as a consequence of Vitali convergence theorem and of the characterization of convergence in measure (Riesz theorem) we obtain the existence of a subsequence (y_{n_k}) of (y_n) which converges to y almost uniformly on $\mathbb{R}^+$. Moreover, y is still nonincreasing a.e. on $\mathbb{R}^+$. Then $y \in Q_r$ and so the set Q_r is closed. Now, in view of Theorem 2.3 the set Q_r is compact in finite measure. To see this it suffices to put $\Omega_c = [0,c] \setminus P$ for any $c \geq 0$, where P denotes a suitable set with $measP = 0$.

Now, we show, that H preserves the monotonicity of functions. Take $x \in Q_r$, then $x(t)$ is a.e. nonincreasing on $\mathbb{R}^+$ and consequently $Kx(t)$ is of the same type in virtue of assumption (iii) and Theorem 2.4. Further, $FKx(t)$ is a.e. nonincreasing on $\mathbb{R}^+$ thanks to assumption (ii). Moreover, from assumption (i) it follows that $Hx = g(t) + FKx(t)$ is also a.e. nonincreasing on $\mathbb{R}^+$. This fact, together with the assertion $H : B_r \to B_r$, gives that H is also a self-mapping of the set Q_r. From the above considerations it follows that H maps continuously Q_r into Q_r.

From now we will assume that X is a nonempty subset of Q_r and the constant $\epsilon > 0$ is arbitrary, but fixed. Then, for an arbitrary $x \in X$ and for a set $D \subset \mathbb{R}^+$, $meas D \leq \epsilon$, we obtain

$$\int_D |(Hx)(t)| dt \leq \int_D \left[|g(t)| + a(t) + b \cdot |x(t)| \cdot \int_\alpha^\beta |u(t,s,x(s))| \, ds \right] dt$$

$$= \|g\|_{L^1(D)} + \|a\|_{L^1(D)} + b \cdot \|x\|_{L^1(D)} \cdot \left\| \int_\alpha^\beta k(t,s) \, ds \right\|_{L^\infty}$$

$$\leq \|g\|_{L^1(D)} + \|a\|_{L^1(D)} + b \cdot \|K_0\|_\infty \cdot \|x\|_{L^1(D)}.$$

Hence, taking into account the equality

$$\lim_{\epsilon \to 0} \left\{ \sup \left[\int_D |g(t)| \, dt + \int_D a(t) \, dt : D \subset \mathbb{R}^+, \ meas D \leq \epsilon \right] \right\} = 0$$

and the definition of $c(X)$ (cf. Section 2), we get

$$(4) \qquad\qquad c(HX) \leq b \cdot \|K_0\|_\infty \cdot c(X).$$

Furthermore, fixing $T > 0$, we get the following estimate

$$\int_T^\infty |(Hx)(t)| dt \leq \int_T^\infty \left[|g(t)| + a(t) + b|x(t)| \int_\alpha^\beta |u(t,s,x(s))| \, ds \right] dt$$

$$\leq \int_T^\infty \left[|g(t)| + a(t) + b|x(t)| \int_\alpha^\beta k(t,s) \, ds \right] dt$$

$$\leq \int_T^\infty |g(t)| \, dt + \int_T^\infty a(t) \, dt + b\|K_0\|_\infty \int_T^\infty |x(t)| \, dt.$$

As $T \to \infty$, the above inequality yields

$$(5) \qquad\qquad d(HX) \leq b \cdot \|K_0\|_\infty \cdot d(X),$$

where $d(X)$ has been defined in Section 2.

Hence, combining (4) and (5) we get

$$\gamma(HX) \leq b \cdot \|K_0\|_\infty \cdot \gamma(X),$$

where γ denotes our measure of noncompactness defined in Section 2.

The inequality obtained above together with the properties of the operator H and the set Q_r established before allow us to use Theorem 2.5 and as a consequence, apply Theorem 2.6. This completes the proof. ■

Remark 3.1. If we assume that the functions g and $t \to u(t, s, x)$ are a.e. nondecreasing and negative then applying the same argumentation, we can show that there exists a solution of our equation being a.e. negative and nondecreasing. Moreover, let us remark, that the monotonicity conditions in the main theorem seems to be restrictive, but they are necessary as claimed in [12, Example 2].

4.   Examples

In order to illustrate the results proved in Theorem 3.1, let us consider the following example:

Let the following equation

$$(6) \qquad\qquad x(t) = e^{-t} + x(t) \int_\alpha^\beta \frac{t}{t^2 + s^2 + (x(s))^2} \ ds$$

be given.

Putting $g(t) = e^{-t}$, $f(t, x) = x$ and $u(t, s, x) = \frac{t}{t^2 + s^2 + x^2}$ it is easy to see, that u is nonicreasing with respect to each variable separately and the integrability condition is also satisfied (i.e. assumptions (i), (ii) and (iii) are satisfied).

Let us take $k(t, s) = \frac{1}{t^2 + s^2}$. Since $\int_\alpha^\beta k(t, s) \ ds = \arctan \frac{\beta}{t} - \arctan \frac{\alpha}{t}$, then

$$\left| \int_\alpha^\beta k(t, s) \ ds \right| \leq |\beta - \alpha|.$$

Thus Assumption (v) holds for K_0 (for sufficiently small parameter b dependent on α and β).

Moreover, given arbitrary $h > 0$ and $|x_2 - x_1| \leq \delta$ we have

$$|u(t,s,x_1) - u(t,s,x_2)|$$

$$\leq \frac{t(x_2^2 - x_1^2)}{(t^2 + s^2 + x_1^2)(t^2 + s^2 + x_2^2)} \leq \frac{2ht\delta}{(t^2 + s^2 + x_1^2)(t^2 + s^2 + x_2^2)}$$

and Assumption (iv) is satisfied.

Taking into account all the above observations and Theorem 3.1 we conclude that the equation (6) has at least one solution $x = x(t)$ defined, integrable and a.e. nonincreasing on $\mathbb{R}^+$.

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