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# A STUDY OF SECOND ORDER DIFFERENTIAL INCLUSIONS WITH FOUR-POINT INTEGRAL BOUNDARY CONDITIONS

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**Abstract**

In this paper, we discuss the existence of solutions for a four-point integral boundary value problem of second order differential inclusions involving convex and non-convex multivalued maps. The existence results are obtained by applying the nonlinear alternative of Leray Schauder type and some suitable theorems of fixed point theory.

**Keywords and phrases:** differential inclusions, four-point integral boundary conditions, existence, nonlinear alternative of Leray Schauder type, fixed point theorems.

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## 1. Introduction

In this paper, we consider the following second order differential inclusion with four-point integral boundary conditions

$$(1.1)\quad \begin{cases} -x''(t) \in F(t, x(t)), \quad 0 < t < 1, \\ x(0) = \alpha \displaystyle\int_0^{\xi_1} x(s)ds, \qquad x(1) = \beta \displaystyle\int_0^{\xi_2} x(s)ds, \quad 0 < \xi_1, \xi_2 < 1, \end{cases}$$

where $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$ and $\alpha,\ \beta \in \mathbb{R}$.

Multi-point boundary conditions arise in a variety of problems of applied mathematics and physics. Nonlocal multi-point problems constitute an important class of boundary value problems and have been addressed by many authors, for instance, see [1, 2, 6, 7, 10, 17, 19, 20, 21, 22, 24, 27, 28, 30, 32, 34, 35].

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to the papers [3, 11] and references therein.

Differential inclusions arise in the mathematical modelling of certain problems in economics, optimal control, stochastic analysis, etc. and are widely studied by many authors, see [4, 5, 8, 9, 14, 29, 31] and the references therein.

The aim of our paper is to establish some existence results for the problem (1.1), when the right hand side is convex as well as nonconvex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. The methods used are standard, however their exposition in the framework of problem (1.1) is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

## 2. Preliminaries

Let us recall some basic definitions on multi-valued maps [16, 23].

For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map $G$ is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in $X$ for all $\mathbb{B} \in P_b(X)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). $G$ is called upper semi-continuous (u.s.c.) on $X$ if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of $x_0$ such that $G(\mathcal{N}_0) \subseteq N$. $G$ is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. $G$ has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator $G$ will be denoted by $FixG$. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

**Definition 2.1.** A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be $L^1$-Carathéodory if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$;

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t) \text{ for all}$$

$$\|x\|_\infty \le \alpha \text{ and for a.e. } t \in [0,1].$$

Note that the multivalued map $F$ is said to be Carathéodory if the conditions (i) and (ii) hold in Definition 2.1.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of $F$ by

$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,1]\}.$$

Let $X$ be a nonempty closed subset of a Banach space $E$ and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. $G$ is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set $B$ in $E$. Let $A$ be a subset of $[0,1] \times \mathbb{R}$. $A$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $A$ belongs to the $\sigma$-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

**Definition 2.2.** Let $Y$ be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say $N$ has a property (BC) if $N$ is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with $F$ as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with $F$.

**Definition 2.3.** Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say $F$ is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let $(X,d)$ be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\big\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\big\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [25]).

**Definition 2.4.** A multivalued operator $N : X \to P_{cl}(X)$ is called:

(a) $\gamma$-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y) \text{ for each } x, y \in X;$$

(b) a contraction if and only if it is $\gamma$-Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

**Lemma 2.1** ([26])**.** *Let $X$ be a Banach space. Let $F : [0,T]\times\mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an $L^1$-Carathéodory multivalued map and let $\Theta$ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator*

$$\Theta \circ S_F : C([0,1], X) \to \mathcal{P}_{cp,c}(C([0,1], X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

*is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.*

**Lemma 2.2** (Nonlinear alternative for Kakutani maps [18])**.** *Let $E$ be a Banach space, $C$ a closed convex subset of $E$, $U$ an open subset of $C$ and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is an upper semicontinuous compact map; where $\mathcal{P}_{cp,c}(C)$ denotes the family of nonempty, compact convex subsets of $C$. Then either*

(i) *$F$ has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

**Lemma 2.3** ([12])**.** *Let $Y$ be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then $N$ has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

**Lemma 2.4** ([15])**.** *Let $(X,d)$ be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

In order to define the solution of (1.1), we consider the following lemma.

**Lemma 2.5.** *For a given $y \in C[0,1]$, the unique solution of the boundary value problem*

$$\text{(2.1)} \quad \begin{cases} x''(t) + y(t) = 0, \quad 0 < t < 1, \\ x(0) = \alpha \int_0^{\xi_1} x(s)ds, \quad x(1) = \beta \int_0^{\xi_2} x(s)ds, \quad 0 < \xi_1, \xi_2 < 1, \end{cases}$$

*is given by*

$$\begin{aligned} x(t) &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right)\int_0^{\xi_1}(\xi_1 - s)^2 y(s)ds \\ &\quad - \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2 y(s)ds \qquad \text{(2.2)} \\ &\quad + \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^1 (1-s)y(s)ds - \int_0^t (t-s)y(s)ds, \end{aligned}$$

*where*

$$\delta = (\beta\xi_2^2 - 2)(\alpha\xi_1 - 1) - \alpha\xi_1^2(\beta\xi_2 - 1) \neq 0.$$

***Proof.*** The general solution of $x''(t) = -y(t)$ can be written as

$$\text{(2.3)} \qquad x(t) = c_1 + c_2 t - \int_0^t (t-s)y(s)ds,$$

where $c_1$ and $c_2$ are arbitrary constants. Using the boundary conditions given by (2.1), we find that

$$\begin{aligned} c_1 &= \frac{\alpha(\beta\xi_2^2 - 2)}{2\delta}\int_0^{\xi_1}(\xi_1 - s)^2 y(s)ds - \frac{\alpha\beta\xi_1^2}{2\delta}\int_0^{\xi_2}(\xi_2 - s)^2 y(s)ds \\ &\quad + \frac{\alpha\xi_1^2}{\delta}\int_0^1 (1-s)^2 y(s)ds, \end{aligned}$$

$$\begin{aligned} c_2 &= -\frac{\alpha(\beta\xi_2 - 1)}{\delta}\int_0^{\xi_1}(\xi - s)^2 y(s)ds + \frac{(\alpha\xi_1 - 1)\beta}{\delta}\int_0^{\xi_2}(\xi_2 - s)^2 y(s)ds \\ &\quad - \frac{2(\alpha\xi - 1)}{\delta}\int_0^1 (1-s)^2 y(s)ds. \end{aligned}$$

Substituting the values of $c_1$ and $c_2$ in (2.3), we obtain (2.2). ∎

**Remark 2.1.** Letting $\alpha = 0$ and replacing $\beta$ by $\alpha$ in the statement of Lemma 2.5, we obtain a unique solution of a three-point integral boundary value problem considered in [33].

**Definition 2.5.** A function $x \in C([0,1])$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) = {} & \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right)\int_0^{\xi_1}(\xi_1 - s)^2 f(s)ds \\ & -\frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2 f(s)ds \\ & +\frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^1(1 - s)f(s)ds - \int_0^t(t - s)f(s)ds. \end{aligned}$$

## 3. Main results

**Theorem 3.1.** *Assume that*

($H_1$) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory and has nonempty compact convex values;*

($H_2$) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,1],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|_\infty)$$
$$\textit{for each} \ \ (t,x) \in [0,1] \times \mathbb{R};$$

($H_3$) *there exists a number* $M > 0$ *such that*

$$\frac{M}{\left(1 + \frac{\gamma_1+\gamma_2}{2|\delta|}\right)\psi(M)\|p\|_{L^1}} > 1, \tag{3.1}$$

*where*

$$\gamma_1 = |\alpha|\left(|\beta\xi_2^2 - 2| + 2|\beta\xi_2 - 1|\right)\xi_1^2,$$
$$\gamma_2 = \left(|\alpha|\xi_1^2 + 2|\alpha\xi_1 - 1|\right)\left(|\beta|\xi_2^2 + 2\right).$$

*Then the boundary value problem* (1.1) *has at least one solution on* $[0,1]$.

***Proof.*** Define the operator

$$\Omega(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right) \int_0^{\xi_1} (\xi_1 - s)^2 f(s)ds \\ -\frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^{\xi_2} (\xi_2 - s)^2 f(s)ds \\ +\frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^1 (1-s)f(s)ds - \int_0^t (t-s)f(s)ds \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$. We will show that $\Omega$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\Omega(x)$ is convex for each $x \in C([0,1],\mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,1]$, we have

$$\begin{aligned} h_i(t) &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right) \int_0^{\xi_1} (\xi_1 - s)^2 f_i(s)ds \\ &\quad - \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^{\xi_2} (\xi_2 - s)^2 f_i(s)ds \\ &\quad + \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^1 (1-s)f_i(s)ds - \int_0^t (t-s)f_i(s)ds, \ i = 1,2. \end{aligned}$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned} &[\omega h_1 + (1-\omega)h_2](t) \\ &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right) \int_0^{\xi_1} (\xi_1 - s)^2 [\omega f_1(s) + (1-\omega)f_2(s)]ds \\ &\quad - \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^{\xi_2} (\xi_2 - s)^2 [\omega f_1(s) + (1-\omega)f_2(s)]ds \\ &\quad + \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^1 (1-s)[\omega f_1(s) + (1-\omega)f_2(s)]ds \\ &\quad - \int_0^t (t-s)[\omega f_1(s) + (1-\omega)f_2(s)]ds. \end{aligned}$$

Since $S_{F,x}$ is convex ($F$ has convex values), then it follows that $\omega h_1 + (1-\omega)h_2 \in \Omega(x)$.

Next, we show that $\Omega(x)$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number $r$, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty \leq r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) = & \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right)\int_0^{\xi_1}(\xi_1 - s)^2 f(s)ds \\ & - \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2 f(s)ds \\ & + \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^1 (1-s)f(s)ds - \int_0^t (t-s)f(s)ds, \end{aligned}$$

and

$$\begin{aligned} |h(t)| \leq & \frac{|\alpha|}{2|\delta|}\left(|\beta\xi_2^2 - 2| + |2(\beta\xi_2 - 1|\right)\int_0^{\xi_1}(\xi_1 - s)^2|f(s)|ds \\ & + \frac{|\beta|}{2|\delta|}\left(|\alpha|\xi_1^2 + 2|\alpha\xi_1 - 1|\right)\int_0^{\xi_2}(\xi_2 - s)^2|f(s)|ds \\ & + \frac{1}{|\delta|}\left(|\alpha|\xi_1^2 + 2|\alpha\xi_1 - 1|\right)\int_0^1(1-s)|f(s)|ds + \int_0^t (t-s)|f(s)|ds \\ \leq & \frac{|\alpha|}{2|\delta|}\left(|\beta\xi_2^2 - 2| + |2(\beta\xi_2 - 1|\right)\xi_1^2\int_0^{\xi_1}|f(s)|ds \\ & + \frac{|\beta|}{2|\delta|}\left(|\alpha|\xi_1^2 + 2|\alpha\xi_1 - 1|\right)\xi_2^2\int_0^{\xi_1}|f(s)|ds \\ & + \frac{1}{|\delta|}\left(|\alpha|\xi_1^2 + 2|\alpha\xi_1 - 1|\right)\int_0^1|f(s)|ds + \int_0^1|f(s)|ds \\ \leq & \left(1 + \frac{\gamma_1 + \gamma_2}{2|\delta|}\right)\psi(\|x\|_\infty)\int_0^1 p(s)ds. \end{aligned}$$

Thus,

$$\|h\|_\infty \le \left(1 + \frac{\gamma_1 + \gamma_2}{2|\delta|}\right) \psi(\|x\|_\infty) \int_0^1 p(s)ds.$$

Now we show that $\Omega$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where $B_r$ is a bounded set of $C([0,1],\mathbb{R})$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned}
&|h(t'') - h(t')| \\
&= \Bigg| -\frac{\alpha(\beta\xi_2 - 1)(t'' - t')}{\delta} \int_0^{\xi_1} (\xi_1 - s)^2 f(s)ds \\
&\quad + \frac{\beta(\alpha\xi_1 - 1)(t'' - t')}{\delta} \int_0^{\xi_2} (\xi_2 - s)^2 f(s)ds \\
&\quad - \frac{2(\alpha\xi_1 - 1)(t'' - t')}{\delta} \int_0^1 (1 - s)f(s)ds \\
&\quad - \int_0^{t''} (t'' - s)f(s)ds + \int_0^{t'} (t' - s)f(s)ds \Bigg| \\
&\le \left| \frac{\alpha(\beta\xi_2 - 1)(t'' - t')}{\delta} \int_0^{\xi_1} (\xi_1 - s)^2 p(s)\psi(\|x\|_\infty)ds \right| \\
&\quad + \left| \frac{\beta(\alpha\xi_1 - 1)(t'' - t')}{\delta} \int_0^{\xi_2} (\xi_2 - s)^2 p(s)\psi(\|x\|_\infty)ds \right| \\
&\quad + \left| \frac{2(\alpha\xi_1 - 1)(t'' - t')}{\delta} \int_0^1 (1 - s)p(s)\psi(\|x\|_\infty)ds \right| \\
&\quad + \left| \int_0^{t'} (t'' - t')p(s)\psi(\|x\|_\infty)ds \right| + \left| \int_{t'}^{t''} (t'' - s)p(s)\psi(\|x\|_\infty)ds \right|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' - t' \to 0$. As $\Omega$ satisfies the above three assumptions, it follows by the Ascoli-Arzelá theorem that $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that $\Omega$ has a closed graph. Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. There exists

$f_n \in S_{F,x_n}$ associated with $h_n \in \Omega(x_n)$ and such that for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2-2)-2(\beta\xi_2-1)t\right)\int_0^{\xi_1}(\xi_1-s)^2 f_n(s)ds \\ &- \frac{\beta}{2\delta}\left(\alpha\xi_1^2-2(\alpha\xi_1-1)t\right)\int_0^{\xi_2}(\xi_2-s)^2 f_n(s)ds \\ &+ \frac{1}{\delta}\left(\alpha\xi_1^2-2(\alpha\xi_1-1)t\right)\int_0^1(1-s)f_n(s)ds-\int_0^t(t-s)f_n(s)ds. \end{aligned}$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_*(t) &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2-2)-2(\beta\xi_2-1)t\right)\int_0^{\xi_1}(\xi_1-s)^2 f_*(s)ds \\ &- \frac{\beta}{2\delta}\left(\alpha\xi_1^2-2(\alpha\xi_1-1)t\right)\int_0^{\xi_2}(\xi_2-s)^2 f_*(s)ds \\ &+ \frac{1}{\delta}\left(\alpha\xi_1^2-2(\alpha\xi_1-1)t\right)\int_0^1(1-s)f_*(s)ds-\int_0^t(t-s)f_*(s)ds. \end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2-2)-2(\beta\xi_2-1)t\right)\int_0^{\xi_1}(\xi_1-s)^2 f(s)ds \\ &- \frac{\beta}{2\delta}\left(\alpha\xi_1^2-2(\alpha\xi_1-1)t\right)\int_0^{\xi_2}(\xi_2-s)^2 f(s)ds \\ &+ \frac{1}{\delta}\left(\alpha\xi_1^2-2(\alpha\xi_1-1)t\right)\int_0^1(1-s)y(s)ds-\int_0^t(t-s)f(s)ds. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t)-h_*(t)\| \\ &= \left\|\frac{\alpha}{2\delta}\left((\beta\xi_2^2-2)-2(\beta\xi_2-1)t\right)\int_0^{\xi_1}(\xi_1-s)^2(f_n(s)-f_*(s))ds\right. \end{aligned}$$

$$
\begin{aligned}
&- \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2(f_n(s) - f_*(s))ds \\
&+ \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{1}(1 - s)(f_n(s) - f_*(s))ds \\
&- \int_0^t (t-s)(f_n(s) - f_*(s))ds\Bigg\| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Thus, it follows by Lemma 2.1 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, then, we have

$$
\begin{aligned}
h_*(t) \;=\; & \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right)\int_0^{\xi_1}(\xi_1 - s)^2 f_*(s)ds \\
&- \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2 f_*(s)ds \\
&+ \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{1}(1 - s)f_*(s)ds - \int_0^t (t-s)f_*(s)ds
\end{aligned}
$$

for some $f_* \in S_{F,x_*}$.

Finally, we discuss a priori bounds of solutions. Let $x$ be a solution of (1.1). Then there exists $f \in L^1([0,1],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$
\begin{aligned}
x(t) \;=\; & \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right)\int_0^{\xi_1}(\xi_1 - s)^2 f(s)ds \\
&- \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2 f(s)ds \\
&+ \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{1}(1 - s)y(s)ds - \int_0^t (t-s)f(s)ds.
\end{aligned}
$$

In view of $(H_2)$, for each $t \in [0,1]$, we obtain

$$\begin{aligned} x(t) \le\ & \frac{|\alpha|}{2|\delta|}\left(|\beta\xi_2^2 - 2| + |2(\beta\xi_2 - 1|\right)\xi_1^2 \int_0^{\xi_1} |f(s)|ds \\ & + \frac{|\beta|}{2|\delta|}\left(|\alpha|\xi_1^2 + 2|\alpha\xi_1 - 1|\right)\xi_2^2 \int_0^{\xi_2} |f(s)|ds \\ & + \frac{1}{|\delta|}\left(|\alpha|\xi_1^2 + 2|\alpha\xi_1 - 1|\right)\int_0^1 |f(s)|ds + \int_0^1 |f(s)|ds \\ \le\ & \left(1 + \frac{\gamma_1+\gamma_2}{2|\delta|}\right)\psi(\|x\|_\infty)\int_0^1 p(s)ds. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|_\infty}{\left(1 + \frac{\gamma_1+\gamma_2}{2|\delta|}\right)\psi(\|x\|_\infty)\|p\|_{L^1}} \le 1.$$

In view of $(H_3)$, there exists $M$ such that $\|x\|_\infty \ne M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\|_\infty < M + 1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of $U$, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by Lemma 2.2, we deduce that $\Omega$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. ∎

As a next result, we study the case when $F$ is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [12] for lower semi-continuous maps with decomposable values.

**Theorem 3.2** *Assume that* $(\mathrm{H}_2), (\mathrm{H}_3)$ *and the following conditions hold:*

$(\mathrm{H}_4)$ $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*

(b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,1]$*;*

($H_5$) *for each* $\sigma > 0$, *there exists* $\varphi_\sigma \in L^1([0,1],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq \varphi_\sigma(t) \quad \textit{for all } \|x\|_\infty \leq \sigma$$
$$\textit{and for a.e. } \quad t \in [0,1].$$

*Then the boundary value problem* (1.1) *has at least one solution on* $[0,1]$.

***Proof.*** It follows from ($H_4$) and ($H_5$) that $F$ is of l.s.c. type. Then from Lemma 2.3, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$(3.2) \qquad \begin{cases} -x''(t) = f(x(t)), \quad 0 < t < 1, \\ x(0) = \alpha \displaystyle\int_0^{\xi_1} x(s)ds, \quad x(1) = \beta \displaystyle\int_0^{\xi_2} x(s)ds, \quad 0 < \xi_1, \xi_2 < 1. \end{cases}$$

Observe that if $x \in C^2([0,1])$ is a solution of (3.2), then $x$ is a solution of the problem (1.1). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right)\int_0^{\xi_1}(\xi_1 - s)^2 f(x(s))ds \\ &\quad - \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2 f(x(s))ds \\ &\quad + \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^1(1-s)f(x(s))ds - \int_0^t (t-s)f(x(s))ds. \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. ∎

Now we prove the existence of solutions for the problem (1.1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [15].

**Theorem 3.3** *Assume that the following conditions hold:*

($H_6$) $F : [0,1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that* $F(.,x) : [0,1] \to P_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$.

($H_7$) $H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x - \bar{x}|$ *for almost all* $t \in [0,1]$ *and* $x, \bar{x} \in \mathbb{R}$ *with* $m \in L^1([0,1], \mathbb{R}^+)$ *and* $d(0, F(t,0)) \leq m(t)$ *for almost all* $t \in [0,1]$.

*Then the boundary value problem* (1.1) *has at least one solution on* $[0,1]$ *if*

$$\left(1 + \frac{\gamma_1 + \gamma_2}{2|\delta|}\right) \|m\|_{L^1} < 1.$$

***Proof.*** Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,1], \mathbb{R})$ by the assumption $(H_6)$, so $F$ has a measurable selection (see Theorem III.6 [13]). Now we show that the operator $\Omega$ satisfies the assumptions of Lemma 2.4. To show that $\Omega(x) \in P_{cl}((C[0,1], \mathbb{R}))$ for each $x \in C([0,1], \mathbb{R})$, let $\{u_n\}_{n \geq 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1], \mathbb{R})$. Then $u \in C([0,1], \mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) = & \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right)\int_0^{\xi_1}(\xi_1 - s)^2 v_n(s)ds \\ & - \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2 v_n(s)ds \\ & + \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^1 (1-s)v_n(s)ds - \int_0^t (t-s)v_n(s)ds. \end{aligned}$$

As $F$ has compact values, we pass to a subsequence to obtain that $v_n$ converges to $v$ in $L^1([0,1], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) \to u(t) = & \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right)\int_0^{\xi_1}(\xi_1 - s)^2 v(s)ds \\ & - \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^{\xi_2}(\xi_2 - s)^2 v(s)ds \\ & + \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right)\int_0^1 (1-s)v(s)ds - \int_0^t (t-s)v(s)ds. \end{aligned}$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega(x), \Omega(\bar{x})) \leq \gamma \|x - \bar{x}\|_\infty \ \text{ for each } \ x, \bar{x} \in C([0,1], \mathbb{R}).$$

Let $x, \bar{x} \in C([0,1], \mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right) \int_0^{\xi_1} (\xi_1 - s)^2 v_1(s) ds \\ &- \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^{\xi_2} (\xi_2 - s)^2 v_1(s) ds \\ &+ \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^1 (1-s) v_1(s) ds - \int_0^t (t-s) v_1(s) ds. \end{aligned}$$

By $(H_7)$, we have

$$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [13]), there exists a function $v_2(t)$ which is a measurable selection for $V$. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) &= \frac{\alpha}{2\delta}\left((\beta\xi_2^2 - 2) - 2(\beta\xi_2 - 1)t\right) \int_0^{\xi_1} (\xi_1 - s)^2 v_2(s) ds \\ &- \frac{\beta}{2\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^{\xi_2} (\xi_2 - s)^2 v_2(s) ds \\ &+ \frac{1}{\delta}\left(\alpha\xi_1^2 - 2(\alpha\xi_1 - 1)t\right) \int_0^1 (1-s) v_2(s) ds - \int_0^t (t-s) v_2(s) ds. \end{aligned}$$

Thus,

$$\begin{aligned}|h_1(t)-h_2(t)| \le\ & \frac{|\alpha|}{2|\delta|}\left(|\beta\xi_2^2-2|+2|\beta\xi_2-1|\right)\int_0^{\xi_1}(\xi_1-s)^2|v_1(s)-v_2(s)|ds\\ &+\frac{|\beta|}{2|\delta|}\left(|\alpha|\xi_1^2+2|\alpha\xi_1-1|\right)\int_0^{\xi_2}(\xi_2-s)^2|v_1(s)-v_2(s)|ds\\ &+\frac{1}{|\delta|}\left(|\alpha|\xi_1^2+2|\alpha\xi_1-1|\right)\int_0^{1}(1-s)|v_1(s)-v_2(s)|ds\\ &+\int_0^t(t-s)|v_1(s)-v_2(s)|ds\\ \le\ & \left(1+\frac{\gamma_1+\gamma_2}{2|\delta|}\right)\int_0^1 m(s)\|x-\overline{x}\|ds.\end{aligned}$$

Hence,

$$\|h_1(t)-h_2(t)\|_\infty \le \left(1+\frac{\gamma_1+\gamma_2}{2|\delta|}\right)\|m\|_{L^1}\|x-\overline{x}\|_\infty.$$

Analogously, interchanging the roles of $x$ and $\overline{x}$, we obtain

$$\begin{aligned}H_d(\Omega(x),\Omega(\bar{x})) \le\ & \gamma\|x-\bar{x}\|_\infty\\ \le\ & \left(1+\frac{\gamma_1+\gamma_2}{2|\delta|}\right)\|m\|_{L^1}\|x-\overline{x}\|_\infty.\end{aligned}$$

Since $\Omega$ is a contraction, it follows by Lemma 2.4 that $\Omega$ has a fixed point $x$ which is a solution of (1.1). This completes the proof. ■

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