\textbf{P-ORDER NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY IN SINGULAR CALCULUS OF VARIATIONS}

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\textbf{Abstract}

This paper is devoted to singular calculus of variations problems with constraint functional not regular at the solution point in the sense that the first derivative is not surjective. In the first part of the paper we pursue an approach based on the constructions of the $p$-regularity theory. For $p$-regular calculus of variations problem we formulate and prove necessary and sufficient conditions for optimality in singular case and illustrate our results by classical example of calculus of variations problem.

\textbf{Keywords:} singular variational problem, necessary condition of optimality, $p$-regularity, $p$-factor operator.

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\section{Introduction}

In the paper we consider the following problem

(1) \[ J_0(x) = \int_{t_1}^{t_2} F(t, x, x') dt \to \min \]
subject to the subsidiary condition

\[(2) \quad J_1(x) = \int_{t_1}^{t_2} G(t, x, x') dt = 0, \]

\[x(t_1) = x_1, \quad x(t_2) = x_2, \quad x \in C^1[t_1, t_2].\]

If \(\text{Im}J'_1(x^*) \neq 0\), where \(x^*(t)\) is a solution to (1)–(2), then necessary condition of Euler-Lagrange

\[(3) \quad \frac{d}{dt}(F_{x'} + \lambda G_{x'}) = F_x + \lambda G_x\]

holds.

In singular (nonregular or degenerate) case when \(J'_1(x^*) = 0\), we can only guarantee that the following equation

\[(4) \quad \frac{d}{dt}(\lambda_0 F_{x'} + \lambda G_{x'}) = \lambda_0 F_x + \lambda G_x\]

holds, where \(\lambda_0^2 + \lambda^2 = 1\), i.e., it may happen that \(\lambda_0 = 0\).

This work is devoted to the investigation of the case mentioned above with necessary and sufficient conditions of optimality.

Let us present an example which illustrate the subject under consideration.

**Example 1.** Consider the problem

\[(5) \quad J_0(x) = \int_0^\pi x^2(t) dt \rightarrow \min\]

subject to

\[(6) \quad J_1(x) = \int_0^\pi \left( m^2 x^2(t) - (x'(t))^2 \right) dt = 0 \]

\[x(0) = x(\pi) = 0, \quad m^2 > 1.\]

Here we have \(F(t, x, x') = x^2, \quad G(t, x, x') = m^2 x^2 - (x')^2\) and \(x^* \equiv 0\) is the solution to (5)–(6) of \(J_1\), i.e., \(J'_1(x^*) = 0\). If we write down the corresponding Euler-Lagrange equation (4) we obtain
\[-\lambda x'' = \lambda_0 x + \lambda m^2 x\]
\[\text{or } \lambda_0 x + \lambda m^2 x + \lambda x'' = 0\]

and we have the following set of solutions:

- if \(\lambda_0 = 0\) (\(\lambda_0 \neq 0\)) then \(x^*(t) = 0\) is a solution to (4),
- if \(\lambda \neq 0\) then \(x'' + \left(\frac{\lambda_0 + \lambda m^2}{\lambda}\right) x = 0\), and for \(\frac{\lambda_0 + \lambda m^2}{\lambda} = m^2\) the function \(x^*(t) = a \sin mt\) is a solution to (4) with \(a \neq 0\), \(m \in \mathbb{N}\).

Hence, the optimality conditions provide a series of spurious solutions \(x^*(t) = a \sin mt\), \(a \neq 0\), \(m \in \mathbb{N}\).

Now we remind the \(p\)-order necessary and sufficient optimality conditions for degenerate optimization problems (see [2]–[5]).

(7) \[\min \varphi(x)\]

(8) \[\text{subject to } f(x) = 0,\]

where \(f : X \to Y\), and \(X, Y\) are Banach spaces, \(\varphi : X \to \mathbb{R}^1\), \(f \in C^{p+1}(X)\), \(\varphi \in C^2(X)\) and at the solution point \(x^*\) to the problem (7)–(8) we have:

(9) \[\text{Im} f'(x^*) \neq Y,\]

i.e., \(f'(x^*)\) is singular.

2. Elements of \(p\)-regularity theory

Let us remain the basic constructions of \(p\)-regularity theory to be used in investigating singular problems in the paper.

The construction of the \(p\)-factor operator (see also in [2]–[5]).

Suppose that the space \(Y\) is decomposed into the direct sum

(10) \[Y = Y_1 \oplus \ldots \oplus Y_p,\]

where \(Y_1 = \text{cl Im} F'(x^*), \ Z_1 = Y\). Let \(Z_2\) be a closed complementary subspace to \(Y_1\) (we assume that such closed complement exists), and let \(P_{Z_2} : Y \to Z_2\) be the projection operator onto \(Z_2\) along \(Y_1\). By \(Y_2\) we mean the closed linear
span of the image of the quadratic map $P_{Z_2} f^{(2)}(x^*) [\cdot]^2$. More generally, define inductively,

$$Y_i = \text{cl lin} P_{Z_i} f^{(i)}(x^*) [\cdot]^i \subseteq Z_i, \quad i = 2, \ldots, p - 1,$$

where $Z_i$ is a chosen closed complementary subspace for $(Y_1 \oplus \cdots \oplus Y_{i-1})$ with respect to $Y$, $i = 2, \ldots, p$ and $P_{Z_i} : Y \to Z_i$ is the projection operator onto $Z_i$ along $(Y_1 \oplus \cdots \oplus Y_{i-1})$ with respect to $Y$, $i = 2, \ldots, p$. Finally, $Y_p = Z_p$. The order $p$ is chosen as the minimum number for which (10) holds. Now, define the following mappings

$$f_i(x) = P_{Y_i} f(x), \quad f_i : X \to Y_i, \quad i = 1, \ldots, p,$$

where $P_{Y_i} : Y \to Y_i$ is the projection operator onto $Y_i$ along $(Y_1 \oplus \cdots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \cdots \oplus Y_p)$ with respect to $Y$, $i = 1, \ldots, p$.

**Definition 1.** Linear operator $\Psi_p(h) \in \mathcal{L}(X, Y_1 \oplus \cdots \oplus Y_p)$, $h \in X$, $h \neq 0$,

$$\Psi_p(h) = f_1'(x^*) + f_2''(x^*)[h] + \ldots + f_p^{(p)}(x^*)[h]^{p-1}$$

is called $p$-factor operator.

**Definition 2.** We say that the mapping $f$ is $p$-regular at $x^*$ along an element $h$, if

$$\text{Im} \Psi_p(h) = Y.$$

**Definition 3.** We say that the mapping $f$ is $p$-regular at $x^*$ if it is $p$-regular along any $h$ from the set

$$H_p(x^*) = \left\{ \bigcap_{k=1}^p \text{Ker} k^k f_k^{(k)}(x^*) \right\} \setminus 0.$$

Here the $k$-kernel of the $k$-order mapping $f_k^{(k)}(x^*)$ is

$$\text{Ker}^k f_k^{(k)}(x^*) = \left\{ \xi \in X : f_k^{(k)}(x^*)[\xi]^k = 0 \right\}.$$
3. **Optimality conditions for p-regular optimization problems**

We define $p$-factor Lagrange function

$$\mathcal{L}_p(x, \lambda, h) = \varphi(x) + \left( \sum_{k=1}^{p} f_{k}^{(k-1)}(x)[h]^{k-1}, \lambda \right),$$

where $\lambda \in Y^*$ and

$$\tilde{\mathcal{L}}_p(x, \lambda, h) = \varphi(x) + \left( \sum_{k=1}^{p} \frac{2}{k(k+1)} f_{k}^{(k-1)}(x)[h]^{k-1}, \lambda \right).$$

**Definition 4.** The mapping $F$ is called strongly $p$-regular at the point $x^*$ if there exists $\gamma > 0$ such that

$$\sup_{h \in H_{\gamma}} \left\| \{\Psi_p(h)\}^{-1} \right\| < \infty,$$

where

$$H_{\gamma} = \left\{ h \in X : \|f_{k}^{(k)}(x^*)[h]^{k}\|_{Y_k} \leq \gamma, \ i = 1, \ldots, p, \ |h| = 1 \right\}.$$

Let us remain the following basic theorem of the $p$-regularity theory ([2]–[5]).

**Theorem 1** (Necessary and sufficient conditions for optimality). Let $X$ and $Y$ be Banach spaces, $\varphi \in C^2(X)$, $f \in C^{p+1}(X)$, $f : X \to Y$, $\varphi : X \to \mathbb{R}$.

Suppose that $h \in H_p(x^*)$ and $f$ is $p$-regular along $h$ at the point $x^*$. If $x^*$ is a local solution to the problem (7)–(8) then there exist multipliers $\lambda^*(h) \in Y^*$ such that

(11) $$\mathcal{L}'_{px}(x^*, \lambda^*(h), h) = 0.$$

Moreover, if $f$ is strongly $p$-regular at $x^*$, there exist $\alpha > 0$ and a multiplier $\lambda^*(h)$ such that (11) is fulfilled and

(12) $$\mathcal{L}'_{pxx}(x^*, \lambda^*(h), h)[h]^2 \geq \alpha \|h\|^2$$

for every $h \in H_p(x^*)$, then $x^*$ is a strict local minimizer to the problem (7)–(8).
Now we are ready to apply this theorem to singular calculus of variations problems. For the sake of simplicity consider the completely degenerate case, namely

\[ f^{(k)}(x^*) = 0, \quad k = 1, \ldots, p - 1 \]

or

\[ J_1^{(k)}(x^*) = 0, \quad k = 1, \ldots, p - 1 \]

and introduce the so-called \( p \)-factor Euler–Lagrange function

\[
H = F + \lambda \sum_{i=0}^{p-1} C^i_{p-1} G^{(p-1)}_{x^{(i)}(x')} h^i(h')^j = F + \lambda G^{(p-1)} h^{p-1},
\]

where

\[
G^{(p-1)}_{x^{(i)}(x')} = G^{(p-1)}_{x^{i} \ldots x^{j}}
\]

and \( i + j = p - 1, \ i = 0, \ldots, p - 1 \).

For \( p = 2 \) we have \( H = F + \lambda (G_x h + G'_x h') \).

**Definition 5.** We say that the problem (1)–(2) is \( p \)-regular at \( x^* \) along \( h \in \text{Ker}^p J_1^{(p)}(x^*) \) if \( J_1^{(p)}(x^*) h^{p-1} \neq 0 \).

The following theorem holds true:

**Theorem 2.** Let \( x^*(t) \) be a solution of the problem (1)–(2) and assume that the problem is \( p \)-regular at \( x^* \) along \( h \in \text{Ker}^p J_1^{(p)}(x^*) \). Then, there exists a constant \( \lambda \) such that the following \( p \)-factor Euler-Lagrange equation

\[
\frac{d}{dt} \left[ F_x' + \lambda \sum_{i=0}^{p-1} C^i_{p-1} G^{(p-1)}_{x^{(i+1)}(x')} h^i(h')^j \right] = 0
\]

holds.
Remark 1. For $p = 2$ this equation transforms to

$$F_x + \lambda \left( G_{xx} h + G_{x'} h' \right) - \frac{d}{dt} \left[ F_{x'} + \lambda (G_{xxx} h' + G_{xx'} h') \right] = 0.$$  

**Proof.** Consider the case $p = 2$. For the first derivative $J_1'(x^*)$ of $J_1(x)$ at the point $x^*$ we have

$$J_1'(x^*) h(t) = \int_{t_1}^{t_2} \left[ G_x(t, x, x') h(t) + G_{x'}(t, x, x') h'(t) \right] dt.$$  

Let us define second derivative $J_1''(x^*)$:

$$J_1'(x^* + \delta x) - J_1'(x^*) = A(x^*) \delta x + \Psi(x^*, \delta x) \cdot \delta x,$$

where $\|\Psi(x^*, \delta x)\| \to 0$ for $\delta x \to 0$.

$$J_1'(x^* + \delta x) \cdot \delta x - J_1'(x^*) \delta x = A(x^*)[\delta x]^2 + \Psi(x^*, \delta x) \|\delta x\|^2.$$  

Here $A(x^*)[\delta x]^2$ is a second variation of $J_1(x)$ at the point $x^*$ and $J_1''(x^*) = A(x^*)$.

For $J_1''(x^*)$ we have

$$J_1''(x^*)[h(t)]^2 = \int_{t_1}^{t_2} \left[ G_{xx} h^2(t) + 2G_{xx'} h(t) h'(t) + G_{xx'}(h'(t))^2 \right] dt = 0.$$  

Hence $h \in \text{Ker}^2 J_1''(x^*)$ is such a function that (16) holds.

The 2-factor operator for the problem (1)–(2) has the following form:

$$J_1''(x^*) h(t) = \int_{t_1}^{t_2} \left[ G_{xx} h(t) + G_{x'} h(t) + G_{xx'} h'(t) + G_{xx'} h'(t) \right] dt$$

$$= h'(t) G_{x'}(\cdot) \big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[ G_{xx} h + G_{xx'} h' - \frac{d}{dt} \left[ G_{xx} h + G_{xx'} h' \right] \right] dt.$$  

For the nonregular case of our problem we have $J_1'(x^*) = 0$, $Y_1 = 0$, $Y_2 = \mathbb{R}$, $f_1(x) = 0$, $f_2(x) = J_1(x)$, $\Psi_2(h)x = J_1''(x^*)[h]x$ and 2-factor Euler–Lagrange function is as follows

$$\mathcal{L}_2(x, \lambda, h) = J_0(x) + \lambda_1 J_1'(x)[h(t)].$$
Applying Theorem 1 we obtain

\[ J'_0(x^*) + \lambda_1 J''_0(x^*)[h(t)] = 0. \tag{17} \]

Substituting \( J''_0(x^*)[h(t)] \) to the explicit form we have

\[ \left[ F_x'(|t|)^{t_2}_{t_1} + \int_{t_1}^{t_2} \left( F_x - \frac{d}{dt} F_{x'} \right) dt \right] + \]

\[ \lambda_1 \left[ G_{xx'} h'(|t|)^{t_2}_{t_1} + \int_{t_1}^{t_2} \left( G_{xx} h + G_{xx'} h' - \frac{d}{dt} \left[ G_{xx'} h + G_{xx'} h' \right] \right) dt \right] = 0. \]

Since the functional on the right hand side of the equation is defined for such elements \( \eta(t) \) that \( \eta(t_1) = \eta(t_2) = 0 \), we obtain

\[ \int_{t_1}^{t_2} \eta(t) \left[ F_x - \frac{d}{dt} F_{x'} \right] + \lambda_1 \left( G_{xx} h + G_{xx'} h' - \frac{d}{dt} \left[ G_{xx'} h + G_{xx'} h' \right] \right) dt = 0. \]

From du Bois-Reymond lemma the equation (13) for \( p = 2 \) follows.

The case \( p > 2 \) can be proved analogously using the explicit form of \( p \)-factor operator \( J^{[p]}_1(x^*)[h(t)]^{p-1} \).

Consider the previous Example 1. We can see that

\[ h(t) = e^{\gamma t} \sin t \in \text{Ker}^2 J''_1(x^*), \quad x^*(t) = 0 \]

for some \( \gamma = \gamma(m) > 0 \) and \( J''_1(x^*) h(t) \neq 0 \). It means that the problem (1)–(2) is 2-regular at \( x^* = 0 \) along \( h(t) \). Hence all conditions of Theorem 2 are fulfilled and equation (13) takes the form:

\[ 2x + \lambda \left( 2m^2 e^{\gamma t} \sin t + 2e^{\gamma t} \left( \gamma^2 \sin t + 2\gamma \cos t - \sin t \right) \right) = 0 \]

or

\[ x = \lambda e^{\gamma t} \left( \sin t - 2\gamma \cos t - \gamma^2 \sin t - m^2 \sin t \right) \]

and \( x(0) = x(\pi) = 0 \). The last condition implies that \( \lambda = 0 \) and \( x^*(t) = 0 \).
4. \( p \)-ORDER SUFFICIENT CONDITIONS FOR OPTIMALITY IN SINGULAR CALCULUS OF VARIATIONS PROBLEMS

For the completely degenerate case sufficient conditions for the problem (1)--(2) have the following form.

**Definition 6.** We say the problem (1)--(2) is strongly \( p \)-regular at \( x^* \) if there exists \( \gamma > 0 \) such that

\[
\left\| \left\{ J_1^{(p)}(x^*)[h]^{p-1} \right\}^{-1} \right\| < \infty
\]

for every \( h \in H_\gamma = \left\{ h \in X : |J_1^{(p)}(x^*)[h]^{p}| \leq \gamma, \|h\| = 1 \right\} \).

**Theorem 3** (\( p \)-order sufficient conditions for optimality). If the problem (1)--(2) is strongly \( p \)-regular at feasible point \( x^* \) and there exist \( \alpha > 0 \) and multipliers \( \lambda(h) \) such that (13) holds and

\[
\left( J_0^{(p)}(x^*) + \frac{2}{p(p+1)} \lambda(h)J_1^{(p+1)}(x^*)h^{p-1} \right) [h]^2 \geq \alpha\|h\|^2
\]

for every \( h \in \text{Ker}^pJ_1^{(p)}(x^*) \), then \( x^* \) is a strict local minimizer to the problem (1)--(2).

The proof of this theorem is an application of Theorem 1 to the problem (1)--(2), where \( f(x) = J_1(x) \) and \( \varphi(x) = J_0(x) \).

However there exists number of applications when strong \( p \)-regularity conditions are not fulfilled. For example

\[
\min x_1^2
\]

subject to

\[
(x_1 - x_2)^2 = 0, \quad x^* = (0,0)^T.
\]

Moreover, in Example 1 with \( h(t) = \sin mt \in \text{Ker}^2J_1''(0) \) we meet the same. For such type of problems we introduce the following definitions.

**Definition 7.** We say that the mapping \( f(x) \) is \( p \)-regular by Cauchy at the point \( x^* \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
g(h,H_p(x^*)) \leq \varepsilon \quad \text{for every} \quad h \in H_\delta.
\]
Remark 2. It is obvious that strong $p$-regularity implies $p$-regularity by Cauchy of the mapping $f(x)$ but not conversely.

Theorem 4. If the problem (1)--(2) is $J_1(x)$-$p$-regular by Cauchy at feasible point $x^*$ and there exist $\alpha > 0$ and multiplier $\lambda(h)$ such that (13) is fulfilled and

$$
\left(J_0''(x^*) + \frac{2}{p(p+1)} \lambda(h)J_1^{(p+1)}(x^*)h^{p-1}\right) [h]^2 \geq \alpha \|h\|^2
$$

$$
\forall h \in \text{Ker}^p J_1^{(p)}(x^*),
$$

then $x^*$ is strict local minimizer to (1)--(2).

The proof is analogous to the proof of Theorem 1, where condition of a strong $p$-regularity is replaced by the condition of $p$-regularity by Cauchy.

In Example 1 we have

$$
J_0(x) = \int_0^\pi x^2 dt,
$$

$$
J_1(x) = \int_0^\pi (m^2 x^2 - (x')^2) dt,
$$

$$
\bar{h}(t) = \sin mt \in \text{Ker}^2 J_1''(0),
$$

but

$$
J_1''(0)\bar{h}(t) = \int_0^\pi (m^2 \sin mt - m \cos mt)' dt = 0.
$$

Hence

$$
J_1''(0)\bar{h}(t)x(t) = \int_0^\pi (mx(t) \cos mt)' dt = 0.
$$

It means that the problem (5)--(6) is not 2-regular along $\bar{h}(t)$. But all conditions of Theorem 3 are fulfilled and

$$
\left(J_0''(0) + \frac{\lambda(\bar{h})}{3} J_1''(0)\bar{h}(t)\right) [\bar{h}(t)]^2 =
$$

$$
= \int_0^\pi (\sin mt)^2 dt \geq \frac{1}{(m+1)^2} \|\sin mt\|^2 = \frac{1}{(m+1)^2} \|\bar{h}(t)\|^2.
$$
Therefore $x^*(t) = 0$ is a strict local minimizer to (5)–(6).

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REFERENCES


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