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PENALTIES, LAGRANGE MULTIPLIERS AND NITSCHE MORTARING

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Abstract

Penalty methods, augmented Lagrangian methods and Nitsche mortaring are well known numerical methods among the specialists in the related areas optimization and finite elements, respectively, but common aspects are rarely available. The aim of the present paper is to describe these methods from a unifying optimization perspective and to highlight some common features of them.

Keywords: augmented Lagrangian, penalty method, domain decomposition, Nitsche mortaring, finite elements.

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1. Introduction

In nonlinear programming penalty methods and augmented Lagrangian methods are widely investigated (cf. [6, 8, 9]) and applied while Nitsche mortaring (cf. [2, 7, 10]) are well known numerical methods among the specialists in the related areas optimization and finite elements, respectively, but common aspects are rarely available. In [14] and a cycle of papers B. Wohlmuth and co-workers duality arguments are applied to construct mortaring coupling. In [11] coupling of non-matching grids by augmented Lagrangians is proposed and studied.

The aim of the present paper is to describe penalty methods, augmented Lagrangian methods and Nitsche mortaring from the optimization point of view just to provide some unified perspective and to highlight some common features.

In Section 1 the basic properties of Poisson equation with homogeneous Dirichlet boundary conditions is briefly described as a simple model of a variational problem. Further, the concept of non-overlapping domain decomposition with two subdomains is given. The original problem is shown to be equivalent to a linearly constrained problem in some broken Sobolev space.

As a central theoretical result of the paper, the convergence of a penalty method applied to the continuous problem arising from domain decomposition is studied in Section 2. Unlike in classical domain decomposition as discussed e.g. in [12] only Dirichlet data are tuned at the boundary connecting the subdomains. The known second condition that the normal derivatives have to coincide in this approach occurs as a natural boundary condition. Further in Section 2 augmented Lagrangian methods are briefly considered as an approximate application of saddle point inequality with additional penalty stabilization.

In Section 3, the idea of Nitsche mortaring is described as an alternative approach to obtain the unknown Lagrangian multipliers using the specific properties of the underlying elliptic boundary value problem is considered. For this purpose the duality between Dirichlet and Neumann boundary conditions is applied. In Section 4, finite elements that discretize the considered problems are briefly discussed. Connections to discontinuous Galerkin methods are sketched. Finally, in Section 5 some numerical results for a simple test problem are given.

2. MODEL PROBLEM, DOMAIN DECOMPOSITION

Let $\Omega \subset \mathbb{R}^2$ be some bounded convex domain with a piecewise sufficiently smooth boundary $\partial\Omega$. As usual we denote by $H^1(\Omega)$ the Sobolev space of functions that possess generalized derivatives of first order and the function as well as these derivatives belong to the Lebesgue space $L_2(\Omega)$. Further, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with zero traces on $\partial\Omega$. With given $f \in L_2(\Omega)$ we consider the variational problem

$$(1) \qquad J(v) := \frac{1}{2}\int_\Omega \nabla v \circ \nabla v - \int_\Omega f\, v \to \min! \qquad \text{s.t.} \quad v \in V := H_0^1(\Omega).$$

Here "$\circ$" stands for the standard scalar product in $\mathbb{R}^2$. The objective functional is convex and Fréchet differentiable. Its derivative is given by

$$\langle J(v), w\rangle := \int_\Omega \nabla v \circ \nabla w - \int_\Omega f\, w \quad \forall v,\, w \in H^1(\Omega).$$

Hence, $u \in V$ forms a solution of (1) iff it satisfies the variational equation

$$\int_\Omega \nabla u \circ \nabla v = \int_\Omega f\, v \qquad \forall v \in V. \tag{2}$$

The Lax-Milgram lemma (cf. [3, 7]) guarantees that (2) has a unique solution $u \in V$. Let us remark that (2) forms the weak formulation of the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3}$$

For an efficient numerical treatment, e.g. to allow parallel computations, the domain Ω can be split into several subdomains. Here we restrict us for the sake of a simpler description to two subdomains Ω_1, $\Omega_2 \subset \Omega$. We consider non-overlapping domain decomposition, i.e.

$$\Omega_1 \cap \Omega_2 = \emptyset, \qquad \overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}.$$

In addition, the inner boundary $\Gamma := (\overline{\Omega}_1 \cap \overline{\Omega}_2)\backslash\partial\Omega$ should be sufficiently regular such that the usual trace theorems (cf. [1, 3, 12]) hold for the subdomains.

Let u be classical solution of problem (3), i.e., $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then domain decomposition as e.g. described in [12] rests on the property that the restrictions $u_j := u|_{\Omega_j}$, $j = 1, 2$ satisfy the elliptic equation in each subdomain

$$-\Delta u_j = f_j \quad \text{in } \Omega_j, \qquad u_j \;=\; 0 \quad \text{on } \overline{\Omega}_j \cap \partial\Omega, \quad j = 1, 2, \tag{4}$$

and the coupling conditions

$$u_1 = u_2 \quad \text{on } \Gamma, \qquad \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2} = 0 \quad \text{on } \Gamma, \tag{5}$$

where n_j denote the outward normals of Ω_j. The conditions (5) are iteratively adjusted while the subproblems (4) can be solved in parallel in each iteration step.

Now, we turn to the weak formulation of domain decomposition (compare [10, 12]). According to the split we define

$$V_j := \{v \in L_2(\Omega) : v|_{\Omega_j} \in H^1(\Omega_j),\, v|_{\partial\Omega} = 0,\, v|_{\Omega_i} \equiv 0,\, i \neq j\},$$

for $j = 1, 2$ and

$$\tilde{V} := V_1 \oplus V_2\,, \qquad G := \left\{ v \in \tilde{V} : v_1|_\Gamma = v_2|_\Gamma \right\}.$$

The space $\tilde{V}$ forms a *broken* Sobolev space and the restrictions to $\partial\Omega$ and Γ, respectively, are understood in the sense of traces of elements of Sobolev spaces (see e.g. [5]).

Lemma 1. *There holds $V \subset \tilde{V}$ and $G = V$.*

Proof. The inclusion is trivial because any $v \in V$ can be represented by $v = v_1 + v_2$ with

$$v_j := \begin{cases} v|_{\Omega_j}\,, & \text{in } \Omega_j \\ 0, & \text{in } \Omega_i \; i \neq j \end{cases} \qquad j = 1,\, 2.$$

Let $v \in \tilde{V}$. With Green's formula we obtain

$$(6) \qquad \sum_{j=1}^{2} \int_{\Omega_j} \nabla v_j \circ \varphi = \sum_{j=1}^{2} \int_\Gamma v_j\,(n_j \circ \varphi) - \int_\Omega v\,(\nabla \circ \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega)^2.$$

As introduced above, n_j denote the outer normals of the subdomains Ω_j. Further, $\mathcal{D}(\Omega)$ are the usual test functions with compact support. Since $n_1 + n_2 = 0$ along Γ equation (6) yields

$$\int_\Omega \nabla v \circ \varphi = \int_\Gamma (v_1 - v_2)\,(n_1 \circ \varphi) - \int_\Omega v\,(\nabla \circ \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega)^2.$$

With $(v_1 - v_2)|_\Gamma$ we obtain the generalized first order differentiability of v and by construction the derivatives belong to $L_2(\Omega)$. Hence $v \in V$. ■

As an immediate consequence of Lemma 1 the unconstrained variational problem (1) is equivalent to the linearly constrained problem

$$(7) \qquad J(v) \to \min! \qquad \text{s.t.} \quad v \in G$$

where $J : \tilde{V} \to \mathbb{R}$ in accordance with the previous definition is extended by

$$J(v) := \frac{1}{2} \sum_{j=1}^{2} a_j(v, v) - \int_\Omega f\, v \qquad \forall v \in \tilde{V},$$

with $a_j : \tilde{V} \times \tilde{V} \to \mathbb{R}$ defined by

$$a_j(v, w) := \int_{\Omega_j} \nabla v_j \circ \nabla w_j \qquad \forall v,\, w \in \tilde{V},\ j = 1, 2.$$

3. PENALTY METHODS

As described above the conditions (5) couple the subproblems (4). Here, however, we apply the standard quadratic penalty method for equality constrained problems to (7), i.e., we include formally only the continuity condition from (5). The second part of (5) is tuned implicitly via the optimality conditions of (7). The quadratic penalty method applied to (7) yields

$$J_s(v) := J(v) + \frac{1}{2s} \int_\Gamma (v_1 - v_2)^2 \to \min! \qquad \text{s.t.} \quad v \in \tilde{V}. \tag{8}$$

Here $s > 0$ denotes the penalty parameter which has to tend to zero. Further, throughout the paper we use the notations

$$[v] := (v_1 - v_2)|_\Gamma, \quad \|v\| := \left(\int_\Omega v^2 \right)^{1/2}, \quad \|[v]\|_\Gamma := \left(\int_\Gamma [v]^2 \right)^{1/2} \qquad \forall v \in \tilde{V}.$$

The penalty functional J_s is Fréchet differentiable with

$$\langle J_s'(v), w \rangle = b_s(v, w) - (f, v) = 0 \qquad \forall v,\, w \in \tilde{V}, \tag{9}$$

where

$$b_s(v, w) := \sum_{j=1}^{2} a_j(v, w) + \frac{1}{s} \int_\Gamma [v][w] \quad \forall v,\, w \in \tilde{V}.$$

The convergence of the considered penalty method is given by

Theorem 1. *For any $s > 0$ the penalty problem* (8) *possesses a unique solution $u^s \in \tilde{V}$ and together with the solution u of* (1) *it satisfies*

$$\lim_{s \to 0+} \|u^s - u\| = 0.$$

Proof. First, using Friedrichs' inequality and trace theorems we can show that

$$((v,w)) := \sum_{j=1}^{2} a_j(v,w) + \int_\Gamma [v][w] \qquad \forall v,\, w \in \tilde{V} \tag{10}$$

defines a scalar product with which $\tilde{V}$ forms a Hilbert space. The norm $|||\cdot|||$ induced by this scalar product yields the continuous embedding $\tilde{V} \hookrightarrow L_2(\Omega)$. Further, trace theorems (cf. [1], [3]) and $H^{1/2}(\Gamma) \hookrightarrow L_2(\Gamma)$ guarantee also the continuous embedding $\tilde{V} \hookrightarrow L_2(\Gamma)$. The related embedding constants we denote by c_Ω and c_Γ, respectively, i.e., we have

$$\|v\| \le c_\Omega\, |||v|||, \qquad \|v\|_\Gamma \le c_\Gamma\, |||v||| \qquad \forall v \in \tilde{V}. \tag{11}$$

In addition, the trace coupling at Γ and the H^1 semi-norm term in each of the subdomains due to the Rellich-Kondrachov theorem (cf. [4]) imply also that the embedding of $\tilde{V}$ into $L_2(\Omega)$ is compact.

By the same argument as for $((\cdot,\cdot))$ we can show that the parametric bilinear form $b_s : \tilde{V} \times \tilde{V} \to \mathbb{R}$ defined by (9) for any $s > 0$ is continuous and $\tilde{V}$-elliptic. Hence, by Lax-Milgram's lemma the variational equation

$$b_s(u^s, v) = (f, v) \qquad \forall v \in V \tag{12}$$

has a unique solution $u^s \in \tilde{V}$. With the convexity of the functional the variational equation (12) is necessary and sufficient for u^s to solve (8).

Now, we turn to the convergence of the method. Since the limit $s \to 0+$ is of interest we assume $s \in (0,1]$ without loss of generality. Using the optimality of u^s and the feasibility $u \in G$, we obtain

$$J_1(u^s) \le J_s(u^s) \le J_s(u) = J(u) \qquad \forall s \in (0,1]. \tag{13}$$

By the definition (10) of the scalar product and the embedding estimate (11) we get

$$\frac{1}{2}|||u^s|||^2 - c_\Omega \|f\|\, |||u^s||| \le \frac{1}{2}((u^s,u^s)) - (f,u^s) \le J(u).$$

Then we obtain

$$\frac{1}{2}(|||u^s||| - c_\Omega \|f\|)^2 \le J(u) + \frac{1}{2}c_\Omega^2\, \|f\|^2.$$

Hence, the family $\{u^s\}$ for $s \to 0+$ is bounded and therefore weakly compact. Let $\{s_k\} \in (0, 1]$ be any sequence converging to zero and denote $u_k := u^{s_k}$. Since $\{u_k\}$ is weakly compact we assume without loss of generality that $\{u_k\}$ is weakly convergent to some $\tilde{u} \in \tilde{V}$, with respect to the scalar product $((\cdot, \cdot))$, i.e.,

$$\lim_{k\to\infty} ((v, u_k)) = ((v, \tilde{u})) \qquad \forall v \in \tilde{V}.$$

By the construction it holds

$$J(v) \le J_s(v) \qquad \forall s > 0,\ v \in \tilde{V}.$$

Using formula (13) we obtain

$$J(u_k) \le J(u), \quad k = 1, 2, \ldots \tag{14}$$

Since J is convex and continuous it is also weakly lower semi-continuous and together with the weak convergence of $\{u_k\}$ this yields

$$J(\tilde{u}) \le J(u).$$

Next, we show that $\tilde{u}$ is feasible for (7), i.e., $\tilde{u} \in G$. We use again (13) to estimate

$$\frac{1}{2s_k} \|u_k\|_\Gamma^2 - (f, u_k) \le J_{s_k}(u_k) \le J(u). \tag{15}$$

By the continuous embedding $\tilde{V} \hookrightarrow L_2(\Gamma)$ the functional $\|\cdot\|_\Gamma^2$ is convex and continuous in $\tilde{V}$, hence also weakly lower semi-continuous. Therefore (15) implies

$$\|\tilde{u}\|_\Gamma \le 0,$$

i.e., $\tilde{u} \in G$. By (14) this yields $\tilde{u} = u$. The uniqueness of the solution of (7) and the weak compactness of the family $\{u^s\}_{s>0}$ imply that the whole family weakly converges, i.e.,

$$\lim_{s\to 0+} ((v, u^s)) = ((v, u)) \qquad \forall v \in \tilde{V}.$$

Finally, the compact embedding of $\tilde{V}$ into $L_2(\Gamma)$ leads to the stated convergence $\lim_{s\to 0+} \|u^s - u\| = 0$. ■

4. AUGMENTED LAGRANGIANS, NITSCHE MORTARING

Before we turn to the concept of augmented Lagrangian methods and the approximation of multipliers, we discuss the regularity of the multiplier λ related to the optimal solution u of (7). Let us formally introduce the corresponding Lagrangian $L(\cdot,\cdot)$ by

$$L(v,\mu) := J(v) + \langle \mu, [v] \rangle \qquad \forall v \in \tilde{V}, \quad \mu \in \Lambda,$$

where $\langle \cdot,\cdot \rangle : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{R}$ denotes the appropriate dual pairing and $\Lambda = H^{-1/2}(\Gamma)$. The used trace space $H^{1/2}(\Gamma)$ is understood as the space of restrictions of elements $v \in \tilde{V}$ and $H^{-1/2}(\Gamma) := H^{1/2}(\Gamma)^*$ is its dual space. However, the multipliers are of low regularity in the general case. Under the assumptions, $f \in L_2(\Omega)$ and Ω is convex, we know that satisfies $u \in V \cap H^2(\Omega)$ the solution u of the constrained problem (7). By integration by parts separately over the subdomains from the formula

$$-\int_{\Omega_1} \Delta u_1 \cdot v_1 - \int_{\Omega_1} f\, v_1 + \int_{\Omega_2} \Delta u_2 \cdot \nabla v_2 - \int_{\Omega_2} f\, v_2 = 0 \quad \forall v \in \tilde{V}$$

with $u\big|_{\partial\Omega} = 0$ we obtain

$$\int_{\Omega_1} \nabla u_1 \cdot \nabla v_1 - \int_{\Omega_1} f\, v_1 + \int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 - \int_{\Omega_2} f\, v_2 - \int_\Gamma \frac{\partial u_1}{\partial n_1} v_1 - \int_\Gamma \frac{\partial u_2}{\partial n_2} v_2 = 0 \qquad \forall v \in \tilde{V}.$$

This is equivalent to

$$\langle J'(u), v \rangle - \int_\Gamma \frac{\partial u_1}{\partial n_1} v_1 - \int_\Gamma \frac{\partial u_2}{\partial n_2} v_2 = 0 \quad \forall v \in \tilde{V}. \tag{16}$$

Since $u \in H^2(\Omega)$, we observe the regularity

$$\frac{\partial u_1}{\partial n_1}, \frac{\partial u_2}{\partial n_2} \in H^{1/2}(\Gamma) \hookrightarrow L_2(\Gamma)$$

of the traces. Further, due to $n_1 = -n_2$ it holds

$$\frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2} = 0.$$

We set $n := n_1$ and denote by $\{\cdot\}$ the averaging operation

$$\left\{\frac{\partial u}{\partial n}\right\} := \frac{1}{2}\left(\frac{\partial u_1}{\partial n} + \frac{\partial u_2}{\partial n}\right) = \frac{1}{2}\left(\frac{\partial u_1}{\partial n_1} - \frac{\partial u_2}{\partial n_2}\right).$$

Therefore (16) can be expressed by

$$\langle J'(u), v\rangle - \int_\Gamma \left\{\frac{\partial u}{\partial n}\right\} [v] = 0 \quad \forall v \in \tilde{V}. \tag{17}$$

This, however, is just the part

$$\langle L_v(u,\lambda), v\rangle = 0 \qquad \forall\, v \in \tilde{V}$$

of the saddle point characterization. The remaining part follows from the feasibility of u for (7). Hence, $\lambda = -\left\{\frac{\partial u}{\partial n}\right\}$ is the multiplier related to u and $\lambda \in L_2(\Gamma)$. This ensures the regularity needed in the following

Theorem 2. *Let $\lambda \in L_2(\Gamma)$ be the Lagrangian multiplier related to the optimal solution u of (7). Then for any $s > 0$ the solution u of (7) is the unique solution of the problem*

$$L(v,\lambda) + \frac{1}{2\,s}\,([v],[v])_\Gamma \to \textit{min!} \qquad \textit{s.t} \quad v \in \tilde{V}. \tag{18}$$

Further u forms a solution of (18) iff it satisfies

$$\langle J'(u), v\rangle + (\lambda, [v]) + s^{-1}\,([u],[v])_\Gamma = 0 \quad \forall v \in \tilde{V}. \tag{19}$$

Proof. The first part follows from the inequality

$$L(u,\lambda) \le L(v,\lambda) \qquad \forall v \in \tilde{V}$$

which is a part of the saddle point inequality and the penalization $\frac{1}{2\,s}\,([v],[v])_\Gamma$ of violations of the feasibility. The second part is simply the well known first order necessary optimality condition that in the considered case is also sufficient due to the convexity. ■

For numerical methods for solving control problems we also refer to [13].

The idea of augmented Lagrangian methods rests upon the approximate recovery of the multiplier λ from penalty approximations. For fixed $s > 0$

we select

$$\lambda^0 := \frac{1}{s}\,[u^s]. \tag{20}$$

The augmented Lagrangian method recursively applies (19) as follows (cf. [6]). Let λ^k be given and let $u^k \in \tilde{V}$ satisfies (19), i.e.,

$$\langle J'(u^k), v\rangle + (\lambda^k, [v]) + s^{-1}\,([u^k], [v])_\Gamma = 0 \quad \forall v \in \tilde{V}. \tag{21}$$

The new λ^{k+1} is defined by

$$\lambda^{k+1} := \lambda^k + s^{-1}\,[u^k],\ k = 0, 1, \dots .$$

The convergence of the method is widely analyzed today and can be found e.g. in [8, 9]. For the convergence of augmented Lagrangian method for the case of finite dimensional problems we also refer to [6].

It follows from Theorem 2 that the recovery of the Lagrangian multiplier combined with the penalty concept leads directly to the wanted solution of the constrained problem (7). However, as a rule the optimal Lagrangian is unknown. The penalty multiplier (20) provides first approximation for λ.

An alternative way to recover the Lagrangian multiplier for the considered type of variational problems which are connected with elliptic boundary value problems rests upon its specific structure. As shown above, the relation

$$\lambda = -\left\{\frac{\partial u}{\partial n}\right\}, \tag{22}$$

holds. Mortaring is a finite element technique which is used to couple different discretizations in the subdomains along common boundaries of subdomains by specific interconnecting conditions called mortar. We have introduced above two subdomains and coupling required at Γ. In Nitsche mortaring this is done by the using of the Langrangian multiplier λ related to the optimal solution u of (7). For this purpose the representation (22) together with the result from Theorem 2 is applied to characterize the optimal solution of the constrained problem (7).

First, we describe continuous Nitsche mortaring, i.e., mortaring of the two subproblems without prior discretization. Using the representation (22) of the Lagrangian multiplier λ (19) turns into the following necessary and sufficient condition

$$\sum_{j=1}^{2} a_j(u,v) - \left(\left\{\frac{\partial u}{\partial n}\right\}, [v]\right)_\Gamma + s^{-1}\left([u],[v]\right)_\Gamma = (f,v) \quad \forall v \in \tilde{V} \tag{23}$$

for $u \in \tilde{V}$ to solve the linearly constrained problem (7) which is equivalent to the original problem (1) due to Lemma 1. Hence, without discretization (23) it is just another weak formulation of the original problem. This variational equation (23) in classical Nitsche mortaring for finite element discretizations is used to couple discrete problems over subdomains. However, unlike the continuous case, the traces $\frac{\partial u_1}{\partial n}$, $\frac{\partial u_2}{\partial n}$ of derivatives of discrete function do not coincide along the common border Γ. Thus, only an approximate adjustment is obtained by the discrete traces. We discuss this in the following section.

5. Finite element discretization

The numerical solution of infinite dimensional problem described as the considered variational problem requires some discretization. Common techniques are finite difference schemes or finite element methods (see [7]). Here we consider the latter one. The finite element methods follow directly from the variational equations if the underlying function space $\tilde{V}$ is replaced by an appropriate finite dimensional space $\tilde{V}_h \subset \tilde{V}$. Since $\tilde{V} \not\subset V$, as a rule this leads to a non-conforming discretization of (2), but in the sense of (7) it is a conforming technique.

Let $\tilde{V} = \operatorname{span}\{\varphi_k\}_{k=1}^N$ with a basis split according to $\tilde{V} = V_1 \oplus V_2$, i.e.,

$$V_{1,h} = \operatorname{span}\{\varphi_k\}_{k=1}^{N_1}, \quad V_{2,h} = \operatorname{span}\{\varphi_k\}_{k=N_1+1}^{N}, \quad \tilde{V}_h = \operatorname{span}\{\varphi_k\}_{k=1}^{N}.$$

The subscript $h > 0$ indicates the discretization and characterizes the approximation properties of $\tilde{V}$ by $\tilde{V}_h$. As a rather weak requirement it is assumed that

$$\lim_{h\to 0+} \inf_{v_h \in \tilde{V}_h} \|v - v_h\| = 0 \qquad \forall v \in \tilde{V}$$

holds. Due to Cea's Lemma (cf. [7, 3]) this provides the convergence of the discretization. Here, we consider the description of the analogies between Lagrangian methods and Nitsche mortaring. For the convergence of these methods we refer to [6, 8, 9] in the case of augmented Lagrangian methods and [2, 10] in the case of mortaring.

In our numerical experiments we choose for both subspaces linear C^0-elements on regular triangulations of Ω_1 and Ω_2, respectively. However, at the inner boundary Γ these triangulations have not to match. The original idea of mortaring is just to provide some conditions to connect the variational equations from different sides of Γ.

The finite element discretization applied to the considered methods leads to the schemes briefly described below. In the case of pure penalties (8) this yields to:

find $u_h^s \in \tilde{V}_h$ such that

$$\langle J_s'(u_h^s), v_h\rangle = \sum_{j=1}^{2} a_j(u_h^s, v_h) + \frac{1}{s}\int_\Gamma [u_h^s][v_h] - (f, v_h) = 0 \quad \forall v_h \in \tilde{V}_h. \tag{24}$$

Due to the bilinearity of the terms containing u_h^s, this is equivalent to a related linear system of Galerkin equations.

Similar to the iteration of the augmented Lagrangian method without discretization given by (21) we obtain the iteration:

find $u_h^k \in \tilde{V}_h$ such that

$$\langle J'(u_h^k), v_h\rangle + (\lambda_h^k, [v_h]) + s^{-1}\,([u_h^k], [v_h])_\Gamma = 0 \quad \forall v_h \in \tilde{V}_h \tag{25}$$

with the update for λ_h^{k+1} defined by

$$\lambda_h^{k+1} := \lambda_h^k + s^{-1}\,[u_h^k],\ k = 0, 1, \ldots .$$

We notice that in (24) as well as in (25) linear systems with symmetric, positive definite system matrices occur. This, however is not automatically the case in Nitsche mortaring. One cause is in the non-symmetric formulation (23). For sufficiently small parameters $s > 0$ the positive definiteness can be guaranteed. We obtain the so-called *incomplete* Nitsche mortaring:

find $u_h \in \tilde{V}_h$ such that

$$\sum_{j=1}^{2} a_j(u_h, v_h) - \left(\left\{\frac{\partial u_h}{\partial n}\right\}, [v_h]\right)_\Gamma + s^{-1}\,([u_h], [v_h])_\Gamma = (f, v_h) \quad \forall v_h \in \tilde{V}_h.$$

A symmetric or non-symnmetric completion is obtained by the additional terms

$$-\left(\left\{\frac{\partial v_h}{\partial n}\right\},\,[u_h]\right)_\Gamma \qquad \text{and} \qquad +\left(\left\{\frac{\partial v_h}{\partial n}\right\},\,[u_h]\right)_\Gamma,$$

to this variational equation, respectively. In the first case symmetric Galerkin systems are obtained, while in the second case one $\tilde{V}$-ellipticity holds for any $s > 0$. The related methods are called symmetric Nitsche mortaring and non-symmetric Nitsche mortaring, respectively.

As mentioned at the beginning of this paper the domain decomposition idea can be extended similarly to an arbitrary, but finite numbers of subdomains. As somehow extreme case the subdomains are just the elements of a finite element triangulation of Ω. In this case Nitsche mortaring is equivalent to discontinuous Galerkin methods as described in [10].

6. NUMERICAL EXPERIMENTS

To illustrate the practical behavior of the considered methods we apply them to the variational formulation of the elliptic boundary value problem

$$(26) \quad -\Delta u(x,y) = \operatorname{sign}(y)\exp(\mathrm{y}) + 3\,x - 2, \quad (x,y) \in \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

with $\Omega := (0,1) \times (-1,1)$. As subdomains we choose

$$\Omega_1 := (0,1) \times (-1,0), \qquad \Omega_2 := (0,1) \times (0,1).$$

Thus, we have $\Gamma := \{(x,0) \in \mathbb{R}^2 : \, x \in (0,1)\}$.

The following graphs were obtained by uniform triangulation of Ω_1 and Ω_2 with the step sizes $h_1 = 1/16$ and $h_2 = 37$, respectively. In the first three figures we selected $s = 0.1$ as penalty parameter. These large step sizes as well as large penalty parameter are purposely chosen make the non-matching discretization and the gap in the case of the penalty method better visible (see Figure 1). By the same reason in the case of the augmented Lagrangian method only one iteration step was performed (see Figure 2).

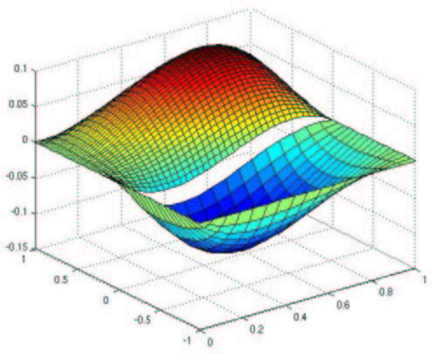

Figure 1. Penalty method.

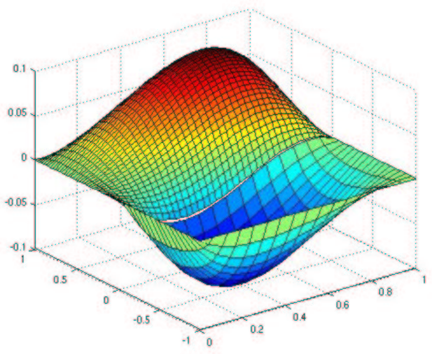

Figure 2. Augmented Lagrangian method.

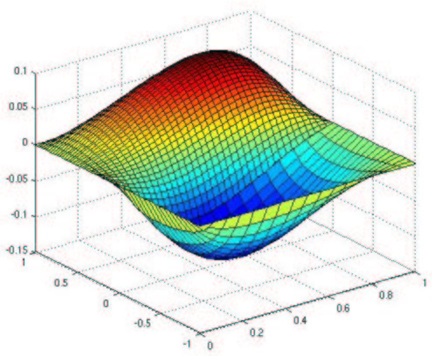

Figure 3. Nitsche mortaring.

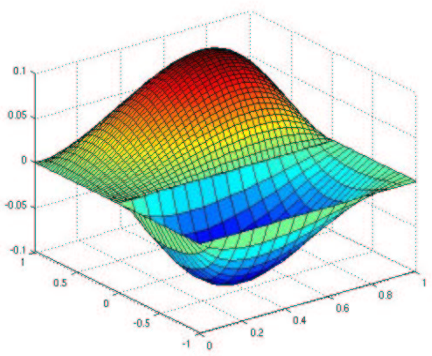

Figure 4. Nitsche mortaring, $s = 10^{-12}$.

Figures 1–3 show the obtained results for these data.

Let us point to the fact of an appropriate adjustment of discretization and penalty parameters. Equality $G = V$ holds in the case without discretization. In the case of non-matching discretization it may happen that the straight forward discretization

$$v_h \in G_h := \{v_h \in \tilde{v}_h : \, [v_h] = 0\}$$

leads to

$$v_h \in G_h \quad \Longrightarrow \quad v_h|_\Gamma \equiv 0.$$

Thus, very small penalty parameters, i.e. $s > 0$, $s \ll 1$, lead to $[u_h^s] \approx 0$ due to $\lim_{s\to 0+}[u_h^s] = 0$. This effect is reflected in Figure 4. To avoid this, the penalties have to decrease with decrease step sizes or as an alternative treatment the penalties have to be approximately evaluated by integration formulas. This effect has been discussed in [5] for penalty methods applied to include non-homogeneous Dirichlet boundary conditions in Galerkin methods.

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REFERENCES

[1] R.A. Adams, Sobolev spaces (Academic Press, Inc., 1975).

[2] R. Becker, P. Hansbo and R. Stenberg, *A finite element method for domain decomposition with non-matching grids*, Math. Model. Numer. Anal. **37** (2003), 209–225.

[3] P.G. Ciarlet, The finite element method for elliptic problems (North-Holland Publ. Co., Amsterdam 1978).

[4] L.C. Evans, Partial differential equations (AMS Publ. Providence, 1998).

[5] C. Grossmann, *Dualität und Strafmethoden bei elliptischen Differentialgleichungen*, Z. Angew. Math. Mech. **64** (1984), 111–121.

[6] C. Grossmann and A. Kaplan, Strafmethoden und modifizierte Lagrange Funktionen in der nichtlinearen Optimierung (Teubner, 1979).

[7] C. Grossmann, H.-G. Roos and M. Stynes, Numerical treatment of partial differential equations (Springer, Berlin, 2007).

[8] K. Ito and K. Kunisch, Lagrange multiplier approach to variational problems and applications (SIAM Publ., 2008).

[9] A. Kaplan and R. Tichatschke, Stable methods for ill-posed variational problems: prox-regularization of elliptic variational inequalities and semi-infinite problems (Akademie Verlag, Berlin 1994).

[10] B. Riviére, Discontinuous Galerkin methods for solving elliptic and parabolic equations (SIAM Publ., 2008).

[11] P. Le Tallec and T. Sassi, *Domain decomposition with nonmatching grids: Augmented Lagrangian approach*, Math. Comput. **64** (1995), 1367–1396.

[12] A. Toselli and O. Widlund, Domain decomposition methods – algorithms and theory (Springer, 2005).

[13] F. Tröltzsch, Optimale Steuerung partieller Differentialgleichungen. Theorie, Verfahren und Anwendungen (Vieweg, Wiesbaden, 2005).

[14] B.I. Wohlmuth, *A mortar finite element method using dual spaces for the Lagrange multiplier*, SIAM J. Numer. Anal. **38** (2000), 989–1012.

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