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THE EXISTENCE OF CARATHÉODORY SOLUTIONS OF HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

We consider the following Darboux problem for the functional differential equation

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = f\left(x, y, u_{(x,y)}, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)\right) \text{ a.e. in } [0,a]\times[0,b],$$

$$u(x,y) = \psi(x,y) \quad \text{on } [-a_0,a]\times[-b_0,b]\backslash(0,a]\times(0,b],$$

where the function $u_{(x,y)} : [-a_0,0]\times[-b_0,0] \to \mathbb{R}^k$ is defined by $u_{(x,y)}(s,t) = u(s+x,t+y)$ for $(s,t)\in[-a_0,0]\times[-b_0,0]$. We prove a theorem on existence of the Carathéodory solutions of the above problem.

Keywords: existence theorem, functional differential equation, hyperbolic equation, Darboux problem, solution in the sense of Carathéodory.

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1. Introduction

Put $I = [0,a]\times[0,b]$, $D = [-a_0,0]\times[-b_0,0]$, $I^* = [-a_0,a]\times[-b_0,b]$, $I_0 = \overline{I^*\backslash I}$. We always assume that $a,b>0$ and $a_0,b_0\in\mathbb{R}_+$, where $\mathbb{R}_+ = [0,+\infty)$. We denote by

(a) $C(I^*,\mathbb{R}^k)$ the space of continuous functions from I^* into $\mathbb{R}^k$ with the usual supremum norm.

(b) $C_x(I,\mathbb{R}^k)$ the space of functions ν of the variables (x,y) defined on I, continuous in $x \in [0,a]$ for almost all $y \in [0,b]$ and measurable in $y \in [0,b]$ for all $x \in [0,a]$ and such that

$$||\nu||_x = \int_0^b \max_{x\in[0,a]} |\nu(x,y)|dy < \infty.$$

(c) $C_y(I,\mathbb{R}^k)$ the space of functions μ of the variables (x,y) defined on I, continuous in $y \in [0,b]$ for almost all $x \in [0,a]$ and measurable in $x \in [0,a]$ for all $y \in [0,b]$ and such that

$$||\mu||_y = \int_0^a \max_{y\in[0,b]} |\mu(x,y)|dx < \infty.$$

(d) $L^1(I,\mathbb{R})$ the space of Lebesgue integrable functions from I into R.

Let $|\cdot|$ denote the maximum norm in $\mathbb{R}^k$. Moreover, $||w||_0$ denotes the usual supremum norm of $w \in C(D,\mathbb{R}^k)$. As in [6] we can verify that $||\cdot||_x$, $||\cdot||_y$ are norms and $(C_x(I,\mathbb{R}^k), ||\cdot||_x)$, $(C_y(I,\mathbb{R}^k), ||\cdot||_y)$ are Banach spaces.

Given a rectangle $J = [a_1,a_2] \times [b_1,b_2]$ contained in I and $u : I \to \mathbb{R}$, let

$$\Delta_J(u) = u(a_1,b_1) - u(a_2,b_1) - u(a_1,b_2) + u(a_2,b_2).$$

A rectangle is called a subrectangle of I if its sides are parallel to the coordinate axes. Let m denote the Lebesgue measure on $\mathbb{R}^2$. We say that $u : I \to \mathbb{R}$ is absolutely continuous if the following two conditions are satisfied:

(I) Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{J\in J} |\Delta_J(u)| < \epsilon,$$

whenever J is a finite collection of pairwise non-overlapping subrectangles of I with

$$\sum_{J\in J} m(J) < \delta.$$

(II) The marginal functions $u(\cdot,b)$ and $u(a,\cdot)$ are absolutely continuous functions of a single variable on $[0,a]$ and $[0,b]$, respectively.

Let $AC(I,\mathbb{R})$ denote the set of absolutely continuous functions on I. In [2] we can find that the following statements are equivalent:

(I) $u \in AC(I,\mathbb{R})$.

(II) There exist $g \in AC([0,a],\mathbb{R})$, $h \in AC([0,b],\mathbb{R})$ and $L \in L^1(I,\mathbb{R})$ such that
$$u(x,y) = g(x) + h(y) + \int_0^x \int_0^y L(s,t)dsdt.$$

Note that if $u \in AC(I,\mathbb{R})$, then $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial x \partial y}$ exist almost everywhere on I. Furthermore, $\frac{\partial u}{\partial x} \in C_y(I,\mathbb{R})$ and $\frac{\partial u}{\partial y} \in C_x(I,\mathbb{R})$.

We consider the problem

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = f\left(x, y, u_{(x,y)}, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)\right) \text{ a.e. in } I, \tag{1}$$

$$u(x,y) = \psi(x,y) \text{ on } I_0, \tag{2}$$

where $f : I \times C(D,\mathbb{R}^k) \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ and $\psi : I_0 \to \mathbb{R}^k$ are given functions. We define $u_{(x,y)} : D \to \mathbb{R}^k$ by the formula $u_{(x,y)}(s,t) = u(s+x, t+y)$ for $(s,t) \in D$. By the solution of the problem we mean a function $u : I^* \to \mathbb{R}^k$ continuous on I^* and absolutely continuous on I which satisfies the differential equation almost everywhere on I and the initial condition everywhere on I_0.

In paper [14] we can find a theorem which extends the Peano existence theorem under Carathéodory conditions for ordinary differential equations to functional equations. In this paper we give an existence theorem which extends this theorem to the Darboux problem (1), (2). Existence theorems of Carathéodory solutions for hyperbolic equations can be found in [3, 5, 6, 12, 13]. Theorems of existence of classical solutions for hyperbolic equations can be found in [3, 15]. Our result will be proved via Schauder's fixed point theorem. Note that existence theorems can also be proved via iterative method (see for instance [8]). In Section 2 we give criteria of relative compactness in $C_x(I,\mathbb{R}^k)$ and $C_y(I,\mathbb{R}^k)$. The Ascoli-Arzéla and Fréchet-Kolomogorov theorems of characterization of the relatively compact subsets of $C(I,\mathbb{R}^k)$ and $L^p(I,\mathbb{R}^k)$ respectively, can be found in [9].

2. Criteria of compactness

This section is devoted to the study of criteria of compactness in $C_x(I,\mathbb{R}^k)$ and $C_y(I,\mathbb{R}^k)$. These results will be of use in the proof of existence theorem. In paper [9] we can find the following proposition:

Proposition 2.1. *Let (X,d) be a complete metric space and let $\{Y_\lambda : \lambda > 0\}$ be a family of relatively compact subsets of X. Assume that $Y \subset X$ is the uniform limit of Y_λ as $\lambda \to 0$, that is, for each $\epsilon > 0$ there is $\lambda_\epsilon > 0$ such that for every $u \in Y$ and $\lambda \in (0,\lambda_\epsilon)$ there exists $u_\lambda \in Y_\lambda$ with $d(u,u_\lambda) < \epsilon$. Then Y is relatively compact.*

This proposition will be of use in the proof of criteria of compactness.

Lemma 2.1. *Let $Z \subset C_y(I,\mathbb{R}^k)$ and the following conditions be satisfied:*

(I) *There exists a constant $c > 0$ such that for all $w \in Z$,*

$$\int_0^a \max_{y\in[0,b]} |w(x,y)|dx \leq c. \tag{3}$$

(II) *For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $w \in Z$,*

$$\int_0^a |w(x,y') - w(x,y)|dx < \epsilon \tag{4}$$

whenever y',$y \in [0,b]$ and $|y' - y| < \delta$.

(III) *Let*

$$\sup_{w\in Z} \int_0^a \max_{y\in[0,b]} |w(x+h,y) - w(x,y)|dy \to 0 \text{ as } h \to 0. \tag{5}$$

Then the set Z is relatively compact in $C_y(I,\mathbb{R}^k)$.

Proof. For a given number $r > 0$ and any function $w \in C_y(I,\mathbb{R}^k)$, we define the function $m_r(w) : \mathbb{R}^2 \to \mathbb{R}^k$ by the formula

$$m_r(w)(x,y) = \frac{1}{2r}\int_{x-r}^{x+r} \overline{w}(z,y)dz,$$

where $\overline{w} = w$ on I and $\overline{w} = 0$ on $\mathbb{R}^2 \backslash I$. For simplicity of notation, we will write w instead of $\overline{w}$.

We note that

$$|m_r(w)(x',y') - m_r(w)(x,y)| \le \frac{1}{2r}\int_{x'-r}^{x'+r} |w(z,y') - w(z,y)|dz$$

$$+\frac{1}{2r}\int_{x-r}^{x+r} |w(z+x'-x,y) - w(z,y)|dz \le \frac{1}{2r}\int_0^a |w(z,y') - w(z,y)|dz$$

$$+\frac{1}{2r}\int_0^a |w(z+x'-x,y) - w(z,y)|dz.$$

This inequality together with (4) and (5) shows that the set $m_r(Z) = \{m_r(w) : w \in Z\}$ is equicontinuous. Moreover, from (3) we have

$$|m_r(w)(x,y)| \le \frac{1}{2r}\int_0^a \max_{y\in[0,b]} |w(z,y)|dz \le \frac{c}{2r}.$$

Hence, the set $m_r(Z)$ is bounded in $C(I,\mathbb{R}^k)$. Now from the Ascoli-Arzeli theorem we conclude that $m_r(Z)$ is a relatively compact subset of $C(I,\mathbb{R}^k)$, and of $C_y(I,\mathbb{R}^k)$. Furthermore,

$$m_r(w)(x,y) - w(x,y) = \frac{1}{2r}\int_{x-r}^{x+r} [w(z,y) - w(x,y)]dz$$

$$= \frac{1}{2r}\int_{-r}^{r} [w(x+z,y) - w(x,y)]dz \text{ for } x,y \in I.$$

By the above,

$$\int_0^a \max_{y\in[0,b]} |m_r(w)(z,y) - w(x,y)|dz \le \frac{1}{2r}\int_0^a\int_{-r}^{r} \max_{y\in[0,b]} |w(x+z,y) - w(x,y)|dz$$

$$\le \sup_{|z|\le r}\int_0^a \max_{y\in[0,b]} |w(x+z,y) - w(x,y)|dz.$$

This shows that Z is the uniform limit (in $C_y(I,\mathbb{R}^k)$) of $m_r(Z)$ as $r \to 0$. From proposition 2.1, it follows that Z is relatively compact in $C_y(I,\mathbb{R}^k)$. ■

Remark 2.1. Exchanging the roles of x and y in Theorem 2.1, we get the corresponding criterion of compactness for $C_x(I,\mathbb{R}^k)$.

Remark 2.2. In paper [11] one can find the theorem about compact sets in the space $L^p(0,T;B)$, where $L^p(0,T;B)$ is the set of Bochner integrable functions $f:(0,T)\to B$ such that $\left(\int_0^T ||f(t)||_B^p\right)^{\frac{1}{p}} < \infty$. Let $w \in C_y(I,\mathbb{R}^k)$ and $\tilde{w}:[0,a]\to C([0,b],\mathbb{R}^k)$ be defined by $\tilde{w} = w(x,\cdot)$ for $x\in[0,a]$. We note that if $w\in C_y(I,\mathbb{R}^k)$ satisfies conditions (3)-(5), then $\tilde{w}$ satisfies assumptions of Theorem 1 from paper [11] for $p=1$, $T=a$ and $B=C([0,b],\mathbb{R}^k)$, where $||\cdot||_{C([0,b],\mathbb{R}^k)}$ is the usual supremum norm.

3. The existence theorem

In this section we prove an existence theorem for the Darboux problem (1), (2). For this purpose some a priori estimates of the solution will be needed. Therefore we first prove some useful lemmas. Let $l_i\in L^1(I,\mathbb{R})$ for $i=1,2,3$ satisfy

$$l_2(x,y)+l_3(x,y)\geq 3l_1(x,y) \text{ for } (x,y)\in I. \tag{6}$$

Moreover, $c_1\geq 3$, $c_2\geq 1$ i $c_3\geq 1$. Define r_1,r_2, $r_3: I\to\mathbb{R}$ by formulas

$$r_1(x,y)=c_1e^{3H(x,y)}, \tag{7}$$

$$r_2(x,y)=\left(\sum_{i=1}^{3}\int_0^y l_i(x,t)dt + c_3\right)r_1(x,y), \tag{8}$$

$$r_3(x,y)=\left(\sum_{i=1}^{3}\int_0^x l_i(s,y)ds + c_2\right)r_1(x,y), \tag{9}$$

where

$$H(x,y)=\sum_{i=1}^{3}\int_0^x\int_0^y l_i(s,t)dsdt + c_3x + c_2y.$$

Lemma 3.1. *Functions r_1, r_2, $r_3: I\to\mathbb{R}$ satisfy the inequalities*

$$\int_0^x \int_0^y \left(c_2 + \int_0^s l_2(z,t)dz\right) r_2(s,t)dsdt \tag{10}$$
$$\leq \frac{1}{9}r_1(x,y) - \frac{c_1}{9} - \frac{1}{3}\int_0^x \int_0^y l_2(s,t)r_1(s,t)dtds,$$

$$\int_0^x \int_0^y \left(c_3 + \int_0^t l_3(s,z)dz\right) r_3(s,t)dsdt \tag{11}$$
$$\leq \frac{1}{9}r_1(x,y) - \frac{c_1}{9} - \frac{1}{3}\int_0^x \int_0^y l_3(s,t)r_1(s,t)dtds,$$

for $(x,y) \in I$.

Proof. We integrate by parts the left-hand side of (10), obtaining

$$c_1 \int_0^x \int_0^y \left(c_2 + \int_0^s l_2(z,t)dz\right)\left(\sum_{i=1}^3 \int_0^t l_i(s,z)dz + c_3\right) e^{3H(s,t)}dsdt$$
$$= \frac{1}{3}\int_0^y \left\{\left[\left(c_2 + \int_0^s l_2(z,t)dz\right) r_1(s,t)\right]_{s=0}^{s=x} - \int_0^x l_2(s,t)r_1(s,t)ds\right\}dt$$
$$= \frac{1}{3}\int_0^y \left\{\left(c_2 + \int_0^x l_2(s,t)ds\right) r_1(x,t) - c_1c_2e^{3c_2t} - \int_0^x l_2(s,t)r_1(s,t)ds\right\}dt$$
$$\leq \frac{1}{3}\int_0^y \left(c_2 + \sum_{i=1}^3 \int_0^x l_i(s,t)ds\right) r_1(x,t)dt - \frac{c_1c_2}{3}\cdot\frac{1}{3c_2}\cdot[e^{3c_2t}]_{t=0}^{t=y}$$
$$- \frac{1}{3}\int_0^y \int_0^x l_2(s,t)r_1(s,t)dtds = \frac{1}{9}[r_1(x,t)]_{t=0}^{t=y} - \frac{c_1}{9}(e^{3c_2y} - 1)$$
$$- \frac{1}{3}\int_0^x \int_0^y l_2(s,t)r_1(s,t)dsdt = \frac{1}{9}r_1(x,y) - \frac{c_1}{9}(e^{3c_3x} + e^{3c_2y} - 1)$$
$$- \frac{1}{3}\int_0^x \int_0^y l_2(s,t)r_1(s,t)dsdt \leq \frac{1}{9}r_1(x,y) - \frac{c_1}{9}$$
$$- \frac{1}{3}\int_0^x \int_0^y l_2(s,t)r_1(s,t)dsdt.$$

This finishes the proof of (10). The proof of (11) is similar. ■

Remark 3.1. *It is easily seen that functions r_1, r_2, $r_3 : I \to \mathbb{R}$ satisfy the inequalities*

$$\tag{12} \begin{aligned} &\int_0^y \left(c_2 + \int_0^x l_2(s,t)ds\right) r_2(x,t)dt \\ &\le \frac{1}{3} r_2(x,y) - \frac{c_1 c_3}{3} - \frac{1}{3}\sum_{i=1}^{3} \int_0^y l_i(x,t) r_1(x,t)dt, \end{aligned}$$

$$\tag{13} \begin{aligned} &\int_0^y \left(c_3 + \int_0^t l_3(x,z)dz\right) r_3(x,t)dt \\ &\le \frac{1}{3} r_2(x,y) - \frac{c_1 c_3}{3} - \frac{1}{3}\int_0^y l_3(x,t) r_1(x,t)dt, \end{aligned}$$

for $(x,y) \in I$.

Lemma 3.2. *Functions r_1, r_2, $r_3 : I \to \mathbb{R}$ satisfy the inequalities*

$$\tag{14} \begin{aligned} r_1(x,y) \ge c_1 + \int_0^x \int_0^y \Big\{ & l_1(s,t) + l_1(s,t) r_1(s,t) \\ & + \left(c_2 + \int_0^s l_2(z,t)dz\right) r_2(s,t) + \left(c_3 + \int_0^t l_3(s,z)dz\right) r_3(s,t) \Big\} dsdt, \end{aligned}$$

$$\tag{15} \begin{aligned} r_2(x,y) \ge c_1 + \int_0^y \Big\{ & l_1(x,t) + l_1(x,t) r_1(x,t) \\ & + \left(c_2 + \int_0^x l_2(z,t)dz\right) r_2(x,t) + \left(c_3 + \int_0^t l_3(x,z)dz\right) r_3(x,t) \Big\} dt, \end{aligned}$$

$$\tag{16} \begin{aligned} r_3(x,y) \ge c_1 + \int_0^x \Big\{ & l_1(s,y) + l_1(s,y) r_1(s,y) \\ & + \left(c_2 + \int_0^s l_2(z,y)dz\right) r_2(s,y) + \left(c_3 + \int_0^y l_3(s,z)dz\right) r_3(s,y) \Big\} ds, \end{aligned}$$

for $(x,y) \in I$.

Proof. From inequalities (10), (11), assumption (6), the fact that $21c_1 \ge 9$, c_2, $c_3 \ge 0$ and the inequality $1 + \alpha \le e^{\alpha}$ for $\alpha \ge 0$, we get that

$$c_1+\int_0^x\int_0^y\Big\{l_1(s,t)+l_1(s,t)r_1(s,t)+\Big(c_2+\int_0^s l_2(z,t)dz\Big)r_2(s,t)$$

$$+\Big(c_3+\int_0^t l_3(s,z)dz\Big)r_3(s,t)\Big\}dsdt\le c_1+\int_0^x\int_0^y l_1(s,t)dsdt$$

$$+\int_0^x\int_0^y l_1(s,t)r_1(s,t)dsdt+\frac{1}{9}r_1(s,t)-\frac{c_1}{9}-\frac{1}{3}\int_0^x\int_0^y l_2(s,t)r_1(s,t)dsdt$$

$$+\frac{1}{9}r_1(s,t)-\frac{c_1}{9}-\frac{1}{3}\int_0^x\int_0^y l_3(s,t)r_1(s,t)dsdt=c_1+\int_0^x\int_0^y l_1(s,t)dsdt$$

$$+\int_0^x\int_0^y\Big[l_1(s,t)-\frac{1}{3}\Big(l_2(s,t)+l_3(s,t)\Big)\Big]r_1(s,t)dsdt+\frac{2}{9}r_1(s,t)-\frac{2c_1}{9}$$

$$\le\frac{7c_1}{9}+\int_0^x\int_0^y l_1(s,t)dsdt+\frac{2}{9}r_1(s,t)\le\frac{7c_1}{9}+\frac{21c_1}{9}\Big(\sum_{i=1}^3\int_0^x\int_0^y l_i(s,t)dsdt$$

$$+\ c_3x+c_2y\Big)+\frac{2}{9}r_1(s,t)=\frac{7c_1}{9}(1+3H(x,y))+\frac{2}{9}r_1(s,t)\le\frac{7c_1}{9}e^{3H(x,y)}$$

$$+\frac{2}{9}r_1(s,t)=r_1(x,y).$$

This finishes the proof of (14). Now, from inequalities (12), (13), the fact that $c_3\ge 1$, $c_1\ge 3$, assumption (6) and the inequality $c_1\le r_1(x,y)$ for $(x,y)\in I$, we have

$$c_1+\int_0^y\Big\{l_1(x,t)+l_1(x,t)r_1(x,t)+\Big(c_2+\int_0^x l_2(z,t)dz\Big)r_2(x,t)$$

$$+\Big(c_3+\int_0^t l_3(x,z)dz\Big)r_3(x,t)\Big\}dt\le c_1+\int_0^y l_1(x,t)dt$$

$$+\int_0^y l_1(x,t)r_1(x,t)dt+\frac{1}{3}r_2(x,y)-\frac{c_1c_3}{3}-\frac{1}{3}\sum_{i=1}^3\int_0^y l_i(x,t)r_1(x,t)dt$$

$$+\frac{1}{3}r_2(x,y)-\frac{c_1c_3}{3}-\frac{1}{3}\int_0^y l_3(x,t)r_1(x,t)dt=c_1-\frac{2c_1c_3}{3}+\int_0^y l_1(x,t)dt$$

$$+\int_0^y\left[l_1(x,t)-\frac{1}{3}\left(\sum_{i=1}^{3}\int_0^y l_i(x,t)+l_3(x,t)\right)\right]r_1(x,t)dt+\frac{2}{3}r_2(x,y)$$

$$\leq c_1c_3-\frac{2c_1c_3}{3}+\frac{c_1}{3}\int_0^y l_1(x,t)dt+\frac{2}{3}r_2(x,y)\leq\frac{c_1}{3}\left(c_3+\sum_{i=1}^{3}\int_0^y l_i(x,t)dt\right)$$

$$+\frac{2}{3}r_2(x,y)\leq\frac{1}{3}\left(c_3+\sum_{i=1}^{3}\int_0^y l_i(x,t)dt\right)r_1(x,y)+\frac{2}{3}r_2(x,y)=r_2(x,y).$$

This finishes the proof of (15). The proof of (16) is similar. ■

Our basic assumptions are the following.

Assumption 3.1. Suppose that the function $f : I\times C(D,\mathbb{R}^k)\times\mathbb{R}^k\times\mathbb{R}^k\to\mathbb{R}^k$ of the variable (x,y,ω,μ,ν) is such that

1° $f(\cdot,\cdot,\omega,\mu,\nu)$ is measurable for all $\omega\in C(D,\mathbb{R}^k)$, $\mu\in\mathbb{R}^k$ and $\nu\in\mathbb{R}^k$.
2° There are $l_i\in L^1(I,\mathbb{R}_+)$ for $i=1,2,3$, modulus of continuity d_f and constants $c_2\geq 1$, $c_3\geq 1$ such that

$$l_2(x,y)+l_3(x,y)\geq 3l_1(x,y)\text{ for }(x,y)\in I, \tag{17}$$

$$|f(x,y,\omega,\mu,\nu)|\leq l_1(x,y)\Big\{1+||\omega||_0\Big\} \tag{18}$$
$$+\left(c_2+\int_0^x l_2(s,y)ds\right)|\mu|+\left(c_3+\int_0^y l_3(x,t)dt\right)|\nu|,$$

$$|f(x,y,\tilde\omega,\tilde\mu,\tilde\nu)-f(x,y,\omega,\mu,\nu)|\leq l_1(x,y)d_f\big(||\tilde\omega-\omega||_0\big) \tag{19}$$
$$+\left(c_2+\int_0^x l_2(s,y)ds\right)|\tilde\mu-\mu|+\left(c_3+\int_0^y l_3(x,t)dt\right)|\tilde\nu-\nu|,$$

for all (ω,μ,ν),$(\tilde\omega,\tilde\mu,\tilde\nu\in C(I^*,\mathbb{R}^k)\times C_y(I,\mathbb{R}^k)\times C_x(I,\mathbb{R}^k)$.

3° There exist a modulus of continuity d such that

$$\int_0^a \int_0^b |f(x+h, y+k, \omega, \mu, \nu) - f(x, y, \omega, \mu, \nu)| dxdy \le d(h) + d(k), \tag{20}$$

for all $(\omega, \mu, \nu) \in C(I^*, \mathbb{R}^k) \times C_y(I, \mathbb{R}^k) \times C_x(I, \mathbb{R}^k)$ such that $|\omega| \le r_1(x, y)$, $|\mu| \le r_2(x, y)$ and $|\nu| \le r_3(x, y)$.

Assumption 3.2. Suppose that:

1° $\psi : I_0 \to \mathbb{R}^k$ is continuous on I_0, $\psi(\cdot, 0)$ and $\psi(0, \cdot)$ are absolutely continuous on $[0, a]$ and $[0, b]$, respectively.

2° There is a constant A such that

$$\operatorname*{ess\,sup}_{x\in[0,a]} \left|\frac{\partial\psi}{\partial x}(x, 0)\right| \le A \quad \text{and} \quad \operatorname*{ess\,sup}_{y\in[0,b]} \left|\frac{\partial\psi}{\partial y}(0, y)\right| \le A. \tag{21}$$

Remark 3.2. From the theorem of continuity in L^p, there exists a modulus of continuity d such that

$$\int_0^a \left|\frac{\partial\psi}{\partial x}(x+h, 0) - \frac{\partial\psi}{\partial x}(x, 0)\right| dx \le d(h),$$

$$\int_0^b \left|\frac{\partial\psi}{\partial y}(0, y+h) - \frac{\partial\psi}{\partial y}(0, y)\right| dy \le d(h).$$

Theorem 3.1. (existence) *Assume that the functions f and ψ satisfy assumptions* 3.1 *and* 3.2, *respectively. Then the problem* (1), (2) *has a solution existing in I. Furthermore, every solution satisfies the estimates*

$$|u(x, y)| \le r_1(x, y), \ \left|\frac{\partial u}{\partial x}(x, y)\right| \le r_2(x, y) \ \textit{and} \ \left|\frac{\partial u}{\partial y}(x, y)\right| \le r_3(x, y),$$

where r_1, r_2, r_3 are defined by (7)–(9) *with* $c_1 = \max\{3, 3||\psi||, A\}$.

Proof. Let $u_1(x, y) = u(x, y)$, $u_2(x, y) = \frac{\partial u}{\partial x}(x, y)$, $u_3(x, y) = \frac{\partial u}{\partial y}(x, y)$ and $\tilde{u} = (u_1, u_2, u_3)$. Then the problem (1), (2) is equivalent to the fixed point equation $u_i = S_i\tilde{u}$ for $i = 1, 2, 3$, where the operators S_i are defined by

$$
\begin{aligned}
\big(S_1\tilde{u}\big)(x,y) &= \psi(x,y) \quad \text{on } I_0,\\
\big(S_1\tilde{u}\big)(x,y) &= \psi(x,0)+\psi(0,y)-\psi(0,0)\\
&\qquad + \int_0^x\int_0^y f(s,t,u_{1(s,t)},u_2(s,t),u_3(s,t))dsdt \quad \text{on } I,\\
\big(S_2\tilde{u}\big)(x,y) &= \frac{\partial\psi}{\partial x}(x,0)+\int_0^y f(x,t,u_{1(x,t)},u_2(x,t),u_3(x,t))dt \text{ on } I,\\
\big(S_3\tilde{u}\big)(x,y) &= \frac{\partial\psi}{\partial y}(0,y)+\int_0^x f(s,y,u_{1(s,y)},u_2(s,y),u_3(s,y))ds \text{ on } I.
\end{aligned}
$$

Let u_1, u_2, u_3 be a solution of our problem and $\rho_1(x,y) = \max\{|u_1(x+s, y+t)| : (s,t)\in D\}$, $\rho_2(x,y) = |u_2(x,y)|$ and $\rho_3(x,y) = |u_3(x,y)|$. Then we have

$$
\rho_1(x,y) \le c_1 \text{ on } I_0,
$$

$$
\rho_1(x,y) \le c_1 + \int_0^x\int_0^y \Big\{l_1(s,t)+l_1(s,t)\rho_1(s,t) + \Big(c_2+\int_0^s l_2(z,t)dz\Big)\rho_2(s,t) + \Big(c_3+\int_0^t l_3(s,z)dz\Big)\rho_3(s,t)\Big\}dsdt \text{ on } I, \tag{22}
$$

$$
\rho_2(x,y) \le c_1 + \int_0^y \Big\{l_1(x,t)+l_1(x,t)\rho_1(x,t) + \Big(c_2+\int_0^x l_2(z,t)dz\Big)\rho_2(x,t) + \Big(c_3+\int_0^t l_3(x,z)dz\Big)\rho_3(x,t)\Big\}dt \text{ on } I, \tag{23}
$$

$$
\rho_3(x,y) \le c_1 + \int_0^x \Big\{l_1(s,y)+l_1(s,y)\rho_1(s,y) + \Big(c_2+\int_0^s l_2(z,y)\Big)\rho_2(s,y) + \Big(c_3+\int_0^y l_3(s,z)dz\Big)\rho_3(s,y)\Big\}ds \text{ on } I. \tag{24}
$$

From (14) and (22), (15) and (23), (16) and (24), we have $\rho_1(x,y)\le r_1(x,y)$ on I^*, $\rho_2(x,y)\le r_2(x,y)$ on I and $\rho_3(x,y)\le r_3(x,y)$ on I.

We consider the Banach space

$$
C(I^*,\mathbb{R}^k)\times C_y(I,\mathbb{R}^k)\times C_x(I,\mathbb{R}^k)
$$

with the norm

$$|u| = \max\{||u_1||, ||u_2||_y, ||u_3||_x\}.$$

Let us consider the set M of all $(u_1, u_2, u_3) \in C(I^*, \mathbb{R}^k) \times C_y(I, \mathbb{R}^k) \times C_x(I, \mathbb{R}^k)$ satisfying inequalities

$$|u_1(x, y)| \leq r_1(x, y), \quad |u_2(x, y)| \leq r_2(x, y), \quad |u_3(x, y)| \leq r_3(x, y),$$

$$\int_0^x |u_2(s + h, y + k) - u_2(s, y)| ds \leq e^{3H(x,y)}(\tilde{d}(h) + \tilde{d}(k)),$$

$$\int_0^y |u_3(x + h, t + k) - u_3(x, t)| dt \leq e^{3H(x,y)}(\tilde{d}(h) + \tilde{d}(k)).$$

Here $\tilde{d} : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity which will be defined later.

We note that from (18) we conclude that there exists a function $m \in L^1(I, \mathbb{R}_+)$ such that

$$|f(x, y, u_{1(x,y)}, u_2(x, y), u_3(x, y))| \leq m(x, y) \tag{25}$$

for $(u_1, u_2, u_3) \in M$. Let $h, k \geq 0$. Then, for $(x, y) \in I$ we have

$$\begin{aligned}
&|u_1(x + h, y + k) - u_1(x, y))| = \Big| \psi(x + h, 0) + \psi(0, y + k) \\
&+ \int_0^{x+h} \int_0^{y+k} f(s, t, u_{1(s,t)}, u_2(s, t), u_3(s, t)) ds dt - \psi(x, 0) - \psi(0, y) \\
&- \int_0^x \int_0^y f(s, t, u_{1(s,t)}, u_2(s, t), u_3(s, t)) ds dt \Big| \leq d_\psi(h) + d_\psi(k) \\
&+ \Big| \int_0^{x+h} \int_y^{y+k} f(s, t, u_{1(s,t)}, u_2(s, t), u_3(s, t)) ds dt \\
&+ \int_x^{x+h} \int_0^y f(s, t, u_{1(s,t)}, u_2(s, t), u_3(s, t)) ds dt \Big| \\
&\leq d_\psi(h) + d_\psi(k) + \int_0^a \max_{y \in [0,b]} \Big| \int_y^{y+k} m(s, t) dt \Big| ds \\
&+ \int_0^b \max_{x \in [0,a]} \Big| \int_x^{x+h} m(s, t) ds \Big| dt,
\end{aligned} \tag{26}$$

where d_ψ is a modulus of continuity of ψ.

For $(x,y) \in I_0$ and $(x+h, y+h) \in I_0$ we get

$$|u_1(x+h, y+k) - u_1(x,y))| = |\psi(x+h, y+k) - \psi(x,y)| \tag{27}$$
$$= |\psi(x+h, y+k) - \psi(x+h, y) + \psi(x+h, y) - \psi(x,y)| \leq d_\psi(h) + d_\psi(k).$$

We note that if $(x,y) \in I_0$ and $(x+h, y+k) \in I$, then

$$\begin{cases} x+h \geq 0 \\ x \leq 0 \end{cases} \tag{28}$$

or

$$\begin{cases} y+k \geq 0 \\ y \leq 0. \end{cases} \tag{29}$$

If (28) is satisfied, then $|x| \leq h$ and $|x+h| \leq h$. If (29) holds, then $|y| \leq k$ and $|y+k| \leq k$.

Let (28) be satisfied, then

$$\begin{aligned}
&|u_1(x+h, y+k) - u_1(x,y))| = \Big|\psi(x+h, 0) + \psi(0, y+k) - \psi(0,0) \\
&+ \int_0^{x+h} \int_0^{y+k} f(s,t,u_{1(s,t)}, u_2(s,t), u_3(s,t))dsdt - \psi(x,y)\Big| \\
&\leq \int_0^{x+h} \int_0^{y+k} m(s,t)dsdt + |\psi(x+h, 0) + \psi(0, y+k) \\
&- \psi(0,0) - \psi(x,y) + \psi(x, y+k) - \psi(x, y+k)| \\
&\leq \int_0^{x+h} \int_0^{b} m(s,t)dsdt + |\psi(x+h, 0) - \psi(0,0)| \\
&+ |\psi(0, y+k) - \psi(x, y+k)| + |\psi(x, y+k) - \psi(x,y)| \\
&\leq \int_0^{x+h} \int_0^{b} m(s,t)dsdt + 2d_\psi(h) + d_\psi(k).
\end{aligned} \tag{30}$$

If (29) holds, then using the fact that $|y| \leq k$ and $|y+k| \leq k$ and the following steps analogous to those above, we obtain

$$\begin{aligned}
&|u_1(x+h, y+k) - u_1(x,y))| \\
&\leq \int_0^{y+k} \int_0^{a} m(s,t)dtds + d_\psi(h) + 2d_\psi(k).
\end{aligned} \tag{31}$$

Define $\tilde{d} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$
\begin{aligned}
\tilde{d}(h) &= 6d(h) + d_f\left(2d_\psi(h) + \int_0^a \max_{y\in[0,b]} \left| \int_y^{y+h} m(x,t)dt \right| dx\right) \\
&+ \int_0^a \max_{y\in[0,b]} \left| \int_y^{y+h} m(x,t)dt \right| dx \\
&+ d_f\left(2d_\psi(h) + \int_0^b \max_{x\in[0,a]} \left| \int_x^{x+h} m(s,y)ds \right| dy\right) \\
&+ \int_0^b \max_{x\in[0,a]} \left| \int_x^{x+h} m(s,y)ds \right| dy.
\end{aligned} \tag{32}
$$

M is the convex, bounded, closed subset of a Banach space. Let $u = (u_1, u_2, u_3)$, $u^n = (u_1^n, u_2^n, u_3^n)$ and $u_1^n \to u_1$ in $C(I^*, \mathbb{R}^k)$, $u_2^n \to u_2$ in $C_y(I, \mathbb{R}^k)$, $u_3^n \to u_3$ in $C_x(I, \mathbb{R}^k)$, that is,

$$||u_1^n - u_1|| \to 0,\ |u_2^n - u_2|_y \to 0 \text{ i } |u_3^n - u_3|_x \to 0 \text{ with } n \to \infty.$$

Then,

$$
\begin{aligned}
&|S\tilde{u}^n - S\tilde{u}| = \max\{||S_1\tilde{u}^n - S_1\tilde{u}||, ||S_2\tilde{u}^n - S_2\tilde{u}||_y, ||S_3\tilde{u}^n - S_3\tilde{u}||_x\} \\
&\le \int_0^a \int_0^b |f(x,y,u^n_{1(x,y)}, u_2^n(x,y), u_3^n(x,y)) \\
&\quad - f(x,y,u_{1(x,y)}, u_2(x,y), u_3(x,y))|dxdy \\
&\le \int_0^a \int_0^b [l_1(x,y) d_f(||u^n_{1(x,y)} - u_{1(x,y)}||_0) \\
&\quad + \left(c_2 + \int_0^x l_2(s,y)ds\right) |u_2^n(x,y) - u_2(x,y)| \\
&\quad + \left(c_3 + \int_0^y l_3(x,t)dt\right) |u_3^n(x,y) - u_3(x,y)|] dxdy \\
&\le \int_0^a \int_0^b l_1(x,y)dxdy \ d_f(||u_1^n - u_1||) \\
&\quad + \int_0^b \left[\left(c_2 + \int_0^a l_2(s,y)ds\right) \int_0^a |u_2^n(x,y) - u_2(x,y)|dx\right] dy
\end{aligned}
$$

$$+\int_0^a\Big[\Big(c_3+\int_0^b l_3(x,t)dt\Big)\int_0^b |u_3^n(x,y)-u_3(x,y)|dy\Big]dx$$

$$\leq \int_0^a\int_0^b l_1(x,y)dxdy\ d_f(||u_1^n-u_1||)$$

$$+\left(c_2 b+\int_0^a\int_0^b l_2(x,y)dxdy\right)\int_0^a \max_{y\in[0,b]}|u_2^n(x,y)-u_2(x,y)|dx$$

$$+\left(c_3 a+\int_0^a\int_0^b l_3(x,y)dxdy\right)\int_0^b \max_{x\in[0,a]}|u_3^n(x,y)-u_3(x,y)|dy,$$

hence

$$\big|S\tilde{u}^n-S\tilde{u}\big|\to 0 \quad \text{if } n\to\infty$$

and that means that the operator S is continuous.

For the rest of this proof we assume that $(u_1,u_2,u_3)\in M$. We will show that the set $S(M)$ is compact. Let $\zeta_1(x,y)=\big(S_1u\big)(x,y)$, $\zeta_2(x,y)=\big(S_2u\big)(x,y)$, $\zeta_3(x,y)=\big(S_3u\big)(x,y)$. Of course, $\zeta_1(x,y)\leq r_1(x,y)$, $\zeta_2(x,y)\leq r_2(x,y)$, $\zeta_3(x,y)\leq r_3(x,y)$ for $(x,y)\in I$ and $\zeta_1(x,y)\leq c$ for $(x,y)\in I_0$. Moreover,

$$\begin{aligned}
\left|\frac{\partial\zeta_1}{\partial x}(x,y)\right| &= \left|\frac{\partial\psi}{\partial x}(x,0)+\int_0^y f(x,z,u_{1(x,z)},u_2(x,z),u_3(x,z))dz\right|\\
&\leq \left|\frac{\partial\psi}{\partial x}(x,0)\right|+\int_0^b\Big\{l_1(x,z)\Big[1+r_1(x,z)\Big]\\
&\quad+\Big(c_2+\int_0^x l_2(s,z)ds\Big)r_2(x,z)\\
&\quad+\Big(c_3+\int_0^z l_3(x,t)dt\Big)r_3(x,z)\Big\}dz,\\
\left|\frac{\partial\zeta_1}{\partial y}(x,y)\right| &\leq \left|\frac{\partial\psi}{\partial y}(0,y)\right|+\int_0^a\Big\{l_1(z,y)\Big[1+r_1(z,y)\Big]\\
&\quad+\Big(c_2+\int_0^z l_2(s,y)ds\Big)r_2(z,y)\\
&\quad+\Big(c_3+\int_0^y l_3(z,t)dt\Big)r_3(z,y)\Big\}dz.
\end{aligned}$$

Therefore, the set $S_1(M)$ is equicontinuous. Now Ascoli-Arzéla theorem guarantees that $S_1(M)$ is a relatively compact subset of $C(I^*, \mathbb{R}^k)$.

From (21) and (25), we get the existence of a constant $C > 0$ such that

$$\int_0^a \max_{y\in[0,b]} |\zeta_2(x,y)|dx \le \int_0^a \left|\frac{\partial\psi}{\partial x}(x,0)\right|dx$$
$$+ \int_0^a \int_0^b |f(x,y,u_{1(x,y)},u_2(x,y),u_3(x,y))|dxdy \le C,$$

which proves that ζ_2 satisfies (3).

From (25), we have

$$\int_0^a |\zeta_2(x,y') - \zeta_2(x,y)|dx < \left|\int_0^a \int_y^{y'} m(x,t)dxdt\right| \to 0 \text{ as } y' \to y,$$

which proves that ζ_2 satisfies (4).

$$\int_0^x \int_0^y |f(s,t,u_{1(s+h,t)},u_2(s+h,t),u_3(s+h,t))$$
$$- f(s,t,u_{1(s,t)},u_2(s,t),u_3(s,t))|dsdt$$
$$\le \int_0^x \int_0^y \Big[l_1(s,t)d_f(||u_{1(s+h,t)} - u_{1(s,t)}||_0)$$
$$+\left(c_2 + \int_0^s l_2(z,t)dz\right)|u_2(s+h,t) - u_2(s,t)|$$
$$+\left(c_3 + \int_0^t l_3(s,z)dz\right)|u_3(s+h,t) - u_3(s,t)|\Big] dsdt$$
$$= \int_0^x \int_0^y l_1(s,t)dsdt\; d_f\left(2d_\psi(h) + \int_0^b \max_{x\in[0,a]}\left|\int_x^{x+h} m(s,t)ds\right|dt\right)$$
$$+ \int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)\int_0^x |u_2(s+h,t) - u_2(s,t)|ds\right]dt$$
$$+ \int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)\int_0^y |u_3(s+h,t) - u_3(s,t)|dt\right]ds$$

$$\leq \int_0^x \int_0^y l_1(s,t)dsdt\ d_f\biggl(2d_\psi(h) + \int_0^b \max_{x\in[0,a]}\biggl|\int_x^{x+h} m(s,t)ds\biggr|dt\biggr)$$

$$+\int_0^y \biggl(c_2 + \int_0^x l_2(s,t)ds\biggr)e^{3H(x,t)}\tilde{d}(h)dt$$

$$+\int_0^x \biggl(c_3 + \int_0^y l_3(s,t)dt\biggr)e^{3H(s,y)}\tilde{d}(h)ds$$

$$\leq \frac{1}{3}e^{3H(x,y)}d_f\biggl(2d_\psi(h) + \int_0^b \max_{x\in[0,a]}\biggl|\int_x^{x+h} m(s,t)ds\biggr|dt\biggr)$$

$$+\frac{1}{3}\bigl(e^{3H(x,y)} - e^{3c_3x}\bigr)\tilde{d}(h) + \frac{1}{3}\bigl(e^{3H(x,y)} - e^{3c_2y}\bigr)\tilde{d}(h)$$

$$\leq \frac{1}{3}e^{3H(x,y)}d_f\biggl(2d_\psi(h) + \int_0^b \max_{x\in[0,a]}\biggl|\int_x^{x+h} m(s,t)ds\biggr|dt\biggr) + \frac{2}{3}e^{3H(x,y)}\tilde{d}(h).$$

It is clear that

$$\int_0^a \max_{y\in[0,b]}|\zeta_2(x+h,y) - \zeta_2(x,y)|dx \leq \int_0^a \biggl|\frac{\partial\psi}{\partial x}(x+h,0) - \frac{\partial\psi}{\partial x}(x,0)\biggr|dx$$

$$+\int_0^a\int_0^b |f(x+h,y,u_{1(x+h,y)},u_2(x+h,y),u_3(x+h,y))$$

$$-f(x,y,u_{1(x+h,y)},u_2(x+h,y),u_3(x+h,y))|dxdy$$

$$+\int_0^a\int_0^b |f(x,y,u_{1(x+h,y)},u_2(x+h,y),u_3(x+h,y))$$

$$-f(x,y,u_{1(x,y)},u_2(x,y),u_3(x,y))|dxdy$$

$$\leq\ 2d(h) + \frac{1}{3}e^{3H(a,b)}d_f\biggl(2d_\psi(h) + \int_0^b \max_{x\in[0,a]}\biggl|\int_x^{x+h} m(s,t)ds\biggr|dt\biggr)$$

$$+\frac{2}{3}e^{3H(a,b)}\tilde{d}(h),$$

which proves that ζ_2 satisfies (5). Now Lemma 2.1 guarantees that $S_2(M)$ is a relatively compact subset of $C_y(I,\mathbb{R}^k)$. Analogously, we show that $S_3(M)$ is a relatively compact subset of $C_x(I,\mathbb{R}^k)$. Hence, $S(M)$ is a relatively compact subset of $C(I^*,\mathbb{R}^k)\times C_y(I,\mathbb{R}^k)\times C_x(I,\mathbb{R}^k)$.

We will show that $S(M)\subset M$. We know that $\zeta_1(x,y)\le r_1(x,y)$, $\zeta_2(x,y)\le r_2(x,y)$, $\zeta_3(x,y)\le r_3(x,y)$ for $(x,y)\in I$ and $\zeta_1(x,y)\le c$ for $(x,y)\in I_0$. It is easy to check that

$$\begin{aligned}
&\int_0^x |\zeta_2(s+h,y+k)-\zeta_2(s,y)|ds \le \int_0^x \left|\frac{\partial\psi}{\partial x}(s+h,0)-\frac{\partial\psi}{\partial x}(s,0)\right|ds\\
&\quad+\int_0^x \Bigg| \int_0^{y+k} f(s+h,t,u_{1(s+h,t)},u_2(s+h,t),u_3(s+h,t))dt\\
&\quad-\int_0^{y+k} f(s,t,u_{1(s+h,t)},u_2(s+h,t),u_3(s+h,t))dt\\
&\quad+\int_0^{y+k} f(s,t,u_{1(s+h,t)},u_2(s+h,t),u_3(s+h,t))dt\\
&\quad-\int_0^{y} f(s,t,u_{1(s+h,t)},u_2(s+h,t),u_3(s+h,t))dt\\
&\quad+\int_0^{y} f(s,t,u_{1(s+h,t)},u_2(s+h,t),u_3(s+h,t))dt\\
&\quad-\int_0^{y} f(s,t,u_{1(s,t)},u_2(s,t),u_3(s,t))dt\Bigg|\\
&\le d(h)+d(h)+\int_0^a \max_{y\in[0,b]}\left|\int_y^{y+k} m(x,t)dt\right|dx\\
&\quad+\frac{1}{3}e^{3H(x,y)}d_f\left(2d_\psi(h)+\int_0^b \max_{x\in[0,a]}\left|\int_x^{x+h} m(s,t)ds\right|dt\right)+\frac{2}{3}e^{3H(x,y)}\tilde d(h)\\
&\le \frac{1}{3}e^{3H(x,y)}\tilde d(h)+\tilde d(k)+\frac{2}{3}e^{3H(x,y)}\tilde d(h)\le(\tilde d(h)+\tilde d(k))e^{3H(x,y)}.
\end{aligned}$$

Arguments similar to those above show that

$$\int_0^y |\zeta_3(x+h,t+k)-\zeta_3(x,t)|dt \le (\tilde d(h)+\tilde d(k))e^{3H(x,y)}.$$

This yields $S(M)\subset M$. Using the Schauder's fixed point theorem, we get the existence of a solution of problem (1), (2). ∎

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