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OPTIMAL CONTROL PROBLEMS WITH UPPER SEMICONTINUOUS HAMILTONIANS

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Abstract

In this paper we give examples of value functions in Bolza problem that are not bilateral or viscosity solutions and an example of a smooth value function that is even not a classic solution (in particular, it can be neither the viscosity nor the bilateral solution) of Hamilton-Jacobi-Bellman equation with upper semicontinuous Hamiltonian. Good properties of value functions motivate us to introduce approximate solutions of equations with such type Hamiltonians. We show that the value function is the unique approximate solution.

Keywords: Hamilton-Jacobi-Bellman equation, Bolza problem, viscosity solution, bilateral solution, monotonic approximation, semicontinuous Hamiltonian.

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1. Introduction

Having a Hamiltonian $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$, we can define a Lagrangian $L : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ by

$$L(t,x,f) := \sup_{p \in \mathbb{R}^l} \ \langle f,p \rangle - H(t,x,p),$$

then we have $L(t,x,f) = H^*(t,x,f)$, where "*" denotes the Legendre-Fenchel transform with respect to the last variable. Analyzing the problems in this paper we concentrate on upper semicontinuous Hamiltonians of linear growth and convex with respect to the last variable. In view of the inverse procedure we can obtain the well known equality $H(t,x,p) = L^*(t,x,p)$.

Given a point $(t_0, x_0) \in [0,T] \times \mathbb{R}^l$, a terminal cost $g : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ and Lagrangian, we consider the generalized problem of Bolza:

$$\mathcal{P}(t_0, x_0): \quad \text{minimize} \quad g(x(T)) + \int_{t_0}^{T} L(t, x(t), \dot{x}(t))dt \quad \text{subject to} \quad x(t_0) = x_0$$

with the minimization carried out over all absolutely continuous arcs $x : [t_0, T] \to \mathbb{R}^l$.

Optimal control problems can be reformulated in Bolza problem: see Clarke [7] or Bardi and Capuzzo-Dolcetta [2]. In section 2 we discuss conditions and facts that we need in further sections of this paper.

Section 3 studies regularity of the *value function* $V : [0,T] \times \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$, defined as the optimal value in $\mathcal{P}(t_0, x_0)$ parameterized by the initial condition. Using techniques from Cesari [6] we show that increasingly convergent sequence of Lagrangians L_n to the Lagrangian L (i.e., $L_n \leq L_{n+1}$ and $L_n \to L$ – pointwise) implies increasing convergence of value functions V_n to value function V, which particularly is an epigraphical convergence. Furthermore, if $x_n(\cdot)$ is optimal trajectory of the value function $V_n(t,x)$, then accumulation points of a sequence $\{x_n(\cdot)\}_{n\in\mathbb{N}}$ are optimal trajectories of the value function $V(t,x)$. From the duality, the previous result can be stated in the following way: decreasing convergence of a sequence of Hamiltonians H_n to the Hamiltonian H, so that it is also hipographical convergence, implies increasing convergence of value functions V_n to the value function V. Similar problems concerning convergence of value functions were investigated by Buttazzo and Dal Maso [5], Briani [4] and Frankowska [9].

In Section 4 we show that upper semicontinuous Hamiltonian H of linear growth and convex with respect to the last variable can be decreasingly approximated by Hamiltonians H_n (i.e., $H_{n+1} \leq H_n$ and $H_n \to H$ – pointwise), which inherit properties of H, besides, they are locally Lipschitz uniformly with respect to p (i.e., $|H(t,x,p) - H(t',x',p)| \leq C(|t-t'| + |x-x'|)(1+|p|)$ for all $p \in \mathbb{R}^l$, $t,t' \in [0,T]$, $x, x' \in B_R := \{x \in \mathbb{R}^l;\ |x| < R\}$). Next, we formulate the above fact in Lagrangian notation, which gives us a version of Antosiewicz-Cellina theorem (Theorem 1.13.1, [1]). Problems of the convergence of value functions and the approximation of Hamiltonian were also studied by Goebel (see [10]), for the case of concave-convex Hamiltonian.

In Section 5 we give examples of upper semicontinuous Hamiltonians, whose value functions are not bilateral or viscosity solutions and even an example of a smooth value function that is not the classic solution (in particular, it can be neither the viscosity nor the bilateral solution). Moreover, we show (not necessarily with an assumption about convexity of Lagrangian) that the value function is locally Lipschitz. Results obtained in sections 3 and 4 allow to conclude that the value function of upper semicontinuous Hamiltonian can be monotonically approximated by locally Lipschitz value functions, which are, in the class of locally Lipschitz functions satisfying boundary condition, unique bilateral (viscosity) solutions of adequate Hamilton-Jacobi-Bellman equations. Assuming that Hamiltonian is continuous, then Barron and Jensen Proposition 3.2 in [3] and our studies imply that the value function is a bilateral solution.

In Section 6 we propose definitions of approximate solutions of Hamilton-Jacobi-Bellman equation, with upper semicontinuous Hamiltonian. Of course, the offered definition includes examples from Section 5. We compare approximate solutions to bilateral, viscosity and classical solutions. Finally, we formulate a theorem about existence and uniqueness of approximate solutions.

2. Preliminaries

Let us introduce conditions needed in this paper which, in fact, are typical of optimal control problems.

1. Conditions responding to Hamiltonian $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$
   - (H1) H is upper semicontinuous,
   - (H2) $H(t,x,p)$ is convex with respect to p for every $t \in [0,T]$, $x \in \mathbb{R}^l$,
   - (H3) $H(t,x,p) \leq C(1+|x|)(|p|+1)$ for every $t \in [0,T]$, $x,p \in \mathbb{R}^l$ and constant $C > 0$,
   - (H4) $H(t,x,p) \geq -C(1+|x|)(|p|+1)$ for every $t \in [0,T]$, $x,p \in \mathbb{R}^l$ and constant $C > 0$.
2. Conditions responding to Lagrangian $L : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$
   - (L1) L is lower semicontinuous,
   - (L2) $L(t,x,f)$ is convex, proper with respect to f for every $t \in [0,T]$, $x \in \mathbb{R}^l$,

(L3) $L(t,x,f) \geq -C(1+|x|)$ for every $t \in [0,T]$, $x, f \in \mathbb{R}^l$ and constant $C > 0$,

(L4) there exists a constant $C > 0$ such that for all $f, x \in \mathbb{R}^l$ and $t \in [0,T]$ the following implication proceeds:

$$|f| > C(1+|x|) \Rightarrow L(t,x,f) = +\infty,$$

(L5) there exists a constant $C > 0$ such that for all $t \in [0,T]$, $x \in \mathbb{R}^l$ there exists $f \in \mathbb{R}^l$ such that $\max\{L(t,x,f), |f|\} \leq C(1+|x|)$.

3. Locally Lipschitz type condition of the Hamiltonian:

(HL) $\forall R > 0\ \exists K > 0\ \forall p \in \mathbb{R}^l\ \forall t, t' \in [0,T]\ \forall x, x' \in B_R$ the following inequality holds $|H(t,x,p) - H(t',x',p)| \leq K(|t-t'| + |x-x'|)$ $(|p|+1)$.

It is not hard to prove the following equivalences: the first one says that (H1)–(H3) are satisfied if and only if (L1)–(L4) are satisfied. The second one says that (H1)–(H4) are satisfied if and only if (L1)–(L5) are satisfied. Moreover, if H satisfies (H1)–(H3) and (HL), then $Q(t,x) := \mathrm{Epi}(L(t,x,\cdot))$ (where L is dual to H) is locally Lipschitz with respect to Hausdorff distance (see Proposition 2.2, [12]).

Assume that $g : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function and Lagrangian $L : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ satisfies (L1), (L3) and (L4), then the value function

$$V(t_0, x_0) = \inf_{x(\cdot) \in AC[t_0,T]\ \ x(t_0) = x_0} g(x(T)) + \int_{t_0}^{T} L(t, x(t), \dot{x}(t))dt$$

is defined from $[0,T] \times \mathbb{R}^l$ in $\mathbb{R} \cup \{+\infty\}$. Moreover modifying proofs in the paper of Plaskacz-Quincampoix [12] we deduce that the value function is lower semicontinuous and has Lipschitz minimizer if Lagrangian satisfies conditions (L1)–(L4).

Subdifferential $D_{-}w(x_0)$ of the lower semicontinuous function $w : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ at $x_0 \in Dom(w)$ is given by

$$D_{-}w(x_0) = \left\{ p \in \mathbb{R}^d \ ;\ \liminf_{x \to x_0} \frac{w(x) - w(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \geq 0 \right\}.$$

Subdifferential $D^{+}w(x_0)$ of the upper semicontinuous function $w : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ at $x_0 \in Dom(w)$ is given by $D^{+}w(x_0) = -D_{-}(-w(x_0))$.

Moreover, properties (L1)–(L4) imply that the set-valued map Q has the following property (cf. Section 10.5 in [6])

$$Q(t,x) = \bigcap_{\varepsilon>0} \overline{\operatorname{conv}}\, Q(t,x,\varepsilon), \tag{1}$$

where

$$Q(t,x,\varepsilon) := \bigcup_{|t-t'|<\varepsilon,\ |x-x'|<\varepsilon} Q(t',x').$$

3. Monotonic convergence of the value function

We use the technique from Cesari [6] to prove the result of monotonic convergence of the value function.

Definition 3.1 (Kuratowski epi/hypo-convergence)**.** A sequence of functions $f_n : \mathbb{R}^l \to [-\infty,+\infty]$, $n = 1,2,\dots$, epi-converges to f (e-lim $f_n = f$ for short) if for every point $x \in \mathbb{R}^l$

(i) $\liminf f_n(x_n) \geq f(x)$ for every sequence $x_n \to x$,
(ii) $\limsup f_n(x_n) \leq f(x)$ for some sequence $x_n \to x$.

We say that a sequence of functions f_n is hypo-convergent to f (h-lim $f_n = f$ for short) if e-lim$(-f_n) = (-f)$.

Let $f_n, f : \mathbb{R}^l \to (-\infty,+\infty]$ be upper (lower) semicontinuous and $f_n \searrow f$ ($f_n \nearrow f$), then h-lim $f_n = f$ (e-lim $f_n = f$). Moreover, if $f_n, f : \mathbb{R}^l \to \mathbb{R}$ are continuous and $f_n \searrow f$, then the sequence of functions f_n is uniformly convergent on a compact set to f. Since $f_n, f : \mathbb{R}^l \to \mathbb{R}$ are convex, we obtain the following equivalence $f_n \searrow f$ if and only if $f_n^* \nearrow f^*$. The sufficient condition in the last equivalence can be proved by using the epigraphical convergence of the sequence $\{f_n^*\}_{n\in\mathbb{N}}$ and super linear growth of the function f_n^*. For details, consult Rockafellar and Wets [15]. These properties give us the following corollary.

Corollary 3.2. *Suppose that H_n, H are dual to L_n, L respectively, then the following properties are equivalent:*

(i) *Hamiltonians H_n, H satisfy* (H1)–(H3) *and $H_n \searrow H$ (this implies* h-lim $H_n = H$),

(ii) *Lagrangians* L_n, L *satisfy* (L1)–(L4) *and* $L_n \nearrow L$ (*this implies* e-$\lim L_n = L$).

Moreover, if H_n, H *are continuous and* $H_n \searrow H$, *then sequence of functions* H_n *is uniformly convergent on a compact set to* H.

Theorem 3.3 (Convergence). *Suppose that Lagrangians* $L_n, L : [0,T]\times\mathbb{R}^l\times\mathbb{R}^l \to \mathbb{R}\cup\{+\infty\}$ *satisfy conditions* (L1)–(L4) *and* $g_n, g : \mathbb{R}^l \to \mathbb{R}\cup\{+\infty\}$ *are lower semicontinuous functions, moreover,* $L_n \nearrow L$ *and* $g_n \nearrow g$. *If* V_n, V *are value functions associated with* L_n, g_n *and* L, g *respectively, then* $V_n \nearrow V$ (*this implies* e-$\lim V_n = V$).

Proof. Fix (t_0, x_0). Monotonicity of L_n, g_n implies that $V_n(t_0,x_0) \leq V_{n+1}(t_0,x_0) \leq V(t_0,x_0)$. Let us define an auxiliary function $W(t_0,x_0) := \lim_{n\to\infty} V_n(t_0,x_0)$, then we have $W \leq V$. To complete the proof it is enough to show that $W = V$. If $W(t_0,x_0) = +\infty$, then $V(t_0,x_0) = +\infty$, so $W(t_0,x_0) = V(t_0,x_0)$. If $W(t_0,x_0)$ is finite, then also $V_n(t_0,x_0)$ is finite. Let every element of the sequence of absolutely continuous functions $x_n : [t_0,T] \to \mathbb{R}^l$ such that $x_n(t_0) = x_0$, be a minimizer of corresponding $V_n(t_0,x_0)$. From the definition of $W(t_0,x_0)$ we obtain

$$g_n(x_n(T)) + \int_{t_0}^{T} L_n(t, x_n(t), \dot{x}_n(t))dt \leq W(t_0,x_0). \tag{2}$$

Next, from monotonicity of L_n, the following inequality is satisfied

$$\int_{t_0}^{T} L_1(t, x_n(t), \dot{x}_n(t))dt \leq \int_{t_0}^{T} L_n(t, x_n(t), \dot{x}_n(t))dt < +\infty.$$

Using assumption (L4) to L_1, we have $|\dot{x}_n(t)| \leq C(1+|x_n(t)|)$ for almost all $t \in [t_0,T]$. By the Gronwall inequality, we have $|x_n(t)| \leq (|x_0| + TC)e^{TC} =: C_1$ for $n \in \mathbb{N}$, $t \in [t_0,T]$ and $|\dot{x}_n(t)| \leq C(1+C_1)$ for $n \in \mathbb{N}$, a.a. $t \in [t_0,T]$. By the Dunford-Pettis criterion (Theorem 0.3.4 in [1]), there exists a subsequence (again denoted by) x_n such that x_n converges uniformely to absolutely continuous function x and $\dot{x}_n$ converges weakly in L^1 to $\dot{x}$. Besides, from (L3) applied to L_1, we have $-C_2(1+C_1) \leq L_n(t, x_n(t), \dot{x}_n(t))$ for $n \in \mathbb{N}$, a.a. $t \in [t_0,T]$. Let $\delta > 0$. Since e-$\lim g_n = g$ and (2) is true, we get $g(x(T))$ is finite and $g(x(T)) - \delta < g_n(x_n(T))$ for almost all $n \in \mathbb{N}$. We can also assume that the following inequality is satisfied:

$$\int_{t_0}^{T} L_n(t, x_n(t), \dot{x}_n(t))dt \leq W(t_0,x_0) - g(x(T)) + \delta.$$

According to Mazur Lemma, there exist non-negative reals $\lambda_{i,N}^k$ such that $\sum_{i=1}^N \lambda_{i,N}^k = 1$ and $\sum_{i=1}^N \lambda_{i,N}^k \dot{x}_{k+i} \to_N \dot{x}$ in L^1 for all $k \in \mathbb{N}$. Then there exists a sequence N_n such that for every $k \in \mathbb{N}$

$$(3) \qquad y_n^k(t) = \sum_{i=1}^{N_n} \lambda_{i,N_n}^k \dot{x}_{k+i}(t) \to_n \dot{x}(t) \quad \text{for a.a. } t \in [t_0, T].$$

Let

$$\eta_n(t) = L_n(t, x_n(t), \dot{x}_n(t)), \quad \eta_n^k = \sum_i^{N_n} \lambda_{i,N_n}^k \eta_{k+i}$$

and

$$\eta^k(t) = \liminf_n \eta_n^k(t), \qquad \eta(t) = \liminf_k \eta^k(t).$$

The following integral can be bounded

$$(4) \qquad \int_{t_0}^T \eta_n^k(t)dt = \sum_i^{N_n} \lambda_{i,N_n}^k \int_{t_0}^T \eta_{k+i}(t)dt \leq W(t_0, x_0) - g(x(T)) + \delta.$$

Using the inequality (4) and Fatou Lemma, we obtain

$$\int_{t_0}^T \eta(t)dt \leq \liminf_k \liminf_n \int_{t_0}^T \eta_n^k(t)dt \leq W(t_0, x_0) - g(x(T)) + \delta.$$

Let $s \in \mathbb{N}$ be fixed. We will show that for almost all $t \in [t_0, T]$ the following inequality is satisfied

$$(5) \qquad \eta(t) \geq L_s(t, x(t), \dot{x}(t)).$$

Let us define $Q_s(t,x) := \mathrm{Epi}(L_s(t,x,\cdot))$. Fix $t \in [t_0,T]$ such that $\eta_n(t)$, $\dot{x}_n(t)$, $y_n^k(t)$, $\eta_n^k(t)$, $\eta^k(t)$, $\eta(t)$ are well defined and finite. For $\varepsilon > 0$ there exists $k_0 \geq s$ such that for every $k \geq k_0$ and all $i \in \mathbb{N}$ we have $|x_{k+i}(t) - x(t)| < \varepsilon$. For $n \geq s$

$$(\dot{x}_n(t), \eta_n(t)) \in Q_s(t, x_n(t)),$$

for $k \geq k_0$, $i \in \mathbb{N}$

$$(\dot{x}_{k+i}(t), \eta_{k+i}(t)) \in Q_s(t, x(t), \varepsilon),$$

for $k \geq k_0$, $n \in \mathbb{N}$

$$(y_n^k(t), \eta_n^k(t)) \in \operatorname{conv} Q_s(t, x(t), \varepsilon),$$

so

$$(\dot{x}(t), \eta^k(t)) \in \overline{\operatorname{conv}}\, Q_s(t, x(t), \varepsilon),$$

so

$$(\dot{x}(t), \eta(t)) \in \overline{\operatorname{conv}}\, Q_s(t, x(t), \varepsilon),$$

ε is arbitrary so

$$(\dot{x}(t), \eta(t)) \in \bigcap_{\varepsilon > 0} \overline{\operatorname{conv}}\, Q_s(t, x(t), \varepsilon).$$

This and (1) imply $\eta(t) \geq L_s(t, x(t), \dot{x}(t))$. Observe that we obtain (5) for almost all $t \in [t_0, T]$. Because $s \in \mathbb{N}$ is also arbitrary, for almost all $t \in [t_0, T]$ and for all $n \in \mathbb{N}$ we obtain inequality

$$\eta(t) \geq L_n(t, x(t), \dot{x}(t)).$$

Then in limit we obtain inequality

$$\eta(t) \geq L(t, x(t), \dot{x}(t)) \quad \text{for a.a. } t \in [t_0, T].$$

Summarizing, we obtain the following inequality

$$g(x(T)) + \int_{t_0}^{T} L(t, x(t), \dot{x}(t))dt \leq W(t_0, x_0) + \delta.$$

Consequently, we have inequality $V(t_0, x_0) \leq W(t_0, x_0) + \delta$. From arbitrariness of δ we get $V(t_0, x_0) \leq W(t_0, x_0)$, so $V(t_0, x_0) = W(t_0, x_0)$. ■

Remark 3.4. Suppose that the assumptions of Theorem 3.3 are satisfied. Then it is not hard to notice from the proof of this theorem that if $x_n(\cdot)$ is optimal trajectory of the value function $V_n(t, x)$, then accumulation points of a sequence $\{x_n(\cdot)\}_{n \in \mathbb{N}}$ are optimal trajectories of the value function $V(t, x)$.

From Theorem 3.3 and Corollary 3.2 we obtain:

Corollary 3.5. *Suppose that Hamiltonians $H_n, H : [0, T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ satisfy conditions* (H1)–(H3) *and $g_n, g : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous functions, moreover, $H_n \searrow H$ and $g_n \nearrow g$. If V_n, V are value* *functions associated with* L_n, g_n *and* L, g *respectively* (*where* L_n, L *are dual to* H_n, H), *then* $V_n \nearrow V$ (*this implies* $\text{e-lim}\, V_n = V$).

4. Approximation of the usc Hamiltonian

Using the technique similar to the one used in the proof of Antosiewicz-Cellina theorem (see Theorem 1.13.1, [1]), we are going to establish the approximation result.

Lemma 4.1. *Let a function* $H : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ *satisfy the following conditions:*

(A) $H(x,p)$ *is convex with respect to* p *for every* $x \in \mathbb{R}^m$,

(B) $|H(x,p)| \leq C(1+|x|)(|p|+1)$ *for every* $x \in \mathbb{R}^m$, $p \in \mathbb{R}^l$ *and constant* $C > 0$,

then we obtain the following inequality $|H(x,p)-H(x,p')| \leq 3C(1+|x|)|p-p'|$ *for every* $x \in \mathbb{R}^m$ *and* $p, p' \in \mathbb{R}^l$.

Proof. Using a similar argument as in the proof of (Theorem 10.4 [14]), we show the conclusion of this lemma. Indeed, let $R > 2$, $p, p' \in B_{R/2}$, $x \in \mathbb{R}^m$ and $p \neq p'$. Put $u = (p'-p)/|p'-p|$, $q = p + Ru$ and $\varepsilon = |p'-p|/R$. So we obtain $p' = (1-\varepsilon)p + \varepsilon q$. We know that H satisfies (A) and (B), so $H(x,p') \leq (1-\varepsilon)H(x,p) + \varepsilon H(x,q)$,

$$\begin{aligned} H(x,p') - H(x,p) &\leq -\varepsilon H(x,p) + \varepsilon H(x,q) \\ &\leq \varepsilon C(1+|x|)(1+|p|+1+|q|) \\ &\leq \varepsilon C(1+|x|)(1+|p|+1+|p|+R) \\ &\leq \varepsilon C(1+|x|)(2+2R) \\ &= C(1+|x|)|p'-p|(2/R+2) \\ &\leq 3C(1+|x|)|p'-p|. \end{aligned}$$

Changing p and p', we obtain $H(x,p) - H(x,p') \leq 3C(1+|x|)|p-p'|$. So $|H(x,p)-H(x,p')| \leq 3C(1+|x|)|p-p'|$ for every $p, p' \in B_{R/2}$ and $x \in \mathbb{R}^m$. From arbitrariness of $R > 2$ we obtain the conclusion of Lemma. ∎

Proposition 4.2. *Let a function* $H : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ *satisfy conditions* (A), (B) *and moreover:*

(D) $\forall x_0 \in \mathbb{R}^m\ \exists R > 0\ \exists K > 0\ \forall p \in \mathbb{R}^l\ \forall x, x' \in B(x_0, R)$ *the following inequality holds* $|H(x,p) - H(x',p)| \leq K|x - x'|(|p| + 1)$.

Then H is locally Lipschitz.

Proof. Let us fix $x_0 \in \mathbb{R}^m$, $p_0 \in \mathbb{R}^l$. Then there exists $K > 0$ and $R > 0$ such that $|H(x,p) - H(x',p)| \leq K|x - x'|(|p| + 1)$ for all $x, x' \in B(x_0, R)$ and $p \in \mathbb{R}^l$. From Lemma 4.1, we obtain $|H(x',p) - H(x',p')| \leq 3C(1 + |x'|)$ $|p - p'|$ for all $p, p' \in \mathbb{R}^l$ and $x' \in \mathbb{R}^m$. Then we have $|H(x,p) - H(x',p')| \leq K(R + |p_0| + 1)|x - x'| + 3C(1 + R + |x_0|)|p - p'|$ for all $x, x' \in B(x_0, R)$ and $p, p' \in B(p_0, R)$. ■

Remark 4.3. Hamiltonian $H : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ given by the following formula

$$H(x,p) = \begin{cases} 0 & ;\ p \leq \frac{1}{|x|},\ x \neq 0 \\ p - \frac{1}{|x|} & ;\ p > \frac{1}{|x|},\ x \neq 0 \\ 0 & ;\ x = 0, \end{cases}$$

is locally Lipschitz and satisfies (A), (B), but it does not satisfy (D). Lagrangian $L : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$, dual to Hamiltonian H is given as

$$L(x,f) = \begin{cases} +\infty & ;\ f \notin [0,1],\ x \neq 0 \\ \frac{1}{|x|} f & ;\ f \in [0,1],\ x \neq 0 \\ 0 & ;\ f = 0,\ x = 0 \\ +\infty & ;\ f \neq 0,\ x = 0. \end{cases}$$

Note that the structure of the Hamiltonian above is so different from the structure of the Hamiltonians satisfying (D), that the multifunction $Q(t,x) := \mathrm{Epi}(L(t,x,\cdot))$ is not even upper semicontinuous with respect to Hausdorff distance.

Theorem 4.4 (Approximation). *Let function $H : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ be upper semicontinuous and satisfy conditions* (A), (B). *Then there exists a family of functions $\{H_n : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}\}_{n \in \mathbb{N}}$, satisfying the following conditions:*

1. $H(x,p) \leq H_{n+1}(x,p) \leq H_n(x,p)$ *for all* $n \in \mathbb{N}$, $x \in \mathbb{R}^m$, $p \in \mathbb{R}^l$,
2. $H_n(x,p) \to H(x,p)$ *for all* $x \in \mathbb{R}^m$, $p \in \mathbb{R}^l$,
3. $\forall n \in \mathbb{N}\ \forall x_0 \in \mathbb{R}^m\ \exists r > 0,\ \exists K > 0\ \forall p \in \mathbb{R}^l,\ \forall x, x' \in B(x_0, r)$ *the following inequality proceeds* $|H_n(x,p) - H_n(x',p)| \leq K|x - x'|(|p| + 1)$,

4. $|H_n(x,p)| \leq 2C(1+|x|)(|p|+1)$ *for all* $; n \in \mathbb{N},\ x \in \mathbb{R}^m,\ p \in \mathbb{R}^l$,
5. $H_n(x,p)$ *is convex with respect to* p *for every* $n \in \mathbb{N},\ x \in \mathbb{R}^m$,
6. H_n *is locally Lipschitz for all* $n \in \mathbb{N}$.

Proof. There exists a family of locally Lipschitz partition of unity $\{\psi^n_\lambda : \mathbb{R}^m \to [0,+\infty)\}_{\lambda \in \Lambda_n}$ such that

(i) $\mathrm{supp}\psi^n_\lambda \subset B(x^n_\lambda, \frac{1}{3^n})$,

(ii) $\sum_{\lambda \in \Lambda_n} \psi^n_\lambda(x) = 1$,

(iii) Let us fix $n \in \mathbb{N}$ and $x \in \mathbb{R}^m$. Then there exist: an open neighbourhood U of a point x and $\lambda^n_1, \lambda^n_2, \ldots, \lambda^n_{s_n} \in \Lambda_n$ such that if $\lambda \in \Lambda_n \setminus \{\lambda^n_1, \lambda^n_2, \ldots, \lambda^n_{s_n}\}$ and $y \in U$, then $\psi^n_\lambda(y) = 0$.

For $\lambda \in \Lambda_n$ we define a function $H^n_\lambda : \mathbb{R}^l \to \mathbb{R}$ by formula

$$H^n_\lambda(p) := \sup_{z \in B(x^n_\lambda, \frac{2}{3^n})} H(z,p).$$

Next let us define $H_n : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ by

$$H_n(x,p) := \sum_{\lambda \in \Lambda_n} \psi^n_\lambda(x) H^n_\lambda(p).$$

Now we show that a family of functions $\{H_n\}_{n \in \mathbb{N}}$ satisfies all the conclusions of our theorem. It is easy to see that for $x \in \mathbb{R}^m$ there exist $\lambda^n_1, \lambda^n_2, \ldots, \lambda^n_{s_n} \in \Lambda_n$ satisfying the equality:

$$H_n(x,p) = \sum_{j=1}^{s_n} \psi^n_{\lambda^n_j}(x) H^n_{\lambda^n_j}(p) \quad \text{for } p \in \mathbb{R}^l. \tag{6}$$

Moreover, $\sum_{j=1}^{s_n} \psi^n_{\lambda^n_j}(x) = 1$ and $\psi^n_{\lambda^n_j}(x) > 0$ for $j \in \{1,2,\ldots,s_n\}$.

The proof of 5 is a consequence of convexity $H^n_\lambda(\cdot)$ and (6).

To prove 4, let us fix $x \in \mathbb{R}^m$. Let $H_n(x,p) = \sum_{j=1}^{s_n} \psi^n_{\lambda^n_j}(x) H^n_{\lambda^n_j}(p)$ be as in (6). Since $x \in \bigcap_{j=1}^{s_n} B(x^n_{\lambda^n_j}, \frac{1}{3^n})$, for $z \in \bigcup_{j=1}^{s_n} B(x^n_{\lambda^n_j}, \frac{2}{3^n})$ we have $|z - x| \leq 1$ ($|z| \leq 1 + |x|$). Therefore for $j \in \{1,2,\ldots,s_n\}$ we obtain

$$
\begin{aligned}
|H^n_{\lambda^n_j}(p)| &\le \sup_{z\in B(x^n_{\lambda^n_j},\frac{2}{3^n})} |H(z,p)| \\
&\le \sup_{z\in B(x^n_{\lambda^n_j},\frac{2}{3^n})} C(1+|z|)(|p|+1) \\
&\le C(2+|x|)(|p|+1) \\
&\le 2C(1+|x|)(|p|+1).
\end{aligned}
$$

That is why we have the inequality $|H_n(x,p)| \le 2C(1+|x|)(|p|+1)$.

To proof 1: We shall determine $x \in \mathbb{R}^m$. Let $H_n(x,p) = \sum_{j=1}^{s_n} \psi^n_{\lambda^n_j}(x) H^n_{\lambda^n_j}(p)$ be as in (6). Since $x \in \bigcap_{j=1}^{s_n} B(x^n_{\lambda^n_j}, \frac{1}{3^n})$, so $H_n(x,p) \ge H(x,p)$. Let $H_{1+n}(x,p) = \sum_{i=1}^{s_{1+n}} \psi^{1+n}_{\lambda^{1+n}_i}(x) H^{1+n}_{\lambda^{1+n}_i}(p)$ be as in (6). We show the following inclusion $B(x^{1+n}_{\lambda^{1+n}_i}, \frac{2}{3^{1+n}}) \subset B(x^n_{\lambda^n_j}, \frac{2}{3^n})$ for $i \in \{1,2,\ldots,s_{1+n}\}$, $j \in \{1,2,\ldots,s_n\}$. Essentially, we know that $x \in \bigcap_{j=1}^{s_n} B(x^n_{\lambda^n_j}, \frac{1}{3^n})$ and $x \in \bigcap_{i=1}^{s_{1+n}} B(x^{1+n}_{\lambda^{1+n}_i}, \frac{1}{3^{1+n}})$. Let us take $z \in B(x^{1+n}_{\lambda^{1+n}_i}, \frac{2}{3^{1+n}})$. Then we obtain

$$
\begin{aligned}
\left|z - x^n_{\lambda^n_j}\right| &\le \left|z - x^{1+n}_{\lambda^{1+n}_i}\right| + \left|x^{1+n}_{\lambda^{1+n}_i} - x\right| + \left|x - x^n_{\lambda^n_j}\right| \\
&< \frac{2}{3^{1+n}} + \frac{1}{3^{1+n}} + \frac{1}{3^n} = \frac{2}{3^n}.
\end{aligned}
$$

Using the inclusion we have inequality

$$H^n_{\lambda^n_j}(p) \ge H^{1+n}_{\lambda^{1+n}_i}(p)$$

for $i \in \{1,2,\ldots,s_{1+n}\}$ $j \in \{1,2,\ldots,s_n\}$. Hence,

$$H^n_{\lambda^n_j}(p) = \sum_{i=1}^{s_{1+n}} \psi^{1+n}_{\lambda^{1+n}_i}(x) H^n_{\lambda^n_j}(p) \ge \sum_{i=1}^{s_{1+n}} \psi^{1+n}_{\lambda^{1+n}_i}(x) H^{1+n}_{\lambda^{1+n}_i}(p) = H_{1+n}(x,p)$$

for $j \in \{1,2,\ldots,s_n\}$. This implies inequality

$$H_n(x,p) = \sum_{j=1}^{s_n} \psi^n_{\lambda^n_j}(x) H^n_{\lambda^n_j}(p) \ge \sum_{j=1}^{s_n} \psi^n_{\lambda^n_j}(x) H_{1+n}(x,p) = H_{1+n}(x,p).$$

To proof 2: Let us fix $x_0 \in \mathbb{R}^m$ and $p_0 \in \mathbb{R}^l$. Since H is upper semi-continuous, $\forall_{\varepsilon>0}\ \exists_{\delta>0}$ such that $|x-x_0|<\delta \Rightarrow H(x,p_0) \le H(x_0,p_0)+\varepsilon$. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{3^{n-1}} < \delta$ for $n \ge n_0$. Let $H_n(x_0,p_0) = \sum_{j=1}^{s_n} \psi^n_{\lambda^n_j}(x_0) H^n_{\lambda^n_j}(p_0)$ be as in (6). Considering that $x_0 \in \bigcap_{j=1}^{s_n} B(x^n_{\lambda^n_j}, \frac{1}{3^n})$, we obtain $\bigcup_{j=1}^{s_n} B(x^n_{\lambda^n_j}, \frac{2}{3^n}) \subset B(x_0,\delta)$ for $n \ge n_0$. Hence, $H^n_{\lambda^n_j}(p_0) \le H(x_0,p_0)+\varepsilon$ for $j \in \{1,2,\ldots,s_n\}$, $n \ge n_0$. Consequently, $H_n(x_0,p_0) \le H(x_0,p_0)+\varepsilon$ for $n \ge n_0$. Using proof 1, we have $H_n(x_0,p_0) \ge H(x_0,p_0)$ so $H_n(x_0,p_0) \to H(x_0,p_0)$.

To proof 3: Let us fix $x_0 \in \mathbb{R}^m$ and $n \in \mathbb{N}$. There exists an open neighbourhood U of a point x_0 and $\lambda^n_1, \lambda^n_2, \ldots, \lambda^n_s \in \Lambda_n$ such that

$$H_n(y,p) = \psi^n_{\lambda^n_1}(y) H^n_{\lambda^n_1}(p) + \psi^n_{\lambda^n_2}(y) H^n_{\lambda^n_2}(p) + \ldots + \psi^n_{\lambda^n_s}(y) H^n_{\lambda^n_s}(p)$$

for $y \in U$, $p \in \mathbb{R}^l$. Furthermore, we can assume that $\psi^n_{\lambda^n_1}, \psi^n_{\lambda^n_2}, \ldots, \psi^n_{\lambda^n_s}$ are Lipschitz in U with constant η. Let $R>0$ be so large that $\bigcup_{j=1}^{s} B(x^n_{\lambda^n_j}, \frac{2}{3^n}) \subset B_R$ and let $k = \eta \cdot s \cdot 2C(1+R)$. Then for $j \in \{1,2,\ldots,s\}$ we have

$$\begin{aligned} |H^n_{\lambda^n_j}(p)| &\le \sup_{x \in B(x^n_{\lambda^n_j}, \frac{2}{3^n})} |H(x,p)| \\ &\le \sup_{x \in B(x^n_{\lambda^n_j}, \frac{2}{3^n})} 2C(1+|x|)(|p|+1) \le 2C(1+R)(|p|+1). \end{aligned}$$

Finally, for $y, y' \in U$ and $p \in \mathbb{R}^l$ we obtain:

$$\begin{aligned} |H_n(y,p) - H_n(y',p)| &\le \sum_{j=1}^{s} |\psi^n_{\lambda^n_j}(y) - \psi^n_{\lambda^n_j}(y')||H^n_{\lambda^n_j}(p)| \\ &\le \sum_{j=1}^{s} \eta |y-y'||H^n_{\lambda^n_j}(p)| \\ &\le \eta \cdot s \cdot 2C(1+R)|y-y'|(|p|+1) \\ &= k|y-y'|(|p|+1). \end{aligned}$$

To proof 6: It is a consequence from Proposition 4.2. ∎

Formulating Theorem 4.4 through the multifunction corresponding to Lagrangian, we get a version of Antosiewicz-Cellina theorem. Moreover, the multifunction in this version of the theorem does not need to be upper semicontinuous in the sense of Hausdorff distance. For instance we can take L from Remark 4.3.

Theorem 4.5. *If $L : \mathbb{R}^m \times \mathbb{R}^l \to (-\infty, +\infty]$ satisfies type conditions* (L1)–(L5), *then there exists a sequence of functions $L_n : \mathbb{R}^m \times \mathbb{R}^l \to (-\infty, +\infty]$ such that*

1. $Q(x) = \mathrm{Epi}(L(x,\cdot))$, $Q_n(x) = \mathrm{Epi}(L_n(x,\cdot))$,
2. $Q_n(x)$ *are convex and closed,*
3. $Q(x) \subset Q_{n+1} \subset Q_n(x), \qquad Q(x) = \bigcap\limits_{n=1}^{\infty} Q_n(x)$,
4. *the map Q_n is locally Lipschitz with respect to Hausdorff distance,*
5. *the Kuratowski limit of sequence $Q_n(x)$ equals $Q(x)$ for each $x \in \mathbb{R}^m$.*

Proof. Let H be a Hamiltonian associated with Lagrangian L satisfying type conditions (H1)–(H4). Then H fulfills the assumptions of Theorem 4.4, so there exists a sequence of Hamiltonians H_n satisfying the conclusion of Theorem 4.4. The sequence of Lagrangians L_n associated with the sequence of Hamiltonians H_n fulfills type conditions (L1)–(L4) and it implies 1. Moreover, from Corollary 3.2 we get $L_n \nearrow L$ and $\text{e-lim}\, L_n = L$, which implies 2 and 4. However, 3 is a consequence of the property presented in preliminaries. ■

Corollary 4.6. *Let $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ satisfy* (H1)–(H4). *Then there exists a family of functions $\{H_n : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}\}_{n\in\mathbb{N}}$ such that:*

1. $H_n \searrow H$,
2. H_n *satisfies conditions (HL),*
3. $|H_n(t,x,p)| \le 2C(1+T)(1+|x|)(|p|+1)$,
4. $H_n(t,x,p)$ *is convex with respect to p for all* $n \in \mathbb{N},\ t \in [0,T],\ x \in \mathbb{R}^l$,
5. H_n *is locally Lipschitz for every $n \in \mathbb{N}$.*

Proof. Let H(t,x,p)=H(0,x,p) for $t < 0$ and $H(t,x,p) = H(T,x,p)$ for $t > T$. Since $|H(t,x,p)| \le C(1+|x|)(|p|+1) \le C(1+|(t,x)|)(|p|+1)$, the assumptions of Theorem 4.4 are satisfied. There exists a family of functions $H_n : \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$, the one which fulfills all the conditions of the conclusion of the above theorem on the set $[0,T] \times \mathbb{R}^l \times \mathbb{R}^l$. We show 3 and 2, the rest is obvious.

The proof of 3: $|H_n(t,x,p)| \leq 2C(1+|(t,x)|)(|p|+1) \leq 2C(1+T+|x|)(|p|+1)$ $\leq 2C(1+T)(1+|x|)(|p|+1)$.

To proof 2: Let us fix $n \in \mathbb{N}$ and $R > 0$. Then from Theorem 4.4 and compactness of the set $[0,T] \times \overline{B}_R$, we obtain finite family of open sets $\{U_i\}_{i=1}^{\tau}$ covering a set $[0,T] \times \overline{B}_R$ and numbers $\{K_i\}_{i=1}^{\tau}$ such that for $i \in \{1 \ldots \tau\}$ the following inequality holds:

$$|H_n(t,x,p) - H_n(t',x',p)| \leq K_i(|t-t'| + |x-x'|)(|p|+1)$$

for $(t,x),\ (t',x') \in U_i,\ \ p \in \mathbb{R}^l$. Next we choose a Lebesgue number $\lambda > 0$ of covering $\{U_i\}_{i=1}^{\tau}$. Let us put $K := \max_{i \in \{1,\ldots\tau\}}\{K_i, 2\frac{2C(1+R)(1+T)}{\lambda}\}$. Then, for $(t,x),\ (t',x') \in [0,T] \times \overline{B}_R$, we obtain: If $|(t,x) - (t',x')| < \lambda$, then $\exists_{i \in \{1,\ldots,\tau\}}$ for which $(t,x),\ (t',x') \in U_i$, hence

$$|H_n(t,x,p) - H_n(t',x',p)| \leq K(|t-t'| + |x-x'|)(|p|+1) \ \text{ for } p \in \mathbb{R}^l.$$

If $|(t,x) - (t',x')| \geq \lambda$, then from proof 3

$$\begin{aligned} |H_n(t,x,p) - H_n(t',x',p)| &\leq |H_n(t,x,p)| + |H_n(t',x',p)| \\ &\leq 2\frac{2C(1+T)(1+R)}{\lambda}\lambda(|p|+1) \\ &\leq K(|t-t'| + |x-x'|)(|p|+1) \ \text{ for } p \in \mathbb{R}^l. \end{aligned}$$

■

5. VALUE FUNCTION AND USC HAMILTONIAN

In this section we provide examples of value functions in Bolza problem that are not bilateral, viscosity and classical solutions and show how to approximate these functions in the case of upper semicontinuous Hamiltonian.

5.1. Locally Lipschitz continuity of the value function

We show that if terminal cost is locally Lipschitz, then the value function is also locally Lipschitz. We also emphasize in the sequel that we do not assume that Lagrangian $L(t,x,\cdot)$ is convex.

Theorem 5.1. *Suppose that Lagrangian* $L : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ *is proper with respect to the last variable and satisfies* (L1), (L3) *and* (L4). *Fix* $R > 0$. *Let* $Q(t,x) = Epi(L(t,x,\cdot))$ *be Lipschitz with respect to Hausdorff distance on* $[0,T] \times B_{(1+R+TC)e^{2TC}}$ *with a constant* $k > 0$. *If* $(x,u) : [\tau,\sigma] \to \mathbb{R}^l \times \mathbb{R}$ *is absolutely continuous function such that* $[\tau,\sigma] \subset [0,T]$, $|x(\tau)| < R$, $(\dot{x}(t), \dot{u}(t)) \in Q(t,x(t))$, *then for* $|x_\tau| < R$, $u_\tau \in \mathbb{R}$ *there exists an absolutely continuous function* $(\overline{x},\overline{u}) : [\tau,\sigma] \to \mathbb{R}^l \times \mathbb{R}$ *such that* $\overline{x}(\tau) = x_\tau$ *and* $\overline{u}(\tau) = u_\tau$. *Moreover,*

1. $(\dot{\overline{x}}(t), \dot{\overline{u}}(t)) \in Q(t, \overline{x}(t))$,
2. $|\dot{\overline{x}}(t) - \dot{x}(t)| + |\dot{\overline{u}}(t) - \dot{u}(t)| \leq (|x_\tau - x(\tau)|)4ke^{kT}$,
3. $|\overline{x}(t) - x(t)| + |\overline{u}(t) - u(t)| \leq (|x_\tau - x(\tau)| + |u_\tau - u(\tau)|)2e^{kT}$.

Theorem 5.1 is a version of Filippov theorem, which can be obtained by little change in the proof of this theorem: we take into consideration the fact that the multifunction comes from Lagrangian with super linear growth.

Lemma 5.2. *Suppose that Lagrangian* $L : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ *is proper with respect to the last variable and satisfies* (L1), (L3) *and* (L4). *Moreover, let* $Q(t,x) = Epi(L(t,x,\cdot))$ *be locally Lipschitz with respect to Hausdorff distance. For* $u_0 \in \mathbb{R}$, $t_0 \in [0,T)$ *and* $(f_0, \eta_0) \in Q(t_0, x_0)$ *there exists a* C^1*-class function* $(x,u) : [t_0,T] \to \mathbb{R}^l \times \mathbb{R}$ *such that* $(x,u)(t_0) = (x_0,u_0)$, $(\dot{x},\dot{u})(t_0^+) = (f_0,\eta_0)$ *and* $(\dot{x}(t),\dot{u}(t)) \in Q(t,x(t))$.

Lemma 5.2 is a version of Proposition 3.14 from the paper of Plaskacz and Quincampoix [12], which can be obtained by taking the multifunction that comes from Lagrangian with super linear growth.

Remark 5.3. From Lemma 5.2 we obtain the following if Lagrangian satisfies the assumptions of Lemma 5.2 and the terminal cost is finite, then the value function is also finite.

Theorem 5.4. *Suppose that Lagrangian* $L : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ *is proper with respect to the last variable and satisfies* (L1), (L3) *and* (L4). *Let* $g : \mathbb{R}^l \to \mathbb{R}$ *be locally Lipschitz and* $Q(t,x) = Epi(L(t,x,\cdot))$ *be locally Lipschitz with respect to Hausdorff distance, then the value function* V *associated with* L, g *is locally Lipschitz.*

Proof. From Lemma 5.2 we obtain a C^1-class function $(z,v) : [0,T] \to \mathbb{R}^l \times \mathbb{R}$ such that $\dot{v}(t) \geq L(t,z(t),\dot{z}(t))$. Let us take non-negative number r such that $|z(t)| < r$ and $|\dot{v}(t)| < r$ when $t \in [0,T]$. Put $R := (1 + r + TC)e^{TC}$. Let k_1 be the Lipschitz constant for the multifunction $Q(\cdot,\cdot)$ on $[0,T] \times B_{(1+R+TC)e^{2TC}}$ and k_2 the Lipschitz constant for $g(\cdot)$ on B_R. To prove Theorem 5.4, it is enough to show that $V(t_1,x_1) - V(t_2,x_2) \leq k(|t_1 - t_2| + |x_1 - x_2|)$ for $t_1,t_2 \in [0,T]$ and $x_1,x_2 \in \overline{B}_r$, where $k = (k_2 + 1)(C(1+R)+1)2e^{k_1T} + \max\{C(1+R),(8k_1e^{k_1T}+1)R\}$. We show this inequality in two steps.

Step 1. The following inequality is true $V(t_1,x_1) - V(t_1,x_2) \leq |x_1 - x_2|$ $(k_2+1)2e^{k_1T}$ for $t_1 \in [0,T]$ and $x_1,x_2 \in \overline{B}_r$. Indeed, fix $t_1 \in [0,T)$, $x_1,x_2 \in \overline{B}_r$ and $\varepsilon > 0$. Let $x : [t_1,T] \to \mathbb{R}^l$ be absolutely continuous such that $x(t_1) = x_2$ and

$$\varepsilon + V(t_1,x_2) \geq g(x(T)) + \int_{t_1}^{T} L(t,x(t),\dot{x}(t))dt.$$

Let us take an absolutely continuous function $(x,u) : [t_1,T] \to \mathbb{R}^l \times \mathbb{R}$ such that $(x,u)(t_1) = (x_2,0)$ and $\dot{u}(t) = L(t,x(t),\dot{x}(t))$. From Theorem 5.1 there exists an absolutely continuous function $(\overline{x},\overline{u}) : [t_1,T] \to \mathbb{R}^l \times \mathbb{R}$ such that $(\overline{x},\overline{u})(t_1) = (x_1,0)$ and $\dot{\overline{u}}(t) \geq L(t,\overline{x}(t),\dot{\overline{x}}(t))$, moreover

$$|\overline{x}(T) - x(T)| + |\overline{u}(T) - u(T)| \leq |x_1 - x_2|2e^{k_1T}.$$

Assumption (L4) implies $|\dot{\overline{x}}(t)| \leq C(1+|\overline{x}(t)|)$. Using Gronwall inequality, we obtain $|\overline{x}(t)| \leq (|x_1| + TC)e^{TC} < R$. Analogously, we show that $|x(t)| < R$. Then $g(\overline{x}(T)) - g(x(T)) \leq k_2|\overline{x}(T) - x(T)|$. Let us prove the inequality

$$\begin{aligned}
&V(t_1,x_1) - V(t_1,x_2)\\
&\leq\ g(\overline{x}(T)) + \int_{t_1}^{T} L(t,\overline{x}(t),\dot{\overline{x}}(t))dt - g(x(T)) - \int_{t_1}^{T} L(t,x(t),\dot{x}(t))dt + \varepsilon\\
&=\ g(\overline{x}(T)) - g(x(T)) + \int_{t_1}^{T} L(t,\overline{x}(t),\dot{\overline{x}}(t))dt - \int_{t_1}^{T} L(t,x(t),\dot{x}(t))dt + \varepsilon\\
&\leq\ k_2|\overline{x}(T) - x(T)| + \overline{u}(T) - u(T) + \varepsilon\\
&\leq\ k_2|x_1 - x_2|2e^{k_1T} + |x_1 - x_2|2e^{k_1T} + \varepsilon = |x_1 - x_2|(k_2+1)2e^{k_1T} + \varepsilon.
\end{aligned}$$

If $t_1 = T$, then $V(t_1, x_1) - V(t_1, x_2) = g(x_1) - g(x_2) \leq k_2|x_1 - x_2|$. From arbitrariness of $\varepsilon > 0$ the proof of Step 1 is complete.

Step 2. The following inequality is true $V(t_1, x_2) - V(t_2, x_2) \leq |t_1 - t_2|$ $((k_2+1)2e^{k_1 T}C(1+R) + \max\{C(1+R), (8k_1 e^{k_1 T} + 1)R\})$ for $t_1, t_2 \in [0, T]$ and $x_2 \in \overline{B}_r$.

Case 1. Let us fix $t_2 < t_1 < T$, $x_2 \in \overline{B}_r$ and $\varepsilon > 0$. Let $x : [t_2, T] \to \mathbb{R}^l$ be absolutely continuous such that $x(t_2) = x_2$ and

$$\varepsilon + V(t_2, x_2) \geq g(x(T)) + \int_{t_2}^{T} L(t, x(t), \dot{x}(t))dt.$$

Condition (L4) implies $|\dot{x}(t)| \leq C(1 + |x(t)|)$. Using Gronwall inequality, we obtain $|x(t)| \leq (|x_2| + TC)e^{TC} < R$ and $|\dot{x}(t)| \leq C(1+R)$. So $x(\cdot)$ is Lipschitz with the constant $C(1+R)$. Condition (L3) implies $-C(1+R) \leq -C(1+|x(t)|) \leq L(t, x(t), \dot{x}(t))$. Let us take an absolutely continuous function $(x, u) : [t_1, T] \to \mathbb{R}^l \times \mathbb{R}$ such that $(x, u)(t_1) = (x(t_1), 0)$ and $\dot{u}(t) = L(t, x(t), \dot{x}(t))$. From Theorem 5.1 there exists an absolutely continuous function $(\overline{x}, \overline{u}) : [t_1, T] \to \mathbb{R}^l \times \mathbb{R}$ such that $(\overline{x}, \overline{u})(t_1) = (x_2, 0)$ and $\dot{\overline{u}}(t) \geq L(t, \overline{x}(t), \dot{\overline{x}}(t))$. Moreover,

$$\begin{aligned}
&|\overline{x}(T) - x(T)| + |\overline{u}(T) - u(T)| \leq |x(t_1) - x_2|2e^{k_1 T} \\
&= |x(t_1) - x(t_2)|2e^{k_1 T} \leq |t_1 - t_2|2e^{k_1 T}C(1+R).
\end{aligned}$$

Then $|x(T)| < R$ and $|\overline{x}(T)| \leq C(|x_2| + TC)e^{TC} < R$ so $g(\overline{x}(T)) - g(x(T)) \leq k_2|\overline{x}(T) - x(T)|$. Let us prove the inequality

$$\begin{aligned}
& V(t_1, x_2) - V(t_2, x_2) \\
\leq\ & g(\overline{x}(T)) + \int_{t_1}^{T} L(t, \overline{x}(t), \dot{\overline{x}}(t))dt - g(x(T)) - \int_{t_2}^{T} L(t, x(t), \dot{x}(t))dt + \varepsilon \\
=\ & g(\overline{x}(T)) - g(x(T)) + \int_{t_1}^{T} L(t, \overline{x}(t), \dot{\overline{x}}(t))dt - \int_{t_2}^{T} L(t, x(t), \dot{x}(t))dt + \varepsilon \\
=\ & g(\overline{x}(T)) - g(x(T)) + \int_{t_1}^{T} L(t, \overline{x}(t), \dot{\overline{x}}(t))dt
\end{aligned}$$

$$
\begin{aligned}
&- \int_{t_1}^{T} L(t, x(t), \dot{x}(t))dt - \int_{t_2}^{t_1} L(t, x(t), \dot{x}(t))dt + \varepsilon \\
&\leq\ k_2|\overline{x}(T) - x(T)| + \overline{u}(T) - u(T) + |t_1 - t_2|C(1 + R) + \varepsilon \\
&\leq\ k_2|t_1 - t_2|2e^{k_1 T}C(1 + R) + |t_1 - t_2|2e^{k_1 T}C(1 + R) + |t_1 - t_2|C(1 + R) + \varepsilon \\
&=\ |t_1 - t_2|((k_2 + 1)2e^{k_1 T}C(1 + R) + C(1 + R)) + \varepsilon.
\end{aligned}
$$

Similarily, we show the inequality when $t_1 = T$. From arbitrariness of $\varepsilon > 0$, the proof of Case 1 is complete.

Case 2. Let us fix $t_1 < t_2 < T$, $x_2 \in \overline{B}_r$ and $\varepsilon > 0$. For a defined C^1-class function $(z, v) : [0, t_2] \to \mathbb{R}^l \times \mathbb{R}$ for which $(z, v)(t_2) = (z(t_2), v(t_2))$ and $\dot{v}(t) \geq L(t, z(t), \dot{z}(t))$ we match from Theorem 5.1 (back in time) an absolutely continuous function $(\overline{z}, \overline{v}) : [0, t_2] \to \mathbb{R}^l \times \mathbb{R}$ such that $(\overline{z}, \overline{u})(t_2) = (x_2, v(t_2))$ and $\dot{\overline{v}}(t) \geq L(t, \overline{z}(t), \dot{\overline{z}}(t))$. Moreover, $|\dot{\overline{v}}(t) - \dot{v}(t)| \leq |z(t_2) - x_2|4k_1 e^{k_1 T}$. Then $|\dot{\overline{v}}(t)| \leq |z(t_2) - x_2|4k_1 e^{k_1 T} + |\dot{v}(t)| \leq 8Rk_1 e^{k_1 T} + R$. So $L(t, \overline{z}(t), \dot{\overline{z}}(t)) \leq (8k_1 e^{k_1 T} + 1)R$. Condition (L4) implies $|\dot{\overline{z}}(t)| \leq C(1 + |\overline{z}(t)|)$. Using Gronwall inequality (back in time), we obtain $|\overline{z}(t)| \leq (|x_2| + TC)e^{TC} < R$ and $|\dot{\overline{z}}(t)| \leq C(1 + R)$. So $\overline{z}(\cdot)$ is Lipschitz with the constant $C(1+R)$. Let $x : [t_2, T] \to \mathbb{R}^l$ be absolutely continuous such that $x(t_2) = x_2$ and

$$\varepsilon + V(t_2, x_2) \geq g(x(T)) + \int_{t_2}^{T} L(t, x(t), \dot{x}(t))dt.$$

Let us take an absolutely continuous function $(y, u) : [t_1, T] \to \mathbb{R}^l \times \mathbb{R}$ such that $(y, u)(t_1) = (\overline{z}(t_1), 0)$ and $\dot{u}(t) = L(t, y(t), \dot{y}(t))$, where $y(t)$ equals $x(t)$ on $[t_2, T]$ and $\overline{z}(t)$ on $[t_1, t_2]$. From Theorem 5.1 there exists an absolutely continuous function $(\overline{y}, \overline{u}) : [t_1, T] \to \mathbb{R}^l \times \mathbb{R}$ such that $(\overline{y}, \overline{u})(t_1) = (x_2, 0)$ and $\dot{\overline{u}}(t) \geq L(t, \overline{y}(t), \dot{\overline{y}}(t))$. Moreover,

$$
\begin{aligned}
&|\overline{y}(T) - y(T)| + |\overline{u}(T) - u(T)| \leq |\overline{z}(t_1) - x_2|2e^{k_1 T} \\
&= |\overline{z}(t_1) - \overline{z}(t_2)|2e^{k_1 T} \leq |t_1 - t_2|2e^{k_1 T}C(1 + R).
\end{aligned}
$$

Then $|\overline{y}(T)| \leq (|x_2| + TC)e^{TC} < R$ and $|y(T)| = |x(T)| \leq (|x_2| + TC)e^{TC}$ $< R$ so $g(\overline{y}(T)) - g(y(T)) \leq k_2|\overline{y}(T) - y(T)|$. Let us prove the inequality

$$
\begin{aligned}
&V(t_1,x_2)-V(t_2,x_2)\\
&\le\ g(\overline{y}(T))+\int_{t_1}^{T}L(t,\overline{y}(t),\dot{\overline{y}}(t))dt-g(x(T))-\int_{t_2}^{T}L(t,x(t),\dot{x}(t))dt+\varepsilon\\
&=\ g(\overline{y}(T))-g(y(T))+\int_{t_1}^{T}L(t,\overline{y}(t),\dot{\overline{y}}(t))dt-\int_{t_2}^{T}L(t,y(t),\dot{y}(t))dt+\varepsilon\\
&=\ g(\overline{y}(T))-g(y(T))+\int_{t_1}^{T}L(t,\overline{y}(t),\dot{\overline{y}}(t))dt\\
&\quad-\int_{t_1}^{T}L(t,y(t),\dot{y}(t))dt+\int_{t_1}^{t_2}L(t,y(t),\dot{y}(t))dt+\varepsilon\\
&\le\ k_2|\overline{y}(T)-y(T)|+\overline{u}(T)-u(T)+\int_{t_1}^{t_2}L(t,\overline{z}(t),\dot{\overline{z}}(t))dt+\varepsilon\\
&\le\ k_2|t_1-t_2|2e^{k_1T}C(1+R)+|t_1-t_2|2e^{k_1T}C(1+R)\\
&\quad+|t_2-t_1|(8k_1e^{k_1T}+1)R+\varepsilon\\
&=\ |t_1-t_2|((k_2+1)2e^{k_1T}C(1+R)+(8k_1e^{k_1T}+1)R)+\varepsilon.
\end{aligned}
$$

Similarly, we show the inequality when $t_2=T$. From arbitrariness of $\varepsilon>0$, the proof of Case 2 is complete. ■

From the properties enlisted in the Preliminaries and Theorem 5.4, we obtain:

Corollary 5.5. *Suppose that* $H:[0,T]\times\mathbb{R}^l\times\mathbb{R}^l\to\mathbb{R}$ *satisfies* (H1)–(H3), (HL) *and* $g:\mathbb{R}^l\to\mathbb{R}$ *is a locally Lipschitz function. Then the value function* V*, associated with* L *and* g (*where* L *is dual to* H), *is locally Lipschitz.*

5.2. Bilateral and viscosity solutions

For $H:[0,T]\times\mathbb{R}^l\times\mathbb{R}^l\to\mathbb{R}$ we give a definition of bilateral solutions of the equation:

$$-U_t+H(t,x,-U_x)=0, \tag{7}$$

which was introduced by Barron, Jensen in [3] (calling them upper semicontinuous solutions) and Frankowska in [9] (calling them lower semicontinuous solutions). The name 'bilateral solutions' comes from [2].

Definition 5.6. We say that a function $U \in LSC([0,T] \times \mathbb{R}^l)$ is a *bilateral solution* of the equation (7), if for all $(t,x) \in Dom(U)$, $t \in (0,T)$, the following inequality holds

$$-p_t + H(t,x,-p_x) = 0, \qquad \forall (p_t,p_x) \in D_-U(t,x).$$

The following definition of the viscosity solutions comes from Crandall and Lions:

Definition 5.7. We say that a function $U \in USC([0,T] \times \mathbb{R}^l)$ is a subsolution of the equation (7), if for all $(t,x) \in Dom(U)$, $t \in (0,T)$, the following inequality holds

$$-p_t + H(t,x,-p_x) \leq 0, \qquad \forall (p_t,p_x) \in D^+U(t,x).$$

Similarly, we say that $U \in LSC([0,T] \times \mathbb{R}^l)$ is a supersolution, if for every $(t,x) \in Dom(U)$, $t \in (0,T)$ the following inequality is satisfied

$$-p_t + H(t,x,-p_x) \geq 0, \qquad \forall (p_t,p_x) \in D_-U(t,x).$$

If $U \in C([0,T] \times \mathbb{R}^l)$ is sub/super-solution, then U is a *viscosity solution* of the equation (7).

Definition 5.8. We say that U satisfies the *boundary condition* with g if $U(T,x) = g(x)$ for every $x \in \mathbb{R}^l$.

Theorem 5.9. *Suppose that $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ satisfies* (H1)–(H3), (HL) *and $g : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function. Then the value function V, associated with L and g (where L is dual to H), is a bilateral solution of the equation* (7).

Theorem 5.9 can be obtained with a few changes of the results in the paper of Plaskacz and Quincampoix [12]. This modification is necessary because we weaken the lower boundary of Lagrangian and strengthen the assumption about super linear growth.

Theorem 5.10 (Viscosity solutions)**.** *Let $U_1, U_2 \in C([0,T] \times \mathbb{R}^l)$ and $U_1(T,x) = U_2(T,x)$ for every $x \in \mathbb{R}^l$. Let U_1 and U_2 be the viscosity solution of*

$$-U_t + H(t,x,-U_x) = 0 \qquad \textit{in } (0,T) \times \mathbb{R}^l,$$

where $H \in C([0,T] \times \mathbb{R}^l \times \mathbb{R}^l)$ *satisfies*

$$|H(t,x,p) - H(t,x,p')| \leq K(1+|x|)|p-p'|$$

for all $t \in [0,T]$ *and* $x, p, p' \in \mathbb{R}^l$ *and*

$$|H(t,x,p) - H(t',x',p)| \leq K_R(|t-t'| + |x-x'|)(1+|p|)$$

for all $p \in \mathbb{R}^l$, $t, t' \in [0,T]$, $x, x' \in B_R$, $R > 0$. *Then* $U_1 = U_2$ *in* $[0,T] \times \mathbb{R}^l$.

Theorem 5.10 can be found in [2], page 182.

Remark 5.11. If $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ satisfies (H1)–(H4), then from Lemma 4.1 we obtain $|H(t,x,p) - H(t,x,p')| \leq 3C(1+T)(1+|x|)|p-p'|$ for all $t \in [0,T]$ and $x, p, p' \in \mathbb{R}^l$. So, if Hamiltonian H satisfies (H1)–(H4), (HL) and U is continuous, then from the results (in local version) of Barron-Jensen (see Theorem 2.3 in [3]), we deduce that U is a viscosity solution if and only if U is a bilateral solution.

From Corollary 5.5, Theorems 5.9, 5.10, Remark 5.11 we obtain the corollary:

Corollary 5.12. *Suppose that* $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ *satisfies* (H1)–(H4), (HL) *and* $g : \mathbb{R}^l \to \mathbb{R}$ *is a locally Lipschitz function. Then in the class of locally Lipschitz functions satisfying boundary condition with g the value function V, associated with L and g* (*where L is dual to H*), *is the unique bilateral* (*equivalently viscosity*) *solution of the equation* (7).

5.3. Examples

Example 5.13. Let $V(t,x)$ be the value function in the Meyer problem given by the differential inclusion

$$\dot{x}(t) \in F(x(t)),$$

where $F : \mathbb{R} \multimap \mathbb{R}$ is given by

$$F(x) = \begin{cases} 1 & ; \quad x > 0 \\ [-1,1] & ; \quad x = 0 \\ -1 & ; \quad x < 0. \end{cases}$$

For the terminal cost $g(x) = x$ the value function $V : [0,T] \times \mathbb{R} \to \mathbb{R}$, $V(t,x) = \inf\{g(x(T)) \,:\, x(\cdot) \in S_F(t,x)\}$ can be easily calculated as

$$V(t,x) = \begin{cases} x + (T-t) & ; \quad x > 0 \\ x - (T-t) & ; \quad x \le 0. \end{cases}$$

Let us observe that at points $(t,x) = (t,0)$, $t \in (0,T)$ subdifferential $D_{-}V(t,x)$ consists of $(1,p_x)$, where $p_x \ge 1$ and the Hamiltonian $H(t,x,p) = \sup_{f\in F(x)} f \cdot p$ is given by

$$H(t,x,p) = \begin{cases} p & ; \quad x > 0 \\ |p| & ; \quad x = 0 \\ -p & ; \quad x < 0. \end{cases}$$

The above Hamiltonian satisfies (H1)–(H4). But the value function V is not a bilateral solution of the HJB equation because $-1 + H(t,0,-p_x) \neq 0$.

Example 5.14. A Hamiltonian $H : [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is given by the formula

$$H(t,x,p) = \begin{cases} p+1 & ; \quad x \ge 0 \\ p-1 & ; \quad x < 0. \end{cases}$$

The above Hamiltonian satisfies (H1)–(H4). Lagrangian $L : [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto [-1,+\infty]$ of this Hamiltonian is given by the formula

$$L(t,x,f) = \begin{cases} -1 & ; \quad f = 1,\ x \ge 0 \\ +\infty & ; \quad f \neq 1 \\ 1 & ; \quad f = 1,\ x < 0. \end{cases}$$

For the terminal cost $g(x)$ equalled 0 for $x < 0$, and x for $x \ge 0$ the value function $V : [0,T] \times \mathbb{R} \to \mathbb{R}$ can be easily calculated as

$$V(t,x) = \begin{cases} |x| & ; \quad t - x \le T \\ T - t & ; \quad t - x > T. \end{cases}$$

For the terminal cost $g(x)$ equalled $+\infty$ for $x < 0$, and x for $x \ge 0$ the value function $V : [0,T] \times \mathbb{R} \to \mathbb{R}$ can be easily calculated as

$$V(t,x) = \begin{cases} |x| & ; \quad t - x \le T \\ +\infty & ; \quad t - x > T. \end{cases}$$

Let us observe in both cases that at points $(t,x)=(t,0)$, $t\in(0,T)$ subdifferential $D_{-}V(t,x)$ consists of $(0,p_x)$ where $p_x\in[-1,1]$. So the value function V is not a bilateral solution of the HJB equation because $-0+H(t,0,-p_x)\neq 0$.

Example 5.15. A Hamiltonian $H:[0,T]\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$ is given by the formula

$$H(t,x,p)=\begin{cases} p+1 & ;\quad x\leq 0\\ p-1 & ;\quad x>0.\end{cases}$$

The above Hamiltonian satisfies (H1)–(H4). Lagrangian $L:[0,T]\times\mathbb{R}\times\mathbb{R}\mapsto[-1,+\infty]$ of this Hamiltonian is given by the formula

$$L(t,x,f)=\begin{cases} -1 & ;\quad f=1,\ x\leq 0\\ +\infty & ;\quad f\neq 1\\ 1 & ;\quad f=1,\ x>0.\end{cases}$$

For the terminal cost $g(x)=-|x|$ the value function $V:[0,T]\times\mathbb{R}\to\mathbb{R}$ can be easily calculated as $V(t,x)=-|x|$. Let us observe that at points $(t,x)=(t,0)$, $t\in(0,T)$ subdifferential $D^{+}V(t,x)$ consists of $(0,p_x)$, where $p_x\in[-1,1]$. So the value function V is not a viscosity solution of the HJB equation because $-0+H(t,0,-p_x)\nleq 0$.

Example 5.16. A Hamiltonian $H:[0,T]\times\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$ is given by the formula

$$H(t,x,p)=\begin{cases} -1 & ;\quad |p|\leq\frac{1}{|t-x|},\ t\neq x\\ T\left(|p|-\frac{1}{|t-x|}\right)-1 & ;\quad |p|>\frac{1}{|t-x|},\ t\neq x\\ 0 & ;\quad t=x.\end{cases}$$

The above Hamiltonian satisfies (H1)–(H4). Lagrangian $L:[0,T]\times\mathbb{R}\times\mathbb{R}\mapsto[0,+\infty]$ of this Hamiltonian is given by the formula

$$L(t,x,f)=\begin{cases} +\infty & ;\quad f\notin[-T,T],\ t\neq x\\ \frac{1}{|t-x|}|f|+1 & ;\quad f\in[-T,T],\ t\neq x\\ 0 & ;\quad f=0,\ t=x\\ +\infty & ;\quad f\neq 0,\ t=x.\end{cases}$$

For the terminal cost $g\equiv 0$ the value function $V:[0,T]\times\mathbb{R}\to\mathbb{R}$ can be easily calculated as a smooth function $V(t,x)=T-t$. Indeed, if we put

$x(\cdot) \equiv x$, then $L(\tau, x(\tau), \dot{x}(\tau)) = 1$ almost surely. Hence, $V(t, x) \le T - t$. Let $x(\cdot)$ be an absolutely continuous function, then the Lebesgue measure of a set $A := \{\tau \in [t, T];\ x(\tau) = \tau,\ \dot{x}(\tau) = 0\}$ equals zero. Assume by contradiction that the Lebesgue measure of A is positive. Then from Lebesgue theorem about the point of density of the set A there exists a point $\tau_0 \in A$ and a sequence $\tau_n \in A$ such that $\tau_n \ne \tau_0$ and $\tau_n \to \tau_0$. Therefore $\lim_n((x(\tau_n) - x(\tau_0))/(\tau_n - \tau_0)) = 1$ and $\dot{x}(\tau_0) = 0$, so we have a contradiction. Hence, for every absolutely continuous function the following inequality holds almost surely $L(\tau, x(\tau), \dot{x}(\tau)) \ge 1$. Finally we see that $V(t, x) \ge T - t$. Let us observe that the smooth value function V at the point (t, t), $t \in (0, T)$ is not a classical solution (in particular, it can be neither a viscosity nor a bilateral solution) of the HJB equation because $-(-1) + H(t, t, 0) \ne 0$.

Remark 5.17. Examples 5.13, 5.14, 5.15, 5.16 show that the value function can be neither the bilateral solution, nor the viscosity solution nor the classical solution. But in examples the value functions satisfy the inequality $-V_t + H(t, x, -V_x) \ge 0$. It is not casual: see Theorem 5.20 at (ii).

5.4. Approximation of the value function

From Examples 5.13, 5.14, 5.15, 5.16 we know that the value function of upper semicontinuous Hamiltonian does not have to be its bilateral, viscosity and classical solution. One observes that this value function can be approximated by value functions that are, in the class of locally Lipschitz functions satisfying boundary condition, unique bilateral (viscosity) solutions of corresponding Hamilton-Jacobi-Bellman equations. The following theorem implies that this is a consequence of Corollaries 3.5, 4.6 and 5.12.

Theorem 5.18. *Suppose that Hamiltonian* $H : [0, T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ *satisfying conditions* (H1)–(H4) *and* $g_n : \mathbb{R}^l \to \mathbb{R}$ *is locally Lipschitz and* $g : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ *is a lower semicontinuous function, moreover* $g_n \nearrow g$. *If* V *is the value function associated with* L *and* g *(where* L *is dual to* H*), then there exists a sequence of functions* $V_n : [0, T] \times \mathbb{R} \to \mathbb{R}$ *and* $H_n : [0, T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ *such that:*

(i) H_n *satisfies assumptions* (H1)–(H4), (HL),

(ii) V_n *is the value function associated with* L_n *and* g_n *(where* L_n *are dual to* H_n*),*

(iii) *in the class of locally Lipschitz functions satisfying boundary condition with* g_n *the value function* V_n *is the unique bilateral* (*equivalently viscosity*) *solution of equation* $-U_t + H_n(t,x,-U_x) = 0$.

(iv) $V_n \nearrow V$ (e-lim $V_n = V$) *and* $H_n \searrow H$ (h-lim $H_n = H$).

We show that in the case of continuous Hamiltonian the value function is its bilateral solution. Let us notice that generally Theorem 5.9 can not be used in the case of continuous Hamiltonians because they are not regular enough (see Remark 4.3).

Proposition 5.19 (Barron and Jensen). *Let* $H_n, H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ *and assume that* H_n *are continuous. Then:*

(i) *If* U_n *is the bilateral* (*viscosity*) *solution of equation* $-U_t + H_n(t,x,-U_x) = 0$ *for every* $n \in \mathbb{N}$, *e-*$\lim U_n = U$ ($U_n \rightrightarrows U$), $H_n \rightrightarrows H$ *uniformly on compact set, then* U *is the bilateral* (*viscosity*) *solution of equality* (7).

(ii) *If* U_n *is the supersolution of* $-U_t + H_n(t,x,-U_x) = 0$ *for every* $n \in \mathbb{N}$, *e-*$\lim U_n = U$ *and h-*$\lim H_n = H$, *then* U *is the supersolution of equality* (7).

Proposition 5.19 comes from the paper of Barron and Jensen (Proposition 3.2, [3]). A consequence of Corollaries 3.2, 3.5, 4.6, Theorem 5.9 and Proposition 5.19 is the following:

Theorem 5.20. *Suppose that* $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ *satisfies* (H1)–(H4) *and* $g : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ *is a lower semicontinuous function. Then:*

(i) *If* H *is a continuous function, then the value function* V, *associated with* L *and* g (*where* L *is dual to* H), *is the bilateral solution of equation* (7).

(ii) *If* H *is upper semicontinuous function, then the value function* V, *associated with* L *and* g (*where* L *is dual to* H), *is the supersolution of equation* (7).

6. Definition of approximate solutions of HJB equation

For upper semicontinuous Hamiltonian $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ we give a definition of an approximate solution of equation (7) $-U_t + H(t,x,-U_x) = 0$.

Definition 6.1. We say that the function $U : [0,T] \times \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ is an *approximate solution* of equation (7) if there exist sequences $U_n : [0,T] \times \mathbb{R}^l \to \mathbb{R}$ and $H_n : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ such that

1. U_n is locally Lipschitz and $U_n \nearrow U$,
2. H_n satisfies assumptions (H1)–(H4), (HL), and $H_n \searrow H$,
3. U_n is a bilateral (equivalently viscosity) solution of $-U_t + H_n(t,x,-U_x) = 0$.

Let us notice that approximate solutions can be interpreted as limits of some bilateral solutions (or viscosity solutions see Remark 5.11). Moreover, Example 5.16 implies that the approximate solution can be neither the bilateral solution nor the viscosity solution nor the classical solution.

In Examples 5.13 and 5.14 one can see that the value function is not the bilateral solution only at points $(t,0)$ of non-continuity of Hamiltonian. In Examples 5.15 the value function is not the viscosity solution only at points $(t,0)$ of non-continuity of Hamiltonian. In Example 5.16 the value function is the classical solution (in particular, it is the viscosity and the bilateral solution) beyond points (t,t) of non-continuity of Hamiltonian. Using arguments as in the proof of Proposition 5.19 we can deduce the general proposition which says when the approximate solution is the bilateral or the viscosity or the classical solution at particular points.

Proposition 6.2 (Compatibility)**.** *The approximate solution U of equation (7) is at a point (t,x), where $t \in (0,T)$,*

(i) *the bilateral solution of equation (7) if H is continuous on a set $\{t\} \times \{x\} \times D_-U(t,x)$,*

(ii) *the viscosity solution of equation (7) if U is continuous and H is continuous on a set $\{t\} \times \{x\} \times D^+U(t,x)$,*

(iii) *the classical solution of equation (7) if U is differentiable and H is continuous at a point $(t,x,\nabla U(t,x))$.*

Furthermore, every approximate solution is the supersolution.

In particular, if Hamiltonian is continuous on the whole domain, then the approximate solution is the bilateral solution and the viscosity solution if U is continuous and the classical solution if U is differentiable. Summarizing, the regularity of Hamiltonian mainly decides when the approximate solution is the bilateral or the viscosity or the classical solution.

Theorem 6.3 (Existence and uniqueness). *Let* $H : [0,T] \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ *satisfy* (H1)–(H4) *and* $g : \mathbb{R}^l \to \mathbb{R} \cup \{+\infty\}$ *be a finite or lower bounded lower semicontinuous function. Then the value function* V*, associated with* L *and* g (*where* L *is dual to* H), *is the unique approximate solution of equation* (7) *in the class functions satisfying boundary condition with* g.

Proof. Due to Theorem 5.18 we conclude that the value function V is the approximate solution satisfying the boundary condition with g because assumptions from Theorem 6.3 allow us to increasingly approximate the function g by locally Lipschitz functions. However, the uniqueness can be obtained in the following way: let U be the approximate solution such that $U(T,x) = g(x)$, then there exist sequences U_n and H_n such that $U_n \nearrow U$ and $H_n \searrow H$. From Corollary 5.12 we have $U_n = V_n$ and from Corollary 3.5 $V_n \nearrow V$. Since $U_n \nearrow V$ and $U_n \nearrow U$, $U = V$. ■

Theoretical results that are obtained in this paper give the method of determining the optimal trajectory in Bolza problem, in which Lagrangian satisfies (L1)–(L5) and a lower semicontinuous terminal cost function is finite or lower bounded. The procedure is: we determine dual Hamiltonian to Lagrangian, next we approximate Hamiltonian and the terminal cost function using such regular functions for which finding optimal trajectories of value functions is possible. From Remark 3.4 we conclude that the accumulation point of optimal trajectories is the optimal trajectory in Bolza problem.

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