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QUADRATIC INTEGRAL EQUATIONS IN REFLEXIVE BANACH SPACE

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Abstract

This paper is devoted to proving the existence of weak solutions to some quadratic integral equations of fractional type in a reflexive Banach space relative to the weak topology. A special case will be considered.

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1. Introduction

Prompted by the application of the quadratic functional integral equations to nuclear physics, these equations have provoked some interest in the literature [2, 7] and [14]. Specifically, the so-called quadratic integral equations of Chandrasekher type can be very often encountered in many applications (cf. [2] and [8]). Some problems in the queueing theory and biology lead to the quadratic functional integral equation of fractional type (cf. e.g. [10] and [13])

$$x(t) = h(t) + g(t, x(t)) I^{\alpha} f(t, x(t)), \; t \in [0,1], \; \alpha > 0, \tag{1}$$

where I^α denotes the standard (Riemann-Liouville) integral operator (*cf.* e.g. [15, 16] and [21]) and f is a real-valued function. The quadratic functional integral equation of type (1) has provoked some interest by many authors (see [6, 9] and [11] for instance). Similar problems are investigated in [4, 5]. In comparison with the existence results in the references, our assumptions ore more natural. While in all other papers the function f is assumed to be real-valued continuous function, here f is supposed to be vector-valued weakly-weakly continuous function. The aim of this paper is to prove the existence of solutions to the quadratic functional integral equation (1) in the Banach space $C[I,E]$ of all E-valued continuous functions, where E is an arbitrary reflexive Banach space. We assume that $f : [0,1] \times E \to E$ is weakly-weakly continuous on $I \times E$ and I^α denotes the fractional Pettis-integral operator of order α (see [20]). Also, we discuss the existence of solutions to the Cauchy problem

$$\text{(2)} \qquad \begin{cases} \dfrac{dx}{dt} &= f(t, D^\beta x(t)), \quad t \in I, \quad 0 < \beta < 1 \\ x(0) &= x_0, \end{cases}$$

with x taking values in E. The proof is based on the fixed point theorem due to O'Regan.

2. Preliminaries

Let $L^1(I)$ be the class of Lebesgue integrable functions on the interval $I = [0,1]$, and E be a reflexive Banach space with norm $\|\cdot\|$ and dual E^*. Let E_w denote the space E where endowed with the weak topology generated by the continuous linear functionals on E and $C[I,E]$ is the Banach space of strong continuous functions $x : I \to E$. Recall that a function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to weakly convergent sequence in E. Furthermore, we recall the following:

Definition 2.1. Let $x : I \to E$. The fractional (arbitrary order) Pettis-integral of x of order $\alpha > 0$ is defined by

$$\text{(3)} \qquad I^\alpha x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds.$$

In the above definition the sign "$\int$" denotes the Pettis integral (see [19]). For the existence of such integral, we refer to [20]. In particular, if x is weakly continuous, then $I^{\alpha}x(\cdot)$ exists as a weakly continuous function on I.

Definition 2.2. Let $x : I \to E$ be a weakly differentiable function and assume that x' is weakly continuous. We define the weak derivative of x of order $\beta \in (0, 1]$ by

$$D^{\beta}\, x(t) := I^{1-\beta}Dx(t), \quad D \text{ denotes the weakly differential operator.}$$

Now, let $r > 0$ be given. Define $B_r := \{x \in C[I, E] : \|x\|_0 \leq r\}$, where $\|\cdot\|_0$ is the usual sup-norm. Let $f(t, x)$ be a function defined on $I \times B_r$, with values in E. Suppose that $f(t, x)$ is weakly-weakly continuous on $I \times B_r$, then

Lemma 2.1.

1. *For each $t \in I$, $f_t = f(t, \cdot)$ is weakly continuous, hence weakly sequentially continuous (cf.* [3]*),*
2. *f is norm bounded, i.e., for any $r > 0$ there exists an M_r such that $\|f(t, x)\| \leq M_r$ for all $(t, x) \in I \times B_r$ (cf.* [22]*).*

Finally, we present some auxiliary results that will be needed in this paper. First, we state a fixed point result (proved in [18]), which was motivated from ideas in [1].

Theorem 2.1. *Let E be a Banach space and let Q be nonempty, bounded, closed, convex and equicontinuous subset of $C[I, E]$. Suppose that $T : Q \to Q$ is weakly sequentially continuous and assume that $TQ(t)$ is relatively weakly compact in E for each $t \in [0, 1]$. Then the operator T has a fixed point in Q.*

The following result is an immediate consequence of the Hahn-Banach Theorem

Proposition 2.1. *Let E be a normed space with $x_0 \neq 0$. Then there exits a $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi x_0 = \|x_0\|$.*

3. Weak solutions to the quadratic integral equation

Since the space of all Pettis integrable functions is not complete, we will restrict our attention to the existence of weak solutions of equation (1) in the space $C[I,E]$. By a weak solution to (1) we mean a weakly continuous function x which satisfies the integral equation (1). This is equivalent to finding $x \in C[I,E]$ with

$$\varphi(x(t)) = \varphi\left(h(t) + g(t,x(t))I^{\alpha}f(t,x(t))\right), \quad t \in [0,1] \text{ for all } \varphi \in E^*.$$

To facilitate our discussion, let us first state the following assumptions

1. $h : [0,1] \to E$ is weakly continuous.
2. $g : [0,1] \times E \to \mathbb{R}$ is weakly-weakly sequentially continuous, such that there exists a constant $\lambda \geq 0$ such that

$$|g(t,x) - g(t,y)| \leq \lambda \left\| x - y \right\|, \quad \text{for all } t \in [0,1] \quad \text{and } x, y \in E.$$

3. $f : [0,1] \times E \to E$ is a weakly-weakly continuous
4. Assume that the inequality $\left\| h \right\|_0 + \frac{(b+\lambda r)M_r}{\Gamma(\alpha+1)} \leq r$ has a positive solution r_0 such that $\frac{\lambda M_{r_0}}{\Gamma(\alpha+1)} < 1$, where $\alpha > 0$, M_r is defined in Lemma 2.1 and $b := \max\limits_{t \in [0,1]} |g(t,0)|$.

Theorem 3.1. *Under the assumptions* (1)–(4), *equation* (1) *has at least one weak solution* $x(\cdot) \in C[I,E]$. *In fact the solution we produce will be norm continuous.*

Proof. Define the operator $T : C[I,E] \to C[I,E]$ by

$$Tx(t) := h(t) + g(t,x(t))I^{\alpha}f(t,x(t), \quad t \in [0,1].$$

First notice that, for every weakly continuous function $x : I \to E$, $f(\cdot, x(\cdot))$, is weakly continuous: To see this we equip E and $I \times E$ with weak topology and note that $t \longmapsto (t, x(t))$ is continuous as a mapping from I into $I \times E$, then $f(\cdot, x(\cdot))$ is a composition of this mapping with f and thus for each weakly continuous $x : I \to E$, $f(\cdot, x(\cdot)) : I \to E$ is weakly continuous. Consequently, the operator T makes sense ([20]). Also, T is well-defined. To see this, let $t_1, t_2 \in [0,1]$ with $t_2 > t_1$. Without loss of generality, assume

$Tx(t_2) - Tx(t_1) \neq 0$. Then there exists (consequence of Proposition 2.1) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t_2) - Tx(t_1)\| = \varphi(Tx(t_2) - Tx(t_1))$. Thus

$$
\begin{aligned}
&\|Tx(t_2) - Tx(t_1)\| = \varphi(Tx(t_2) - Tx(t_1))) \\
&\leq \varphi(h(t_2) - h(t_1)) + \varphi(g(t_2, x(t_2))I^\alpha f(t_2, x(t_2)) - g(t_1, x(t_1))I^\alpha f(t_1, x(t_1))) \\
&\leq \varphi(h(t_2) - h(t_1)) + \varphi(g(t_2, x(t_2))I^\alpha f(t_2, x(t_2)) - g(t_1, x(t_1))I^\alpha f(t_2, x(t_2))) \\
&\quad + \varphi(g(t_1, x(t_1))I^\alpha f(t_2, x(t_2)) - g(t_1, x(t_1))I^\alpha f(t_1, x(t_1))) \\
&\leq \varphi(h(t_2) - h(t_1)) + |g(t_2, x(t_2)) - g(t_1, x(t_1))| \, \varphi(I^\alpha \, |f(t_2, x(t_2)))| \\
&\quad + \frac{|g(t_1, x(t_1))|}{\Gamma(\alpha)} \varphi \left(\int_0^{t_2} \frac{f(\tau, x(\tau))}{(t_2 - \tau)^{1-\alpha}} d\tau - \int_0^{t_1} \frac{f(\tau, x(\tau))}{(t_2 - \tau)^{1-\alpha}} d\tau \right) \\
&\quad + \frac{|g(t_1, x(t_1))|}{\Gamma(\alpha)} \varphi \left(\int_0^{t_1} \frac{f(\tau, x(\tau))}{(t_2 - \tau)^{1-\alpha}} d\tau - \int_0^{t_1} \frac{f(\tau, x(\tau))}{(t_1 - \tau)^{1-\alpha}} d\tau \right) \\
&\leq \|h(t_2) - h(t_1)\| + \frac{|g(t_2, x(t_2)) - g(t_1, x(t_1))|)}{\Gamma(\alpha)} \int_0^{t_2} \frac{|\varphi(f(\tau, x(\tau)))|}{(t_2 - \tau)^{1-\alpha}} d\tau \\
&\quad + \frac{|g(t_1, x(t_1))|}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1} \right| |\varphi(f(\tau, x(\tau)))| d\tau \\
&\quad + \frac{|g(t_1, x(t_1))|}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} |\varphi(f(\tau, x(\tau)))| d\tau \\
&\leq \|h(t_2) - h(t_1)\| + \frac{\lambda \, \|x(t_2) - x(t_1)\|}{\Gamma(\alpha)} M_r \int_0^{t_2} \frac{1}{(t_2 - \tau)^{1-\alpha}} d\tau \\
&\quad + \frac{|g(t_1, x(t_1)) - g(t_1, 0)| + |g(t_1, 0)|}{\Gamma(\alpha)} M_r \int_0^{t_1} \left| (t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1} \right| d\tau \\
&\quad + \frac{|g(t_1, x(t_1)) - g(t_1, 0)| + |g(t_1, 0)|}{\Gamma(\alpha)} M_r \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} d\tau.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|Tx(t_2) - Tx(t_1)\| \leq \|h(t_2) - h(t_1)\| \\
&+ \frac{M_r}{\Gamma(\alpha+1)} \left[\lambda \, \|x(t_2) - x(t_1)\| + (b + \lambda \, \|x\|_0) \left(2(t_2 - t_1)^\alpha + |t_2^\alpha - t_1^\alpha| \right) \right].
\end{aligned}
\tag{4}
$$

This estimation shows that Tx is norm continuous, i.e., T maps $C[I,E]$ into itself. Define the closed, convex, bounded, equicontinuous subset of $Q \subset C[I,E]$ by

$$Q := \Bigg\{ x \in C[I,E] : \|x\|_0 \le r_0, \forall t_1,\ t_2 \in [0,1] \text{ we have } \|x(t_2) - x(t_1)\| \\ \le \frac{1}{1 - \frac{\lambda M_{r_0}}{\Gamma(\alpha+1)}} \left[\|h(t_2) - h(t_1)\| + \frac{(b+\lambda r_0)M_{r_0}}{\Gamma(\alpha+1)} (2(t_2-t_1)^\alpha + |t_2^\alpha - t_1^\alpha|) \right] \Bigg\}.$$

We claim that $T : Q \to Q$ is weakly sequentially continuous and $TQ(t)$ is weakly relatively compact. Once the claim is established, then Theorem 2.1 guarantees a fixed point of T, and hence the problem (1) has a solution in $C[I,E]$. We begin by showing that $T : Q \to Q$. To see this, take $x \in Q$, $t \in [0,1]$. Without loss of generality, assume $Tx(t) \neq 0$. Then there exists (consequence of Proposition 2.1) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tx(t))$. Notice also that, since $\|x\|_0 \le r_0$, then Lemma 2.1 guarantees the existence of a constant M_{r_0} with

$$\|f(t, x(t)\| \le M_{r_0} \quad \text{for all } \ t \in [0,1] \ \text{ and for all } \ x \in Q.$$

Keeping in mind the definition of Q and using inequality (4) we have

$$\|Tx(t_2) - Tx(t_1)\| \\ \le \frac{1}{1 - \frac{\lambda M_{r_0}}{\Gamma(\alpha+1)}} \left[\|h(t_2) - h(t_1)\| + \frac{(b+\lambda r_0)M_{r_0}}{\Gamma(\alpha+1)} (2(t_2-t_1)^\alpha + |t_2^\alpha - t_1^\alpha|) \right].$$

Secondly, we show that $\|Tx\|_0 = sup_{t\in[0,1]} \|Tx(t)\| \le r_0$ for any $x \in Q$. To see this, look at $Tx(t)$ for $t \in [0,1]$. Without loss of generality, we may assume that $Tx(t) \neq 0$ for all $t \in [0,1]$. Then there exists (consequence of Proposition 2.1) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tx(t))$. Thus

$$\|Tx(t)\| \le \|h\|_0 + \frac{(b+\lambda r_0)M_{r_0}}{\Gamma(\alpha+1)}.$$

In the view of our assumptions, we obtain $\|Tx\|_0 \le r_0$. Thus $T : Q \to Q$. Next, we notice that $TQ(t)$ is a bounded subset of E, then the condition that $TQ(t)$ is relatively weakly compact is automatically satisfied, since E is a reflexive Banach space [22]. Finally, we will show that T is weakly sequentially continuous. To see this, let (x_n) be a sequence in Q and let $x_n(t) \to x(t)$ in E_w for each $t \in [0,1]$. Recall [17], that a sequence (x_n) is weakly convergent in $C[I,E]$ iff it is weakly pointwise convergent in E. Fix $t \in I$. From the weak sequential continuity of $f(t,\cdot)$ and $g(t,\cdot)$, the Lebesgue dominated convergence theorem for the Pettis-integral ([12], Corollary 4) implies for each $\varphi \in E^*$ that $\varphi(Tx_n(t)) \to \varphi(Tx(t))$ a.e. on I, $Tx_n(t) \to Tx(t)$ in E_w. So $T : Q \to Q$ is weakly sequentially continuous. The proof is now completed. ■

Definition. Let us recall that a function $x : I \to E$ is called a solution of (2) if

(a) x is weakly differentiable and x' is weakly continuous,

(b) $x(0) = x_0$,

(c) $\frac{dx}{dt} = f(t, D^\beta x(t))$, $t \in I$.

Theorem 3.2. *If the assumptions of Theorem* 3.1 *are satisfied with* $g \equiv 1$, $\alpha = 1 - \beta, \lambda = 1$ *and* $h \equiv 0$, *then Cauchy problem* (2) *has at least one solution* x *in the space* $C[I,E]$.

Proof. Putting $\alpha = 1-\beta, h \equiv 0$, $g \equiv 1$ and $\lambda = 1$ in the equation (1) and considering $y : I \to B_r$ to be a solution, we deduce that

(5) $$y(t) = I^{1-\beta} f(t, y(t)), \ t \in I.$$

Operating by I^β on both sides and using (Lemma 2.1. in [20]), we get

$$I^\beta y(t) = I^1 f(t, y(t)), \ t \in I.$$

Since $f(\cdot, y(\cdot))$ is weakly continuous on I and the integral of a weakly continuous function is weakly differentiable with respect to the right endpoint of the integration interval and its derivative equals the integrand at that point (cf. [17]). Therefore,

$$\frac{d}{dt} I^\beta y(t) = f(t, y(t)).$$

Now, set

(6) $$x(t) = x_0 + I^\beta y(t) = x_0 + I^1 f(t, y(t)),$$

then $x(\cdot)$ is weakly differentiable and

$$x(0) = x_0, \quad \frac{dx}{dt} = f(t, y(t)),$$

since $f(\cdot, y(\cdot))$ is weakly continuous on I, $I^{1-\beta}\frac{dx}{dt}$ exists. Moreover, we have

$$D^\beta x(t) = I^{1-\beta}\frac{dx}{dt} = I^{1-\beta} f(t, y(t)) = y(t).$$

Then any solution to equation (5) will be a solution to the Cauchy problem (2), this solution given by equation (6). This completes the proof. ∎

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