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EXISTENCE RESULTS FOR ϕ-LAPLACIAN DIRICHLET BVP OF DIFFERENTIAL INCLUSIONS WITH APPLICATION TO CONTROL THEORY

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Abstract

In this paper, we study ϕ-Laplacian problems for differential inclusions with Dirichlet boundary conditions. We prove the existence of solutions under both convexity and nonconvexity conditions on the multi-valued right-hand side. The nonlinearity satisfies either a Nagumo-type growth condition or an integrably boundedness one. The proofs rely on the Bonhnenblust-Karlin fixed point theorem and the Bressan-Colombo selection theorem respectively. Two applications to a problem from control theory are provided.

Keywords and phrases: differential inclusions, boundary value problem, fixed point, compact, convex, nonconvex, decomposable, continuous selection, controllability.

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1. Introduction

The aim of this paper is to study the existence of solutions to the Dirichlet boundary value problem:

$$\begin{cases} -(\phi(x'))'(t) \in F(t,x), & 0<t<1 \\ x(0)=x(1)=0, \end{cases} \tag{1.1}$$

where $F : J \times \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^+)$ is a multi-function, $J := [0,1]$, and $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$. The model case corresponds to the so-called p-Laplacian nonlinear operator

$$\phi(s) = \phi_p(s) = \begin{cases} |s|^{p-2}s, & \text{for } s \neq 0 \\ 0, & \text{for } s = 0, \end{cases}$$

where $p > 1$ is a real number. By $\mathbb{R}^+$ we mean the set of nonnegative real numbers and $\mathcal{P}(\mathbb{R})$ will denote the class of all non-empty subsets of $\mathbb{R}$. AC will refer to the set of absolutely continuous functions.

Definition 1.1. A function $x \in AC\,(J,\mathbb{R}^+)$ is said to be a solution of Problem (1.1) if $\phi(x') \in AC(J,\mathbb{R})$ and there exists $v \in L^1(J,\mathbb{R}^+)$ with $v(t) \in F(t,x(t))$ for a.e. $t \in J$ such that $-(\phi(x'))'(t) = v(t)$ for a.e. $t \in J$ and $x(0) = x(1) = 0$.

During the last few years, second-order boundary value problem for ordinary and functional differential equations corresponding to $\phi(s) = \phi_2(s) = s$, with various conditions (periodic, nonlinear, integral conditions, etc..) have attracted the attention of many mathematicians and are still intensively studied. Indeed, these problems arise in different areas of physics, mechanics, and more generally in applied mathematics. The foundation of the general theory of such problems is deeply investigated in the literature (see for instance the monographs by Bernfeld and Lakshmikantham [14], Henderson [28] or Mawhin [39]).

The case of second order boundary value problem for differential inclusions has been studied in [24] where the multi-function satisfies a Bernstein-Nagumo condition.

Very recently, Benchohra *et al.* [10] have studied some 3-point boundary value problem associated with a differential inclusion $x''(t) \in F(t,x(t))$

where F is a nonempty compact valued multi-valued mapping which is integrably bounded. This study is extended in [46] to perturbed differential inclusion $Lx \in F(t, x(t))$, where F satisfies Carathéodory conditions and L is a second-order Sturm-Liouville differential operator of the form $Lx = -x'' + qx' + rx$. The existence of positive solutions is obtained under sub-linear growth condition on the upper semi-continuous function F with respect to the second argument.

The general differential inclusion $x'' \in F(t, x, x')$ is considered in [21] and [22], where the authors prove some inequalities in order to ensure a priori estimates for solutions; an existence principle is then derived in case of compact, convex valued nonlinearity (see also [26]).

In [40, 41, 42], N.S. Papageorgiou *et al.* have considered p-Laplacian problems associated with Dirichlet, Neumann and periodic boundary conditions (see also [8] and [32]). They offer various results in the case of the nonlinearity of the form $Ax(t) + F(t, x(t), x'(t))$, where the operator A is maximal monotone and F obeys some Hartmann conditions.

A class of p-Laplacian m-point problems is discussed in [10] under Carathéodory Bressan-Colombo conditions on the multi-valued nonlinearity $F = F(t, x)$.

An extension to ϕ-Laplacian differential equations with periodic boundary conditions has been recently given by Rachunkova and Tvrdy (see [44] and the references therein).

Notice that most of these works consider the convex-valued right-hand side nonlinearity. The general problem of a nonconvex nonlinearity which is a composition of two convex functions is investigated in [43] where some existence results are obtained with applications to differential inclusion problems.

Our goal in this work is to complement and extend some of these results by giving some existence results to Problem (1.1), where the right-hand side is either convex or nonconvex and ϕ is a general nonlinear differential operator. Our proofs are essentially based on the Bonhnenblust-Karlin fixed point theorem (in the convex case) and on the Bressan-Colombo selection theorem (in the nonconvex case). A particular attention will be given to some problems from control theory.

The paper is organized as follows. We first collect some background material and basic results from multi-valued analysis in Section 2. In order to formulate Problem (1.1) as an equivalent integral problem in the Banach space of continuous functions, our existence results need a fixed point formulation which is developed in Section 3. In Section 4, the existence result for the case of the nonlinear multi-valued mapping F has compact, convex values is proved under Nagumo-type growth condition. An example illustrates the existence theorem. Section 5 is devoted to the compact, nonconvex case. The applicability of the obtained results, in both the convex and the nonconvex cases, to a problem from control theory is presented in Section 6.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multi-valued analysis which are used throughout this paper. $C(J,\mathbb{R})$ will denote the Banach space of all continuous functions from J into $\mathbb{R}$ with Tchebyshev norm

$$\|x\|_\infty = \sup\{|x(t)|,\ t \in J\}.$$

$L^1(J,\mathbb{R})$ refers to the Banach space of measurable functions $x : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable; it is normed by

$$|x|_1 = \int_0^1 |x(s)|ds.$$

If (X,d) is a metric space, the following notations will be used throughout this paper:

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$.
- $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y$ has the property "p"$\}$, where p could be: *cl*=closed, *b*=bounded, *cp*=compact, *cv*=convex, etc. Thus
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$.
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$.
- $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$.
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$.
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

Let $(X, \|\cdot\|)$ be a separable Banach space and $F : J \to \mathcal{P}_{cl}(X)$ be a multi-valued map. F is called measurable provided for every open $U \subset X$, the set $F^{+1}(U) = \{t \in J,\ F(t) \subset U\}$ is Lebesgue measurable in J. We have

Lemma 2.1 (see [2, 17, 27]). *F is measurable if and only if for each $x \in X$, the function $\zeta : J \to [0, +\infty)$ defined by*

$$\zeta(t) = \mathrm{dist}(x, F(t)) = \inf\{\|x - y\|,\ y \in F(t)\}, \quad t \in J$$

is Lebesgue measurable.

Let $(X, \|\cdot\|)$ be a Banach space and $F : X \to \mathcal{P}(X)$ be a multi-valued map. We say that F has a *fixed point* if there exists $x \in X$ such that $x \in F(x)$. The fixed point set of F will be denoted by $Fix\, F$.

F has *convex (closed) values* if $F(x)$ is convex (closed) for all $x \in X$.

F is *totally bounded* if $F(A) = \bigcup_{x \in A}\{F(x)\}$ is bounded in X for each bounded set A of X, i.e.,

$$\sup_{x \in A}\{\sup\{\|y\|,\ y \in F(x)\}\} < \infty.$$

Let (X, d) and (Y, ρ) be two metric spaces and $F : X \to \mathcal{P}_{cl}(Y)$ a multi-valued mapping. F is said to be *lower semi-continuous* (*l.s.c. for short*) if the inverse image of V by F

$$F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$$

is open for any open set V in Y. Similarly, F is *l.s.c.* if the core of V by F

$$F^{+1}(V) = \{x \in X,\ F(x) \subset V\}$$

is closed for any closed set V in Y.

Likewise, the map F is called *upper semi-continuous (u.s.c. for short)* on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Similarly, F is *u.s.c.* if the set $F^{+1}(V)$ is open for any open set V in Y.

F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup\{F(x),\ x \in A\} \subset K.$$

F is compact if $F(X)$ is relatively compact. It is called locally compact if for each $x \in X$, there exists $U \in \mathcal{V}(x)$ such that $F(U)$ is relatively compact.

By the graph of F we mean the set $\mathcal{G}r(F) = \{(x, y) \in X \times Y,\ y \in F(x)\}$ and recall

Lemma 2.2 (see [18] or [19], Proposition 1.2)**.** *If $F : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $\mathcal{G}r(F)$ is a closed subset of $X \times Y$, i.e., for every sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in F(x_n)$, then $y_* \in F(x_*)$. Conversely, if F has nonempty compact values, is locally compact and has a closed graph, then it is u.s.c.*

Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multi-valued function and denote

$$\|F(t, x)\| := \sup\{|v| : v \in F(t, x)\}.$$

Definition 2.1. F is said integrably bounded if there exists $p \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| \le p(t) \ \text{ for a.e. } t \in J \ \text{ and each } x \in \mathbb{R}.$$

Definition 2.2. F is called a multi-valued Carathéodory function if

(a) The function $t \mapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$.

(b) For a.e. $t \in J$, the map $x \mapsto F(t, x)$ is upper semi-continuous.

It is further an L^1-Carathéodory if it is locally integrably bounded, i.e., for each positive r, there exists some $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| \le h_r(t) \ \text{ for a.e. } t \in J \ \text{ and all } |x| \le r.$$

For more details on multi-valued maps, we refer the reader to the books by Aubin and Cellina [3], Aubin and Frankowska [4], Deimling [19], Górniewicz [27], Hu and Papageorgiou [29, 30], Kamenskii *et al.* [31], Kisielewicz [33] and Tolstonogov [47].

3. Fixed point formulation

The following results will be useful in the sequel. The first two lemmas are from [13, 12]. The third one is rather classical.

Lemma 3.1. *For any function $h \in L^1(0,1)$ positive almost everywhere, the problem of seeking $0 < t < 1$ such that*

$$\int_0^t \phi^{-1}\left(\int_s^t h(\tau)d\tau\right)ds = \int_t^1 \phi^{-1}\left(\int_t^s h(\tau)d\tau\right)ds \tag{3.1}$$

has uniquely one solution $\theta = \theta(h)$ such that $0 < \theta < 1$.

Lemma 3.2. *Consider the boundary value problem*

$$\begin{cases} -(\phi(x'))'(t) = h(t), & 0 < t < 1 \\ x(0) = x(1) = 0, \end{cases} \tag{3.2}$$

where $h \in L^1(0,1)$ is positive almost everywhere. Then Problem (3.2) *has a unique solution $x \in C^1(J, \mathbb{R}^+)$ given by*

$$x(t) = \begin{cases} \displaystyle\int_0^t \phi^{-1}\Big(\int_s^\theta h(\tau)d\tau\Big)ds, & \text{if } 0 \le t \le \theta < 1 \\ \displaystyle\int_t^1 \phi^{-1}\Big(\int_\theta^s h(\tau)d\tau\Big)ds, & \text{if } 0 < \theta \le t \le 1, \end{cases} \tag{3.3}$$

where $\theta = \theta(h)$ is as given in Lemma 3.1.

The following result is known as Grönwall-Bihari Theorem:

Lemma 3.3 [6]**.** *Let $I = [a,b]$ and let u, $g\colon I \to \mathbb{R}$ be positive real continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h\colon \mathbb{R} \to (0,+\infty)$ such that*

$$u(t) \le c + \int_a^t g(s)h(u(s))\,ds, \quad \forall\, t \in I.$$

Then we have

$$u(t) \le H^{-1}\left(\int_a^t g(s)\,ds\right), \quad \forall\, t \in I$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_a^b g(s)\,ds.$$

Here H^{-1} refers to the inverse of the function $H(u) = \int_c^u \frac{dy}{h(y)}$ for $u \ge c$.

Let $F : J \times \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^+)$ be a multi-function and consider the operator $N : C(J,\mathbb{R}^+) \longrightarrow \mathcal{P}\left(C(J,\mathbb{R}^+)\right)$ defined by $N(x) = \{y\}$ with

$$\forall\, t \in J, \quad y(t) = \begin{cases} \displaystyle\int_0^t \phi^{-1}\Big(\int_s^\theta v(\tau)d\tau\Big)ds, & \text{if } 0 \le t \le \theta < 1 \\ \displaystyle\int_t^1 \phi^{-1}\Big(\int_\theta^s v(\tau)d\tau\Big)ds, & \text{if } 0 < \theta \le t \le 1, \end{cases}$$

where

$$v \in S_{F,x} = \{v \in L^1(J,\mathbb{R}^+) : \ v(t) \in F(t,x(t)), \ \text{ a.e. } \ t \in J\}$$

and $\theta = \theta(v)$ is as defined in Lemma 3.1. The set $S_{F,x}$ known as the set of selection functions, is closed. It is convex if and only if $F(t,x(t))$ is convex for a.e. $t \in J$.

Remark 3.1. When F is an L^1-Carathéodory multi-valued mapping, we know from the result due to Lasota and Opial [36] that for each $x \in C(J,\mathbb{R})$, the set $S_{F,x}$ is nonempty. Thus, we can define a multi-operator

$$\begin{aligned} S_F : C(J,\mathbb{R}^+) &\to \mathcal{P}(C(J,\mathbb{R}^+)) \\ x &\mapsto S_F(x) = S_{F,x}. \end{aligned}$$

Moreover, From Lemma 3.2, x is a fixed point of N if and only if x is a solution to Problem (1.1).

4. The convex case

4.1. Existence result

We will make use of the following lemma in the proof of our first existence theorem. The second one is known as the Bohnenblust-Karlin fixed point theorem.

Lemma 4.1 [17, 36]**.** *Let X be a Banach space, $F : [a,b] \times X \longrightarrow \mathcal{P}_{cp,cv}(X)$ an L^1-Carathéodory multi-valued map and let Γ be a linear continuous mapping from $L^1([a,b],X)$ to $C([a,b],X)$. Then the operator*

$$\begin{aligned} \Gamma \circ S_F : C([a,b],X) &\longrightarrow \mathcal{P}_{cp,cv}(C([a,b],X)) \\ x &\longmapsto (\Gamma \circ S_F)(x) := \Gamma(S_{F,x}) \end{aligned}$$

is a closed graph operator in $C([a,b],X) \times C([a,b],X)$.

Lemma 4.2 [15]**.** *Let X be a Banach space, $C \in \mathcal{P}_{cp,cv}(X)$ and $T : C \to \mathcal{P}_{cl,cv}(C)$ be an upper semi-continuous multivalued operator such that the set $T(C)$ is relatively compact in X. Then T has a fixed point in C.*

Our main existence result in this section is:

Theorem 4.3. *Assume $F : J \times \mathbb{R}^+ \to \mathcal{P}_{cp,cv}(\mathbb{R}^+)$ is a multi-valued L^1-Carathéodory mapping such that $0 \notin F(.,.)$ and*

$$(\mathcal{H}_1) \quad \begin{cases} \textit{There exists a continuous nondecreasing function} \\ \psi : [0,\infty) \mapsto (0,\infty) \ \textit{ and } \ p \in L^1(J,\mathbb{R}^+) \ \textit{ such that} \\ \|F(t,x)\| \le p(t)\psi(|x|) \ \textit{ for a.e. } \ t \in J, \ \textit{ all } \ x \in \mathbb{R}, \ \textit{ and} \\ \exists\, R_0 > 0, \quad \psi(R_0) \le \dfrac{\phi(R_0)}{|p|_1}. \end{cases}$$

Then Problem (1.1) *has at least one positive solution. Moreover, the set of all solutions is compact.*

Remark 4.1. It is obvious that any integrably bounded multi-function satisfies $(\mathcal{H}_1)$.

Remark 4.2. It is clear that any solution x is in $AC(J,\mathbb{R}^+)$. Moreover, if ϕ is odd, then

$$\phi^{-1}\left(\int_t^\theta v(s)\,ds\right) = -\phi^{-1}\left(\int_\theta^t v(s)\,ds\right),$$

where $v \in S_{F,x}$. Thus, differentiating y in the definition of the mapping N we get $x \in C^1(J,\mathbb{R}^+)$.

***Proof of Theorem* 4.3.** Consider the convex subset of $C(J,\mathbb{R}^+)$

$$K = \left\{ \begin{array}{c} w \in C(J,\mathbb{R}^+),\ w(0) = 0,\ w \text{ is nondecreasing, and} \\ 0 \le w(t) - w(s) \le \psi(R_0) \displaystyle\int_s^t p(\tau)d\tau, \ \text{ for all } 0 \le s \le t \le 1 \end{array} \right\}.$$

It is clear that for every $w \in K$, $\|w\|_\infty \le R_1 := |p|_1\psi(R_0)$ and K is compact by Ascoli-Arzéla Lemma. Furthermore, any element $w \in K$ is absolutely continuous. Thus, we can define

$$K \xrightarrow{S} C(J,\mathbb{R})$$

such that $x = S(w)$ is a unique solution of the problem

$$\begin{cases} -(\phi(x'))'(t) = w'(t), & t \in [0,1] \\ x(0) = x(1) = 0 \end{cases}$$

that is

$$x(t) = \begin{cases} \displaystyle\int_0^t \phi^{-1}\left(w(\theta) - w(s)\right) ds, & \text{if } 0 \le t \le \theta < 1 \\ \displaystyle\int_t^1 \phi^{-1}\left(w(s) - w(\theta)\right) ds, & \text{if } 0 < \theta \le t \le 1, \end{cases} \tag{4.1}$$

where $\theta = \theta(w)$ satisfies, by Lemma 3.1, the C^0-matching relation:

$$\int_0^\theta \phi^{-1}\left(w(\theta) - w(s)\right) ds = \int_\theta^1 \phi^{-1}\left(w(s) - w(\theta)\right) ds.$$

Finally define the multivalued map

$$G : C(J, \mathbb{R}^+) \to \mathcal{P}(C(J, \mathbb{R}^+))$$

by

$$G(x) = \{y \in C(J, \mathbb{R}^+),\ y(t) = \int_0^t v(s)\, ds \text{ for some } v \in S_{F,x}\}.$$

Next, the properties of the mapping $G' = G \circ S$ are studied.

Claim 1. $G'(K) \subset K$. Let $w \in K$ and $y \in G(w)$; then there exist $x \in C(J, \mathbb{R})$ and $v \in S_{F,x}$ such that

$$y(t) = \int_0^t v(s)\, ds,\ t \in [0,1].$$

It is clear that y is a nondecreasing function and, for $0 \le s \le t \le 1$,

$$y(t) - y(s) = \int_s^t v(\tau)\, d\tau \le \int_s^t \|F(\tau, x(\tau))\|\, d\tau \le \int_s^t p(\tau)\psi(|x(\tau)|)\, d\tau.$$

(4.1) yields the estimate

$$\|x\|_\infty \le \phi^{-1}(R_1)$$

and then

$$0 \leq y(t) - y(s) \leq \psi(\phi^{-1}(R_1)) \int_s^t p(\tau) d\tau.$$

It follows from the definition of R_0 and R_1 that

$$\psi(\phi^{-1}(R_1)) \leq \psi(R_0),$$

showing that $N(K) \subset K$.

Claim 2. $G'(w)$ is convex for each $w \in K$. Indeed, if $y_1,\ y_2 \in G'(w)$, then there exist $x \in C(J, \mathbb{R})$ and $v_1, v_2 \in S_{F,x}$ such that for each $t \in [0, 1]$ we have

$$y_i(t) = \int_0^t v_i(s)\, ds, \quad i = 1, 2.$$

Let $0 \leq \alpha \leq 1$. Then for each $t \in [0, 1]$ we have

$$(\alpha y_1 + (1 - \alpha) y_2)(t) = \int_0^t [\alpha v_1(s) + (1 - \alpha) v_2(s)]\, ds.$$

Since $S_{F,x}$ is convex (because F has convex values), then

$$\alpha y_1 + (1 - \alpha) y_2 \in G'(w).$$

Claim 3. G' maps bounded sets into bounded sets in $C(J, \mathbb{R}^+)$. Indeed, it is enough to show that there exists a positive constant ℓ such that for each $w \in B_r = \{w \in C(J, \mathbb{R}^+) : \|w\|_\infty \leq r\}$, one has $\|G'(w)\|_\infty \leq \ell$. Let $w \in B_r$ and $y \in G'(w)$; then there exist $x \in C(J, \mathbb{R})$ and $v \in S_{F,x}$ such that for each $t \in J$ we have

$$y(t) = \int_0^t v(s)\, ds,\ \ t \in [0, 1]. \tag{4.2}$$

Using $(\mathcal{H}_1)$ and noting that ψ is nondecreasing, we have $\|x\|_\infty \leq \phi^{-1}(r)$ and then for each $t \in J$

$$|y(t)| \leq \int_0^t |v(s)|\, ds \leq \int_0^1 p(s) \psi(|x(s)|)\, ds \leq |p|_1 \psi(\phi^{-1}(r)).$$

Claim 4. G' maps bounded sets into equicontinuous sets of $C(J, \mathbb{R}^+)$. Let B_r be the ball centered at the origin and of radius r in $C(J, \mathbb{R}^+)$; we prove that the family set $\{G'w,\ w \in B_r\}$ is relatively compact. As in Claim 3, it is clear that this set is bounded. To check that it is equicontinuous, let $t_1,\ t_2 \in J$ be such that $t_1 < t_2$. From $(\mathcal{H}_1)$ we have

$$|y(t_2) - y(t_1)| \le \psi(\phi^{-1}(r)) \int_{t_1}^{t_2} p(s)\,ds$$

and the right-hand side tends to zero as $t_2 - t_1 \to 0$.

Claim 5. G' is *upper semi-continuous.* Thanks to Lemma 2.2, it suffices to prove that G' has a closed graph. Let $w_n \longrightarrow w_*$, $y_n \in G'(w_n)$ and $y_n \longrightarrow y_*$ as $n \to \infty$. We claim that $y_* \in G'(w_*)$. Indeed, $y_n \in G'(w_n)$ means that there exist $x_n \in C(J, \mathbb{R})$ and $v_n \in S_{F,x_n}$ such that for each $t \in J$

$$y_n(t) = \int_0^t v_n(s)\,ds, \quad t \in [0,1].$$

We must prove that there exists $v_* \in S_{F,x_*}$ such that for each $t \in J$

$$y_*(t) = \int_0^t v_*(s)\,ds, \quad t \in [0,1].$$

Consider the continuous linear operator

$$\begin{aligned} \Gamma : L^1(J,\mathbb{R}) &\longrightarrow C(J,\mathbb{R}) \\ u &\mapsto \Gamma u \end{aligned}$$

such that

$$(\Gamma u)(t) = \int_0^t u(s)ds, \quad t \in [0,1].$$

By Lemma 4.1, the operator $\Gamma \circ S_F$ has a closed graph and the definition of G' yields

$$y_n \in \Gamma(S_{F,x_n}) = (\Gamma \circ S_F)(x_n).$$

Moreover, the operator S is continuous (see the proof of Claim 1 in Theorem 5.1). Then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and so there exists an $M \ge 0$ such that

$$\|x_n\|_\infty \le M, \quad \forall\, n \in \mathbb{N}.$$

Hence

$$|v_n(t)| \le p(t)\psi(M), \qquad \text{for a.e. } t \in J \text{ and all } n \in \mathbb{N}$$

and $v_n \to v_*$ a.e. in $\mathbb{R}$, as $n \to +\infty$. By the Lebesgue dominated convergence theorem, $\lim_{n\to\infty} y_n(t) = y_*(t),\ t \in J$. Since $x_n \longrightarrow x_*$, we finally deduce from the continuity of F and Γ that

$$y_* \in \Gamma(S_{F,x_*}) = (\Gamma \circ S_F)(x_*),$$

ending our claim.

From Claims 3–5, G' is completely continuous and hence has nonempty compact values. To sum up, the multi-valued map $G' : K \to \mathcal{P}_{cl,cv}(K)$ satisfies all conditions of Lemma 4.2 and therefore has a fixed point w in K. It follows that $x = S(w)$ is a fixed point of N, hence a solution to Problem (1.1) in $S(K)$. Conversely, if x is a solution to Problem (1.1), then w defined by $w(t) = \int_0^t x(s)ds$ is a fixed point of the mapping G' and lies in K. Since K is compact and S is continuous, the set $S(K)$ is compact and the last statement of the theorem follows. ■

Hereafter, a more precise uniform estimate of the set of solutions is provided.

Proposition 4.4. *Assume that the last restriction in Assumption* $(\mathcal{H}_1)$ *is replaced by*

$$\int_0^{+\infty} \frac{ds}{\phi^{-1}\left(|p|_1\psi(s)\right)} > 1.$$

Then the set of solutions of Problem (1.1) *is uniformly bounded by* $\Upsilon^{-1}(1)$, *where*

$$\Upsilon(s) := \int_0^s \frac{d\tau}{\left(\phi^{-1} \circ |p|_1\psi\right)(\tau)}.$$

Proof. Let $x \in C(J, \mathbb{R}^+)$ be a possible solution of the differential inclusion $x \in N(x)$. Then there exists $v \in S_{F,x}$ such that for each $t \in J$ we have by Assumption $(\mathcal{H}_1)$

$$|v(t)| \leq \|F(t, x(t))\| \leq p(t)\psi(|x(t)|), \quad a.e.\ t \in J.$$

Assuming $0 \leq t \leq \theta$, we find that

$$\begin{aligned} |x(t)| &\leq \int_0^t \phi^{-1}\left(\int_s^\theta p(\tau)\psi(|x(\tau)|)\, d\tau\right) ds \\ &\leq \int_0^t \phi^{-1}\left(|p|_1\psi(m(s))\right) ds, \end{aligned}$$

where

$$m(s) := \max\{|x(\tau)|, \quad s \le \tau \le \theta\}, \quad \text{for } s \in [0, \theta].$$

Therefore,

$$m(t) \le \int_0^t \left(\phi^{-1} \circ \tilde{\psi}\right)(m(s))ds, \ \forall\, t \in [0, \theta]$$

with $\tilde{\psi} = |p|_1 \psi$. By the nonlinear Grönwall-Bihari inequality (Lemma 3.3), we infer the bound

$$m(t) \le \Upsilon_1^{-1}(t) \le M := \Upsilon_1^{-1}(\theta), \quad \forall\, t \in [0, \theta],$$

where $\Upsilon_1(s) := \int_0^s \frac{d\tau}{(\phi^{-1} \circ \tilde{\psi})(\tau)}$. In a similar way we deal with the case $\theta \le t \le 1$, and arrive at the estimate

$$\sup\{|x(t)| : t \in J\} \le \Upsilon_2^{-1}(\theta),$$

where

$$\Upsilon_2(s) := \int_s^1 \frac{d\tau}{\left(\phi^{-1} \circ \tilde{\psi}\right)(\tau)},$$

ending the proof of the proposition. ■

4.2. Example

Consider the p-Laplacian differential inclusion ($p > 1$) with Dirichlet boundary conditions

$$\left\{ \begin{array}{l} -(|x'|^{p-2}x')'(t) \in F(t, x), \quad 0 < t < 1 \\ x(0) = x(1) = 0 \end{array} \right. \tag{4.3}$$

where $F : [0, 1] \times \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^+)$ is a multi-valued map defined by

$$F(t, x) = \{v \in \mathbb{R}^+ : f_1(t, x) \le v \le f_2(t, x)\},$$

and $f_1, f_2 : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ are single-valued functions. We assume that for each $t \in [0, 1]$, $f_1(t, .)$ is lower semi-continuous (i.e, the set $\{x \in \mathbb{R} : f_1(t, x) > \mu\}$ is open for each $\mu \in \mathbb{R}$) and that for each $t \in [0, 1]$, $f_2(t, .)$ is upper semi-continuous (i.e., the set $\{x \in \mathbb{R} : f_1(t, x) < \mu\}$ is open for each

$\mu \in \mathbb{R}$). Assume further that there exist $p \in L^1([0,1], \mathbb{R}^+)$ and $\sigma \in [0, +\infty)$ such that

$$0 \leq \max(f_1(t,x), f_2(t,x)) \leq p(t)u^\sigma, \quad \text{for a.e. } t \in [0,1] \text{ and all } x \in \mathbb{R}^+.$$

It is clear that F is compact convex valued and upper semi-continuous (see [19]). Assumption $(\mathcal{H}_1)$ is fulfilled whenever either $\sigma \neq p-1$ or $\sigma = p-1$ and $|p|_1 \leq 1$. Notice that this condition on σ is less restrictive than $0 \leq \sigma < \frac{1}{1-q}$ with $\frac{1}{p} + \frac{1}{q} = 1$, which implies the hypothesis in Proposition 4.4. With such a choice of the parameter σ, all conditions of Theorem 4.3 are met and then Problem (4.3) has at least one solution $x \in C(J, \mathbb{R}^+)$.

5. The nonconvex case

We first recall some definitions (see e.g. [4]). Let E be a Banach space and A a subset of $J \times E$.

Definition 5.1. A is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$, where I is Lebesgue measurable in J and D is Borel measurable in E.

Definition 5.2. A subset $A \subset L^1(J, E)$ is decomposable if for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$ we have:

$$u\chi_I + v\chi_{J \setminus I} \in A,$$

where χ stands for the characteristic function.

Let $F : J \times E \to \mathcal{P}(E)$ be a multi-valued map with nonempty closed values. Assign to F the multi-valued operator $\mathcal{F} : C(J, E) \to \mathcal{P}(L^1(J, E))$ defined by $\mathcal{F}(y) = S_{F,y}$ and let $\mathcal{F}(t, y) = S_{F,y}(t),\ t \in J$. The operator $\mathcal{F}$ is called the Nemytskii operator associated to F.

Definition 5.3. Let $F : J \times E \to \mathcal{P}(E)$ be a multi-valued function with nonempty compact values. We say that F is of lower semi-continuous type (*l.s.c.* type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Our main result in this section is

Theorem 5.1. *Assume the multi-valued map* $F : J \times \mathbb{R}^+ \longrightarrow \mathcal{P}_{cp}(\mathbb{R}^+)$ *is integrably bounded, satisfies* $0 \notin F(\cdot, \cdot)$ *and*

$$(\mathcal{H}_2) \qquad \begin{cases} (a) & (t,x) \mapsto F(t,x) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ measurable;} \\ (b) & x \mapsto F(t,x) \text{ is lower semi-continuous for a.e. } t \in J. \end{cases}$$

Then Problem (1.1) *has at least one positive solution.*

The following three auxiliary results are fundamental in the sequel; the first one is a selection theorem due to Bressan and Colombo (also called Fryszkowski Selection Theorem). The third one is the classical Nonlinear Alternative of Leray and Schauder for single-valued mappings.

Lemma 5.2 (see [3, 16, 19, 29])**.** *Let* X *be a separable metric space and let* E *be a Banach space. Then every l.s.c. multi-valued operator* $N : X \to \mathcal{P}_{cl}(L^1(J,\ E))$ *with closed decomposable values has a continuous selection, i.e., there exists a continuous single-valued function* $f : X \to L^1(J,E)$ *such that* $f(x) \in N(x)$ *for every* $x \in X$.

Lemma 5.3 (see [19, 25])**.** *Let* $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *be an integrably bounded multi-valued function satisfying* $(\mathcal{H}_2)$. *Then* F *is of lower semi-continuous type.*

Lemma 5.4 [20]**.** *Let* X *be a Banach space and* $C \subset X$ *a nonempty bounded, closed, convex subset. Assume* U *is an open subset of* C *with* $0 \in U$ *and let* $G : \bar{U} \to C$ *be a a continuous compact map. Then*

(a) *either there is a point* $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u = \lambda G(u)$,
(b) *or* G *has a fixed point in* $\overline{U}$.

***Proof of Theorem* 5.1.** From Lemmas 5.2 and 5.3, there exists a continuous selection function $f : C(J,\mathbb{R}^+) \to L^1(J,\mathbb{R}^+)$ such that $f(x)(t) \in \mathcal{F}(t,x)$ for every $x \in C(J,\mathbb{R}^+)$ and a.e. $t \in J$. Next, consider the boundary value problem for an autonomous ϕ-Laplacian ordinary differential equation:

$$\begin{cases} -(\phi(x'))'(t) = f(x)(t), & \text{a.e. } t \in J \\ x(0) = x(1) = 0. \end{cases} \tag{5.1}$$

Clearly, if $x \in C(J,\mathbb{R}^+)$ is a solution to Problem (5.1), then x is a solution to Problem (1.1). Problem (5.1) is then reformulated as a fixed point problem for the operator $A : C(J, \mathbb{R}^+) \to C(J, \mathbb{R}^+)$ defined by

$$(5.2) \qquad (Ax)(t) = \begin{cases} \displaystyle\int_0^t \phi^{-1}\Big(\int_s^\theta f(x(\tau))\,d\tau\Big)\,ds, & \text{if } 0 \le t \le \theta < 1 \\ \displaystyle\int_t^1 \phi^{-1}\Big(\int_\theta^s f(x(\tau))\,d\tau\Big)\,ds, & \text{if } 0 < \theta \le t \le 1, \end{cases}$$

where $\theta = \theta(x)$ is as defined by (3.1). Hereafter, the main properties of A are investigated. In three steps, we first check that A is completely continuous.

Claim 1. A is continuous. We give a direct proof. Further proofs can be found in [1]-Lemma 3, [37]-Lemma 3.1, [38]-Lemma 2.1, or in [48]-Lemma 2.5. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to some limit x in $C(J, \mathbb{R}^+)$ and let $v_n(\cdot) = f(x_n(\cdot))$. By continuity of the selection f, $v_n(\cdot) \to v(\cdot) = f(x(\cdot))$ a.e., as $n \to +\infty$ and then

$$\forall\, s \in (0, \theta_n),\ 0 \le \lim_{n\to\infty} \int_s^{\theta_n} |v_n(\tau) - v(\tau)|\,d\tau \le \lim_{n\to\infty} \int_0^1 |v_n(\tau) - v(\tau)|\,d\tau = 0,$$

where $\theta_n = \theta(x_n)$ is as defined by (3.1).

Since $0 < \theta_n < 1$, then θ_n converges, up to a subsequence, to some limit $\theta_* \in [0, 1]$. Assume $0 < \theta_* < 1$. By the Lebesgue dominated convergence theorem, the integral $\int_0^t \phi^{-1}(\int_s^{\theta_n} v_n(\tau_n)\,d\tau)ds$ converges to $\int_0^t \phi^{-1}(\int_s^{\theta_*} v(\tau)\,d\tau)ds$ because ϕ is a homeomorphism. The same holds for the second term in (5.2) with $\theta = \theta_n$. In addition,

$$A(x)(t) = \begin{cases} \displaystyle\int_0^t \phi^{-1}\Big(\int_s^{\overline{\theta}} f(x(\tau))\,d\tau\Big)\,ds, & \text{if } 0 \le t \le \overline{\theta} < 1 \\ \displaystyle\int_t^1 \phi^{-1}\Big(\int_{\overline{\theta}}^s f(x(\tau))\,d\tau\Big)\,ds, & \text{if } 0 < \overline{\theta} \le t \le 1, \end{cases}$$

where $\overline{\theta} = \overline{\theta}(x)$ is uniquely defined by (3.1). Since

$$\int_0^{\theta_n} \phi^{-1}\Big(\int_s^{\theta_n} f(x_n(\tau))\,d\tau\Big) = \int_{\theta_n}^1 \phi^{-1}\Big(\int_{\theta_n}^s f(x_n(\tau))\,d\tau\Big)\,ds,$$

it follows, by the Lebesgue dominated convergence theorem, that

$$\int_0^{\theta_*} \phi^{-1}\Big(\int_s^{\theta_*} f(x(\tau))\,d\tau\Big)\,ds = \int_{\theta_*}^1 \phi^{-1}\Big(\int_{\theta_*}^s f(x(\tau))\,d\tau\Big)\,ds.$$

By uniqueness of $\overline{\theta}$, it follows that $\theta_* = \overline{\theta}$, proving the continuity of A. Now, assume $\theta_* = 0$. Then

$$\int_0^1 \phi^{-1}\left(\int_0^s f(x(\tau))\, d\tau\right) ds = 0 \Rightarrow \phi^{-1}\left(\int_0^t f(x(s))ds\right) = 0,\ t \in [0,1]$$
$$\Rightarrow f(x(\cdot)) = 0 \ \text{ a.e on } \ [0,1].$$

This is contradiction with $0 \notin F(.,.)$. Analogously, we check that $\theta_* \neq 1$, ending the proof of our claim.

Claim 2. A maps bounded into bounded sets. Let B be a bounded subset of $C(J,\mathbb{R}^+)$ and $u \in B$. Then $\|Ax\| \leq M = \phi^{-1}(|p|_1)$, where $|f(x(t))| \leq |p(t)|$; this implies the boundedness of $A(B)$.

Claim 3. The set $\{Au,\ u \in B\}$ is equicontinuous. For this, let t_1, $t_2 \in J$ and distinguish between four cases taking into account the relative position of t_1, t_2 with respect to θ. We only consider $0 \leq t_1$, $t_2 \leq \theta < 1$, in which case we have

$$\begin{aligned}|(Ax)(t_1) - (Au)(t_2)| &= \left|\int_{t_1}^{t_2} \psi\left(\int_s^\theta f(x(\tau))d\tau\right) ds\right| \\ &\leq |t_1 - t_2|\phi^{-1}(|p|_1).\end{aligned}$$

Letting $|t_1 - t_2| \to 0$, the claim follows.

With Claims 1–3, the Arzéla-Ascoli Lemma implies that A is completely continuous.

Claim 4. Uniform a priori bounds.
For every fixed point $x = \lambda A(x)$ with $\lambda \in (0,1)$, we have, as in Claim 2,

$$\|x\|_\infty \leq M = \phi^{-1}(|p|_1).$$

Let

$$U = \{x \in C(J,\mathbb{R}^+) :\ \|u\|_\infty < M+1\}.$$

From the choice of U, there is no solution $x \in \partial U$ such that $x = \lambda A(x)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type (Lemma 5.4), we deduce that A has a fixed point x in U, which is a solution to Problem (1.1). ■

6. APPLICATION TO CONTROL THEORY

Many boundary value problems of controllability may be described by nonlinear differential equations of the form

$$\text{(6.1)} \qquad \begin{cases} -(\phi(x'))'(t) = f(t, x(t), u(t)), & 0 < t < 1, \\ x(0) = x(1) = 0 \\ u \in U \end{cases}$$

with constrained control u. Here $f : J \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is a single-valued function measurable in t, continuous in x, u which is not identically zero. The time-varying set of constraints function $U : J \to \mathcal{P}_{cp}(\mathbb{R})$ is a measurable multi-valued function. By $u \in U$, we mean $u(t) \in U(t)$, for a.e. $t \in J$. Problem (6.1) is solved if there exists a control function u for which the problem admits a solution. If we define the multi-function

$$\text{(6.2)} \qquad F(t, x) = \{f(t, x, u),\ u \in U\}$$

then the control problem (6.1) coincides with the set of Carathéodory solution of (1.1) with right-hand side given by (6.2).

The controllability of ordinary differential equations and boundary value problems were investigated by many authors (see [5, 7, 9, 11, 35] and the references therein). It has many applications, mainly in optimal control and economy. Moreover, the first motivation of the study of the concept of differential inclusions comes from the development of some studies in control theory. For more information about the relation between differential inclusions and control theory, we refer the reader to [4, 23, 34, 45, 47] and the references therein.

Hereafter, we apply the existence results obtained in Sections 4 and 5 to study the ϕ-Laplacian boundary value problem, that is Problem (1.1):

$$\text{(6.3)} \qquad \begin{cases} -(\phi(x'))'(t) \in F(t, x(t)), & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases}$$

with F given by (6.2).

6.1. The convex control case

We will need the following auxiliary result in order to prove our main controllability theorem.

Lemma 6.1 [4]**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finitemeasurable space, X a complete separable metric space and $F : \Omega \to \mathcal{P}(X)$ a measurable set-valued map with closed images. Consider a Carathéodory set-valued map G from $\Omega \times X$ to a complete separable metric space Y. Then the map*

$$\Omega \ni \omega \mapsto \overline{G(\omega, F(\omega))} \in \mathcal{P}(Y)$$

is measurable.

While this result characterizes the measurability, the following lemma is a measurable selection result (Filippov's Theorem). It is crucial in the proof that the control system coincide with the differential inclusion problem.

Lemma 6.2 (see [4], Theorem 8.2.10)**.** *Consider a complete σ-finite measurable space (Ω, A, μ). Let X, Y be two complete separable metric spaces. Let $F : X \to \mathcal{P}(Y)$ be a measurable set-valued map with closed nonempty values and $g : \Omega \times Y \to Y$ a Carathéodory map. Then for every measurable map $h : \Omega \to Y$ satisfying*

$$h(\omega) \in g(\omega, F(\omega)) \quad \text{for a.e.} \quad \omega \in \Omega,$$

there exists a measurable selection $f(\omega) \in F(\omega)$ such that

$$h(\omega) = g(\omega, f(\omega)) \quad \text{for a.e.} \quad \omega \in \Omega.$$

Our first controllability existence result is

Theorem 6.3. *Assume that U and f satisfy the following hypotheses:*

($\mathcal{H}_3$) *$U : J \to \mathcal{P}_{cv,\,cp}(\mathbb{R})$ is a measurable multi-function and has a compact image.*

($\mathcal{H}_4$) *The function f is not identically zero and is linear in the third argument, i.e., there exist L^1-Carathéodory functions $f_i : J \times \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1, 2)$ such that for a.e. $t \in J$,*

$$f(t, x, y) = f_1(t, x)y + f_2(t, x), \ \forall\, (x, y) \in \mathbb{R}^2, \ y \geq 0.$$

($\mathcal{H}_5$) *There exist $k \in L^1(J, (0, +\infty))$ and a continuous nondecreasing function ψ such that*

$$0 \leq f(t, x, y) \leq k(t)\psi(|x|), \ \text{for a.e.} \ \ t \in J, \ \ \text{all} \ x \in \mathbb{R}^+, \ \ \text{all} \ y \in U$$

and

$$\exists\, R_0 > 0, \quad \psi(R_0) \leq \frac{\phi(R_0)}{|k|_1}\cdot$$

Then the control boundary value problem (6.1) *has at least one solution.*

Proof. **Claim 1.** Clearly, the map $t \mapsto F(t,\cdot)$ is a measurable multifunction. From Assumptions $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$, we have $F(\cdot,\cdot) \in \mathcal{P}_{cv}(\mathbb{R}^+)$. Using the compactness of U and the continuity of f, we can easily show that $F(\cdot,\cdot) \in \mathcal{P}_{cp}(\mathbb{R}^+)$. Therefore $F(\cdot,\cdot) \in \mathcal{P}_{cp,\,cv}(\mathbb{R}^+)$.

Claim 2. The selection set of F is not empty. Since U is a measurable multifunction and has a compact image, then $\overline{F(t,x)} = F(t,x)$. Let $x \in \mathbb{R}$. From Assumptions $(\mathcal{H}_3)$–$(\mathcal{H}_5)$, the map $(t,u) \mapsto f(t,x,u)$ is L^1-Carathéodory. Hence from Lemma 6.1, $F(\cdot,x)$ is measurable.

Claim 3. The map $x \mapsto F(\cdot,x)$ is an *u.s.c.* multifunction. Arguing by contradiction, assume that $F(t,\cdot)$ is not *u.s.c.* at some point x_0. Then there exists an open neighborhood W of $F(t,x_0)$ in $\mathbb{R}$ such that for every open neighborhood V at x_0 in $\mathbb{R}$ there exists $x_1 \in V$ such that $F(t,x_1) \not\subset U$. Let

$$V_n = \{x \in \mathbb{R},\ |x - x_0| < 1/n\},\ n = 1,2,\ldots$$

Then for each $n = 1,2,\ldots,$ there exist some points $x_n \in V_n$ and $y_n \in F(t,x_n)$. Hence, there exist $u_n \in U$ such that $y_n = f(t,x_n,u_n)$ and $y_n \notin W$, $\forall\, n \in \mathbb{N}$. Since $\{u_n,\ n \geq 1\} \subset U$, there exists a subsequence $(u_{n_m})_{m\geq 1}$ such that u_{n_m} converges to some limit u. By continuity of f and the convergence of x_{n_m} to x_0, the sequence y_{n_m} converges to y, where $y_{n_m} = f(t,x_{n_m},u_{n_m})$ and $y = f(t,x_0,u)$; this implies that $y \in F(t,x_0) \subset W$; but this contradicts the assumption that $y_n \notin W$ for each n.

Finally, from $(\mathcal{H}_4)$, we deduce that F is an L^1-Carathéodory multifunction and $(\mathcal{H}_5)$ corresponds to Assumption $(\mathcal{H}_1)$. Therefore, all conditions of Theorem 4.3 are fulfilled and then Problem (6.3) has at least one solution.

Claim 4. The solutions of the differential inclusion (6.3) and those of the control problem (6.1) defined on the time interval J do coincide. Let x be a solution of Problem (6.3). Then there exists a single-valued selection $g \in S_{F,x}$ such that

$$-(\phi(x'))'(t) = g(t),\ \text{ a.e. } t \in J, \text{ and } x(0) = x(1) = 0.$$

We shall show that there exists $u \in U$ such that

$$g(t) = f(t,x(t),u(t)),\ \text{ a.e. } \text{ in } J. \tag{6.4}$$

Define the function $\Psi(t,u) = f(t,x(t),u)$. Then Ψ is measurable in t and continuous on u. Moreover, for almost every $t \in J$, $g(t) \in \Psi(t,U(t)) := f(t,x(t),U(t)) := \{f(t,x(t),u(t)),\ u \in U\}$. From Lemma 6.2, we deduce the existence of some $u \in U$ satisfying (6.4). Conversely, let x be a function satisfying the control problem, i.e., for some $u \in U$ we have

$$-(\phi(x'))'(t) = f(t,x(t),u(t)), \quad x(0) = x(1) = 0.$$

Then x is solution of Problem (6.3), and the proof of the theorem is completed. ∎

6.2. The nonconvex control case

In this final sub-section, we derive a second existence result for Problem (6.1) with a nonconvex-valued right-hand side. First, some preliminaries are needed.

Let (X,d) be a metric space induced from the normed space $(X,\|.\|)$. Consider the mapping $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$, called Hausdorff distance, defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,\,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized complete metric space (see [33]).

Definition 6.1. A multi-valued operator $G : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x,y), \quad \text{for each } x,\ y \in X,$$

(b) a contraction if it is γ-Lipschitz with $0 < \gamma < 1$.

Also, the continuity with respect to the metric H_d is defined in a natural manner.

The following auxiliary lemma is concerned with measurability for two-variable multi-functions:

Lemma 6.4 [29]**.** *Let (Ω, A) be a measurable space, X, Y two separable metric spaces and let $F : \Omega \times X \to \mathcal{P}_{cl}(Y)$ be a multi-function such that*

(a) *for every* $x \in X$, $\omega \mapsto F(\omega, x)$ *is measurable,*

(b) *for a.e.* $\omega \in \Omega$, $x \mapsto F(\omega, x)$ *is either continuous or* H_d*-continuous.*

Then the mapping $(\omega, x) \mapsto F(\omega, x)$ *is measurable.*

Our contribution is the following:

Theorem 6.5. *Assume that* U *and* f *satisfy the following hypotheses:*

($\mathcal{H}_6$) $U : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is a measurable multi-function.*

($\mathcal{H}_7$) *There exists* $k \in L^1(J, (0, +\infty))$ *such that for a.e.* $t \in J$, *all* $x \in \mathbb{R}$ *and all* $u \in U$,

$$|f(t, x, u) - f(t, y, u)| \le k(t)|x - y|.$$

($\mathcal{H}_8$) *There exists* $p \in L^1(J, (0, +\infty))$ *such that*

$$|f(t, x, u)| \le p(t), \ \ \textit{for a.e. } \ t \in J, \ \ \textit{all } \ x \in \mathbb{R} \textit{ and all } \ u \in U.$$

Then the solution set of Problem (6.1) *is not empty.*

Proof. **Claim 1.** $F(t, \cdot)$ is a k-Lipschitz. Clearly, $F(\cdot, x)$ is a measurable multi-function for any fixed x and we have $F(\cdot, \cdot) \in \mathcal{P}_{cp}(\mathbb{R}^+)$. To prove that $F(t, \cdot)$ is a k-Lipschitz for a.e. $t \in J$, let $x, y \in \mathbb{R}$ and $h \in F(t, x)$. Then there exists $u \in U$ such that $h(t) = f(t, x, u)$. From Assumption ($\mathcal{H}_7$), we have the estimates

$$\begin{aligned}
d(h, F(t, y)) &= \inf_{z \in F(t,y)} |h - z| \\
&= \inf_{v \in U} |f(t, x, u) - f(t, y, v)| \\
&\le |f(t, x, u) - f(t, y, u)| \\
&\le k(t)|x - y|.
\end{aligned}$$

By an analogous relation obtained by interchanging the roles of x and y, we find that for each $l \in F(t, y)$,

$$d(F(t, x), l) \le k(t)|x - y|$$

and hence

$$H_d(F(t, x), F(t, y)) \le k(t)|x - y|, \ \text{ for each } \ x, y \in \mathbb{R}.$$

Claim 2. The map $x \mapsto F(t,x)$ is *u.s.c.* Since $F(t,\cdot)$ is a k-Lipschitz, it is H_d-continuous and from Lemma 6.4 the two-variable multi-function $(t,x) \mapsto F(t,x)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable. Now, let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$; then there exists a $\delta > 0$ such that for every $x \in \mathbb{R}$ with $|x - x_0| \leq \delta$, we have

$$F(t,x) \subset F(t,x_0) + (-\varepsilon,\varepsilon), \quad \text{for a.e. } \ t \in J.$$

Hence, $F(t,\cdot)$ is an $\varepsilon-\delta$ *l.s.c.* multi-valued function (see [19], Definition 1.2). Since $F(\cdot,\cdot) \in \mathcal{P}_{cp}(\mathbb{R})$, then $F(t,\cdot)$ is in fact *u.s.c.* (see [19], Proposition 1.1).

Finally, notice that F is integrably bounded by Assumption $(\mathcal{H}_8)$. Consequently, all the conditions of Theorem 5.1 are met and the solution set of Problem 6.3 is not empty. ∎

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