Discussiones Mathematicae
Differential Inclusions, Control and Optimization 29 (2009) 113–126

EXISTENCE OF SOLUTIONS OF THE DYNAMIC CAUCHY PROBLEM ON INFINITE TIME SCALE INTERVALS

Ireneusz Kubiaczyk

and

Aneta Sikorska-Nowak

Faculty of Mathematics and Computer Science
Adam Mickiewicz University, Poznań, Poland

e-mail: kuba@amu.edu.pl, anetas@amu.edu.pl

Abstract

In the paper, we prove the existence of solutions and Carathéodory's type solutions of the dynamic Cauchy problem

$$x^{\Delta}(t) = f(t, x(t)), \quad t \in T,$$
$$x(0) = x_0,$$

where T denotes an unbounded time scale (a nonempty closed subset of R and such that there exists a sequence (x_n) in T and $x_n \to \infty$) and f is continuous or satisfies Carathéodory's conditions and some conditions expressed in terms of measures of noncompactness. The Sadovskii fixed point theorem and Ambrosetti's lemma are used to prove the main result. The results presented in the paper are new not only for Banach valued functions, but also for real-valued functions.

Keywords: Cauchy dynamic problem, Banach space, measure of noncompactness, Carathéodory's type solutions, time scales, fixed point.

2000 Mathematics Subject Classification: 34G20, 34A40, 39A13.

1. Introduction

A time scale (or measure chain) was introduced by Hilger in his Ph. D. thesis in 1988 in order to unify discrete and continuous analysis [21]. Since the time Hilger formed the definitions of a derivative and integral on a time scale, several authors have extended them on various aspects of the theory [1, 7, 9, 10, 12, 13]. The time scale has been shown to be applicable to any field that can be described by means of discrete or continuous models. In recent years there have been many research activities on dynamic equations in order to unify the results concerning difference equations and differential equations [1–3, 6, 12, 13, 18].

The Cauchy differential equation $x'(t) = f(t, x(t))$ and the Cauchy difference equation $\Delta x(t) = f(t, x(t))$ have been widely studied by many authors [4, 5, 14–17, 20, 26]. However, the dynamic equations in Banach spaces constitute quite a new research area and Carathéodory's type solutions are new even in the real case.

Time scale boundary value problems on a finite interval have received a lot of attention in the literature. This paper discusses time scale boundary value problems on an infinite time scale interval.

In the paper, we prove the existence of solution and the existence of Carathéodory's type solution of the dynamic Cauchy problem

$$(1.1) \qquad \begin{aligned} &x^{\Delta}(t) = f(t, x(t)), \quad t \in T, \\ &x(0) = x_0, \end{aligned}$$

where T denotes an unbounded time scale (nonempty closed subset of R such that there exists a sequence (x_n) in T and $x_n \to \infty$). The function f, with values in a Banach space, satisfies some regularity conditions expressed in terms of the Kuratowski measure of noncompactness. Our results will be proved using the fixed point theorem of Sadovskii (see [25], Theorem 3.4.3.)

We were motivated by interesting papers found in the literature [4, 16, 20]. Their authors present results which guarantee the existence of one or more solutions for particular cases of (1.1). The result of the paper extends the above results.

The notion of a time scale allows us to treat in a unified manner differential equations, integral equations and difference equations. For example, if $T = \mathrm{N}$ we have an existence theorem for the corresponding difference equations.

2. Preliminaries

To understand the so-called dynamic equation and follow this paper easily, we present some preliminary definitions and notations of time scale which are very common in the literature (see [1, 12, 13, 21–23] and references therein).

A time scale T is a nonempty closed subset of real numbers R, $(0 \in T)$, with the subspace topology inherited from the standard topology of R. Thus R;Z;N and the Cantor set are examples of time scales while Q and (0;1) are not time scales.

If a, b are points in T, then we denote $[a, b] = \{t \in T : a \leq t \leq b\}$ and $I_a = \{t \in T : 0 \leq t \leq a\}$. Other types of intervals are approached similarly. By a subinterval I_b of I_a we mean the time scale subinterval.

Definition 2.1. The *forward jump operator* $\sigma : T \to T$ and the *backward jump operator* $\rho : T \to T$ are defined by $\sigma(t) = \inf\{s \in T : s > t\}$ and $\rho(t) = \sup\{s \in T : s < t\}$, respectively. We put $\inf \emptyset = \sup T$ (i.e., $\sigma(M) = M$ if T has a maximum M) and $\sup \emptyset = \inf T$ (i.e., $\rho(m) = m$ if T has a minimum m).

The jump operators σ and ρ allow the classification of points in the time scale in the following way: t is called *right dense*, *right scattered*, *left dense*, *left scattered*, *dense* and *isolated* if $\sigma(t) = t$, $\quad \sigma(t) > t, \rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively.

Definition 2.2. We say that $k : T \to E$ is *right – dense continuous* (*rd – continuous*) if k is continuous at every right – dense point $t \in T$ and $\lim_{s \to t^-} k(s)$ exists and is finite at every left – dense point $t \in T$.

Next, we define the so – called Δ-*derivative* and Δ-*integral*.

Definition 2.3. Fix $t \in T$. Let $f : J \to E$. Then we define $f^{\Delta}(t)$ by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

Let us mention that Δ-derivative satisfies

(i) $f^{\Delta} = f'$ is the usual derivative if $T = \mathrm{R}$ and
(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $T = \mathrm{Z}$.

Hence, the time scale allows us to unify the treatment of differential and difference equations (and not only these ones).

Definition 2.4. If $F^{\Delta}(t) = f(t)$ then we define the Δ-*integral* by

$$\int_a^t f(\tau)\Delta\tau = F(t) - F(a), \quad a \in T.$$

The above notion, specific for time scales, is important in view of the existence of antiderivatives.

Remark 2.5 [11] (Existence of antiderivatives)**.** Every rd-continuous function has an antiderivative.

The Kuratowski measure of noncompactness (see [11]) is the fundamental tool employed in the paper.

For any bounded subset A of E we denote by $\alpha(A)$ *the Kuratowski measure of noncompactness* of A, i.e., the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameters smaller than ε.

The properties of the measure of noncompactness α are:

(i) if $A \subset B$, then $\alpha(A) \le \alpha(B)$;
(ii) $\alpha(A) = \alpha(\bar{A})$, where $\bar{A}$ denotes the closure of A;
(iii) $\alpha(A) = 0$ if and only if A is relatively compact;
(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;
(v) $\alpha(\lambda A) = |\lambda|\alpha(A)$ $(\lambda \in R)$;
(vi) $\alpha(A + B) \le \alpha(A) + \alpha(B)$;
(vii) $\alpha(convA) = \alpha(A)$, where $conv(A)$ denotes the convex extension of A;
(viii) $\alpha(A) < \delta(A)$, where $\delta(A) = \sup_{x,y\in A}\{\|x - y\|\}$.

Let $C(T, E)$ denote the set of all continuous functions from T to E endowed with the topology of almost uniform convergence (i.e., uniform convergence on each closed bounded subset of T).

We will need the following lemmas.

Lemma 2.6 [24]**.** *Let E_1, E_2 be bounded subsets of the Banach space E. If $\|E_1\| = \sup\{\|x\| : x \in E_1\} < 1$, then*

$$\alpha(E_1 + E_2) \le \alpha(E_2) + \|E_1\|\,\alpha(K(E_2, 1)),$$

where $K(E_2, 1) = \{x : D(E_2, x) < 1\}$ and $D(E_2, x) = \inf\{\|x - y\| : y \in E_2\}$.

The lemma below is an adaptation of the corresponding result of Ambrosetti ([8]).

Lemma 2.7. *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let $H(t) = \{h(t) \in E, h \in H\}$, for $t \in I_a$ and $H(I_a) = \bigcup_{t \in I_a} H(t)$. Then*

$$\alpha_C(H) = \sup_t \in I_a \alpha(H(t)) = \alpha(H(I_a)),$$

where $\alpha_C(H)$ denotes the measure of noncompactness in $C(I_a, E)$ and the function $t \mapsto \alpha(H(t))$ is continuous.

Proof. I. First, we prove the equality: $\sup_{t \in I_a} \alpha(H(t)) = \alpha(H(I_a))$. Since $H(t) \subset H(I_a)$ by the first property of measure of noncompactness, $\alpha(H(t)) \leq \alpha(H(I_a))$ and consequently

$$\sup_{t \in I_a} \alpha(H(t)) \leq \alpha(H(I_a)). \tag{2.1}$$

By the strong equicontinuity of H, we deduce that for $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - s| < \delta \Rightarrow \|u(t) - u(s)\| < \varepsilon$ for $t, s \in I_a$ and for all $u \in H$.

We divide the interval $I_a = \{t \in T : 0 \leq t \leq a\}$ in the following way: $t_0 = 0$,

$$t_1 = \sup_{s \in I_a} \{s : s \geq t_0, s - t_0 < \delta\}, \quad t_2 = \sup_{s \in I_a} \{s : s > t_1, s - t_1 < \delta\}, \ \ldots \ ,$$

$$t_n = \sup_{s \in I_a} \{s : s > t_{n-1}, s - t_{n-1} < \delta\}.$$

Since T is closed, so $t_i \in I_a$. If some $t_{i+1} = t_i$, then
$t_{i+2} = \inf\{t \in I_a : t > t_{i+1}\}$. As

$$u(t) = u(t_i) + u(t) - u(t_i) \in u(t_i) + \varepsilon K(0,1),$$

where $K(0,1) = \{x : \|x\| < 1\}$, we have

$$u(t) \in \bigcup_{i=1}^{n} H(t_i) + \varepsilon K(0,1) \quad \text{and} \quad H(I_a) \subset \bigcup_{i=1}^{n} H(t_i) + \varepsilon K(0,1).$$

By the properties of the measure of noncompactness and Lemma 2.6, we obtain

$$\begin{aligned}\alpha(H(I_a)) &\le \alpha\left(\bigcup_{i=1}^{n} H(t_i)\right) + \|\varepsilon K(0,1)\| \cdot \alpha\left(K\left(\bigcup_{i=1}^{n} H(t_i), 1\right)\right) \\ &< \sup_{t_i \in I_a} \alpha\left(H(t_i)\right) + \varepsilon\alpha\left(K(H(I_a),1)\right) \\ &\le \sup_{t \in I_a} \alpha(H(t)) + \varepsilon\alpha(K(H(I_a),1)).\end{aligned}$$

Since the above inequality holds for any $\varepsilon > 0$, we have

$$\alpha(H(I_a)) \le \sup_{t\in I_a} \alpha(H(t)). \tag{2.2}$$

Hence, from (2.1) and (2.2), we conclude that $\alpha(H(I_a)) = \sup_{t\in I_a} \alpha(H(t))$.

II. The proof of the equality $\alpha_C(H) = \sup_{t\in I_a} \alpha(H(t))$ is similar to the proof of Lemma 2.1 of Ambrosetti (see [8]), where we choose points t_i as in part I of our proof.

III. Now we prove that the function $t \mapsto \alpha(H(t))$ is continuous. Let $v(t) = \alpha(H(t))$. Because $H(t) \subset H(t)\dot{-}H(s)\dot{+}H(s) \subset H(t)\dot{-}H(s) + H(s)$, where

$$H(t)\dot{-}H(s)\dot{+}H(s) = \{y(t) : y(t) = y(t) - y(s) + y(s) : y \in H\}.$$

By the property (vi) of the measure of noncompactness, we have

$$\alpha(H(t)) \le \alpha(H(t)\dot{-}H(s)) + \alpha(H(s)).$$

This implies, by (viii), that

$$\begin{aligned}|\alpha(H(t)) - \alpha(H(s))| &\le \alpha\left(H(t)\dot{-}H(s)\right) \le \delta\left(H(t)\dot{-}H(s)\right) \\ &= \sup_{x,y\in H} \left\{\|(x(t)-x(s)) - (y(t)-y(s))\|\right\} \\ &\le \sup_{x,y\in H} \left\{\|x(t)-x(s)\| + \|y(t)-y(s)\|\right\}.\end{aligned}$$

By equicontinuity of H, we obtain the continuity of $v(t)$.

Let us denote by S_∞ the set of all nonnegative real sequences. For $\xi = (\xi_n) \in S_\infty$, $\eta = (\eta_n) \in S_\infty$, we write $\xi < \eta$ if $\xi_n \le \eta_n$ (i.e., $\xi_n \le \eta_n$, for $n = 1, 2, \ldots$) and $\xi \ne \eta$.

Let X be a closed convex subset of $C(T, E)$ and let ϕ be a function which assigns to each nonempty subset Z of X a sequence $\phi(Z) \in S_\infty$ such that

$$\phi(\{z\} \cup Z) = \phi(Z), \text{ for } z \in X, \tag{2.3}$$

$$\phi(\overline{conv} Z) = \phi(Z), \tag{2.4}$$

$$\text{if } \phi(Z) = \emptyset \text{ (the zero sequence), then } \overline{Z} \text{ is compact.} \tag{2.5}$$

In the proof of the main theorem, we will apply the following results.

Theorem 2.8 [25]**.** *If $F : X \to X$ is a continuous mapping satisfying $\phi(F(Z)) < \phi(Z)$ for arbitrary nonempty subset Z of X with $\phi(Z) > 0$, then F has a fixed point in X.*

Theorem 2.9 (Mean Value Theorem)**.** *If the function $f : I_a \to E$ is Δ-integrable, then*

$$\int_{I_b} f(t)\Delta t \in \mu_\Delta(I_b) \cdot \overline{conv} f(I_b),$$

where I_b is an arbitrary subinterval of I_a and $\mu_\Delta(I_b)$ is the Lebesgue Δ-measure of I_b.

See [13, 19] for the definition and basic properties of the Lebesgue Δ-measure and the Lebesgue Δ-integral.

3. Existence of solutions

In this section we assume that $f : T \times E \to E$ is a continuous function.

By *a solution of* (1.1) we understand a function $x \in C(T, E)$ such that $x(0) = x_0$, and $x(\cdot)$ satisfies (1.1) for all $t \in T$.

For such solutions, problem (1.1) is equivalent to the integral problem

$$x(t) = x_0 + \int_0^t f(s, x(s))\Delta s, \ t \in T. \tag{3.1}$$

Theorem 3.1. *Let $G : T \times [0, \infty) \to [0, \infty)$ be a continuous function nondecreasing in the second variable and such that for every continuous, locally bounded function $u : [0, \infty) \to [0, \infty)$, $G(x, u(x))$ is continuous. Moreover, let $L : T \times [0, \infty) \to [0, \infty)$ be a function such that for each continuous function $u : [0, \infty) \to [0, \infty)$ the mapping $x \mapsto L(x, u)$ is continuous and $L(x, 0) \equiv 0$ on T. If the following conditions*

(c_1) *$f : T \times E \to E$ is continuous,*

(c_2) *$\|f(x, p)\| \le G(x, \|p\|)$, for $x \in T$ and $p \in E$,*

(c_3) *$\alpha(f(I \times W)) \le \sup\{L(x, \alpha(W)) : x \in I\}$, for any compact subinterval I of T and each nonempty bounded subset W of E,*

(c_4) *the integral inequality $g(x) \ge \int_0^x G(u, g(u))\Delta u$ has a continuous, locally bounded solution g_0 existing on T,*

(c_5) *$\int_0^\infty L(x, r)\Delta x < r$, for all $r > 0$*

hold, then there exists a solution z of (1.1) such that $\|z(t) - x_0\| \le g_0(t)$ on T.

Proof. Denote by X the set of all $z \in C(J, E)$ with $\|z(t) - x_0\| \le g_0(t)$ on T and

$$\|z(x_1) - z(x_2)\| \le \left| \int_{x_1}^{x_2} G(u, g_0(u))\Delta u \right|, \quad \text{for} \quad x_1, x_2 \in T.$$

The set X is a nonempty, closed, convex and almost equicontinuous subset of $C(T, E)$. Moreover, as g_0 is locally bounded , the set X is bounded on each closed, bounded subsets of T.

We define a mapping F of X into itself as follows

$$F(z)(x) = x_0 + \int_0^x f(u, z(u))\Delta u, \quad \text{for} \quad x \in T.$$

Since f is continuous, then F is continuous on X with the topology of almost uniform convergence (i.e., uniform convergence on each closed bounded subsets of T).

Let n be a positive integer and $I_n = [0, a_n] \cap T$, where T denotes a time scale and $a_n \in T$, where $a_n \to \infty$ if $n \to \infty$. Let Z be a nonempty subset of X and $W_n = Z(I_n) = \bigcup\{Z(x) : x \in I_n\}$. Note that since X is bounded, then W_n is bounded.

For any given $\varepsilon > 0$ there exists $\delta > 0$ such that $u', u" \in I_n$ with $|u' - u"| < \delta$ imply $|L(u', \alpha(W_n)) - L(u", \alpha(W_n))| < \varepsilon$.

We divide the interval I_n into m parts $0 = x_0^n < x_1^n < \ldots < x_m^n = a_n$ in such a way that:

$$x_0^n = 0, x_1^n = \sup_{s \in I_n} \{s : s \ge x_0^n, s - x_0^n < \delta\},$$

$$x_2^n = \sup_{s \in I_n} \{s : s > x_1^n, s - x_1^n < \delta\}, \ldots,$$

$$x_m^n = \sup_{s \in I_n} \{s : s > x_{m-1}^n, s - x_{m-1}^n < \delta\}.$$

Since T is a closed subset of R, then $x_i^n \in I_n$. If some $x_{i+1}^n = x_i^n$, then $x_{i+2}^n = \inf\{x \in T : x > x_{i+1}^n\}$.

Let for $x \in I_k^n = [x_{k-1}^n, x_k^n] \cap T$, $k = 1, 2, \ldots, m$, σ_j^n, $j = 1, 2, \ldots, k-1$ be choosen in I_j^n in such a way that

$$L(\sigma_j^n, \alpha(W_j^n)) = \max\{L(x, \alpha(W_j^n) : x \in I_j^n, j = 1, 2, \ldots, k-1\}$$

and let σ_k^n be choosen in $[x_{k-1}^n, x] \cap T$ with

$$L(\sigma_k^n, \alpha(W_k^n)) = \max\{L(x, \alpha(W_k^n) : x \in [x_{k-1}^n, x] \cap T\},$$

where $W_j^n = Z(I_j^n)$, $W_k^n = Z([x_{k-1}^n, x] \cap T)$, $j = 1, 2, \ldots, k-1$, $k = 1, 2, \ldots, m$.

Using the Mean Value Theorem (Theorem 2.9), the assumption (c_3) and Lemma 2.7, we obtain

$$\begin{aligned}
&\alpha(F(Z)(x)) = \\
&= \alpha\left(\left\{\int_0^x f(u, z(u))\Delta u : z \in Z\right\}\right) = \alpha\left(\int_0^x f(u, Z(u))\Delta u\right) \\
&= \alpha\left(\sum_{j=0}^{k-1} \int_{I_j^n} f(u, Z(u))\Delta u + \int_{x_{k-1}^n}^x f(u, Z(u))\Delta u\right) \\
&\le \alpha\left(\sum_{j=0}^{k-1} \mu_\Delta(I_j^n)\overline{conv}(f(I_j^n \times W_j^n))\right. \\
&\qquad \left. + \mu_\Delta([x_{k-1}^n, x] \cap T)\overline{conv}(f([x_{k-1}^n, x] \cap T \times W_k^n))\right) \\
&\le \sum_{j=0}^{k-1} \mu_\Delta(I_j^n) L(\sigma_j^n, \alpha(W_j^n)) + \mu_\Delta([x_{k-1}^n, x] \cap T) L(\sigma_k^n, \alpha(W_k^n)) \\
&\le \sum_{j=0}^{k-1} \int_{I_j^n} L(u, \alpha(W_j^n))\Delta u + \sum_{j=0}^{k-1} \int_{I_j^n} \left| L(\sigma_j^n, \alpha(W_j^n) - L(u, \alpha(W_j^n))\right| \Delta u
\end{aligned}$$

$$+ \int_{x_{k-1}^n}^{x} L(u, \alpha(W_k^n))\Delta u + \int_{x_{k-1}^n}^{x} \left| L(\sigma_k^n, \alpha(W_k^n)) - L(u, \alpha(W_k^n)) \right| \Delta u$$
$$< \int_0^x L(u, \alpha(W_n))\Delta u + \varepsilon x = \varepsilon x + \int_0^x L(u, \sup\{\alpha(Z(x)) : x \in I_n\})\Delta u.$$

As $\varepsilon > 0$ is arbitrary, this implies that

$$\begin{aligned} &\sup\{\alpha(F(Z)(x) : x \in I_n\} \leq \\ &\leq \int_0^x L(u, \sup\{\alpha(Z(x)) : x \in I_n\})\Delta u \\ &\leq \int_0^\infty L(u, \sup\{\alpha(Z(x)) : x \in I_n\})\Delta u \\ &< \sup\left\{\alpha(Z(x)) : x \in I_n\right\}, \text{ for } \alpha(Z(x)) > 0. \end{aligned} \tag{3.2}$$

If $\alpha(Z(x)) = 0$ then, we have $\alpha(F(Z)(x)) = 0$ because $L(x, 0) = 0$.

Define $\phi(Z) = \left(\sup_{x \in I_1} \alpha(Z(x)), \sup_{x \in I_2} \alpha(Z(x)), \ldots\right)$ for any nonempty subset Z of X.

Evidently, $\phi(Z) \in S_\infty$. By the properties of α, the function ϕ satisfies conditions (2.3)–(2.4) listed above. From (3.2), our assumption on L and inequality (c_5), it follows that $\phi(F(Z)) < \phi(Z)$ whenever $\phi(Z) > 0$. If $\phi(Z) = 0$, then for each $x \in T$, $\alpha(Z(x)) = 0$. By Arzela-Ascoli theorem the set Z is compact. This means that the condition (2.5) is satisfied. Thus, all assumptions of Sadovskii's fixed point theorem (see [25]) have been satisfied, F has a fixed point in X and the proof is complete.

Remark 3.2. In particular, the function $L(u, r) = l(u)\varphi(r)$, where $\int_0^\infty l(u)\Delta u \leq 1$ and $0 < \varphi(r) < r$, $r > 0$, satisfies conditions from the Theorem 3.1.

4. Existence of Carathéodory's type solutions

In this section we assume, that $f : T \times E \to E$ is a Carathéodory function.

Investigating the existence of solutions of (1.1), we can consider the so-called Carathéodory's type solutions. We recall that a function $f : T \times E \to E$ is a *Carathéodory function* if for each $x \in E$, $f(t, x)$ is μ_Δ measurable in $t \in T$ and for almost all $t \in T$, $f(t, x)$ is continuous with respect to x.

By a *Carathéodory's type solution* of (1.1) we understand a function $x \in C(T, E)$ such that $x(0) = x_0$, and $x(\cdot)$ satisfies (1.1) μ_Δ a.e. in T. For such solutions problem (1.1) is equivalent to the integral problem

$$x(t) = x_0 + \int_0^t f(s, x(s))\Delta s, \quad \mu_\Delta \text{ a.e. on } T, \tag{4.1}$$

where integral is taken in the sense of Δ-Lebesgue.

See [13, 19] for definitions and basic properties of the Lebesgue Δ-measure and the Lebesgue Δ-integral.

To verify the equivalence, let a continuous function $x : T \to E$ be a solution of the problem (1.1). Since $\int_A f(s, x(s))\Delta s = 0$ (see [19]), where $A = \{t \in T : x^\Delta \neq f(t, x(t))\}$ $(\mu_\Delta(A) = 0)$, by the properties of the Δ-Lebesgue integral, we have

$$\begin{aligned}
\int_0^t f(s, x(s))\Delta s &= \int_A f(s, x(s))\Delta s + \int_{I_t - A} x^\Delta(s)\Delta s \\
&= \int_0^t x^\Delta(s)\Delta s = x(t) - x(0) = x(t) - x_0, \quad \mu_\Delta \text{ a.e. on } T,
\end{aligned}$$

which means that the function x is the Carathéodory type solution of the problem (4.1).

Now, let the function x be a solution of the problem (4.1). Then, by the properties of the Δ-Lebesgue integral and the Δ-derivative, we obtain that if F is a function such that $F^\Delta(t, x(t)) = f(t, x(t)), \mu_\Delta$ a.e. then

$$\begin{aligned}
x^\Delta(t) &= \left(x_0 + \int_0^t f(s, x(s))\Delta s\right)^\Delta = \left(\int_0^t f(s, x(s))\Delta s\right)^\Delta \\
&= (F(t, x(t)) - F(0, x(0)))^\Delta = F^\Delta(t, x(t))
\end{aligned}$$

μ_Δ a.e.

Concluding, the function x is the Carathéodory's type solution of the problem (1.1).

Similarly as in Theorem 3.1, we can prove the following theorem.

Theorem 4.1. *Let $G : T \times [0, \infty) \to [0, \infty)$ be a continuous function non-decreasing in the second variable and such that for every continuous locally bounded function $u : [0, \infty) \to [0, \infty), G(x, u(x))$ is continuous. Moreover, let $L : T \times [0, \infty) \to [0, \infty)$ be a function such that for each continuous* *function* $u : [0,\infty) \to [0,\infty)$ *the mapping* $x \mapsto L(x,u)$ *is continuous and* $L(x,0) \equiv 0$ *on* T. *If the following conditions*

(A_1) $f : T \times E \to E$ *is a Carathéodory function,*

(A_2) $\|f(x,p)\| \le G(x,\|p\|)$, *for* $x \in T$ *and* $p \in E$,

(A_3) $\alpha(f(I \times W)) \le \sup\{L(x,\alpha(W)) : x \in I\}$, *for any compact subinterval* I *of* T *and each nonempty bounded subset* W *of* E,

(A_4) *the integral inequality* $g(x) \ge \int_0^x G(u,g(u))\Delta u$ *has a continuous, locally bounded solution* g_0 *existing on* T,

(A_5) $\int_0^\infty L(x,r)\Delta x < r$, *for all* $r > 0$

hold, then there exists a Carathéodory's type solution z *of* (1.1) *such that* $\|z(t) - x_0\| \le g_0(t)$ *on* T.

Proof. The proof is the same as in the case of continuous function f (Theorem 3.1).

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Received 3 August 2009