Discussiones Mathematicae
Differential Inclusions, Control and Optimization 29 (2009) 107–111

ON THE EXISTENCE OF SOLUTIONS OF AN INTEGRO-DIFFERENTIAL EQUATION IN BANACH SPACES

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Dedicated to Professor Michał Kisielewicz on the occasion of his 70th birthday

Consider the Cauchy problem

$$x^{(m)}(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \tag{1}$$

$$x(0) = 0, x'(0) = \eta_1, \ldots, x^{(m-1)}(0) = \eta_{m-1} \tag{2}$$

in a Banach space E, where $m \geq 1$ is a natural number. We assume that $D = [0, a]$, $B = \{x \in E : \| x \| \leq b\}$ and $f : D \times B \to E$, $g : D^2 \times B \to E$ are bounded continuous functions. Let

$$m_1 = \sup\{\|f(t, x)\| : \ t \in D, \ x \in B\}$$

$$m_2 = \sup\{\|g(t, s, x)\| : \ t, s \in D, \ x \in B\}.$$

We choose a positive number d such that $d \leq a$ and

$$\sum_{j=1}^{m-1} \| \eta_j \| \frac{d^j}{j!} + m_1 \frac{d^m}{m!} + m_2 \frac{d^{m+1}}{m!} \leq b. \tag{3}$$

Let $J = [0, d]$. Denote by $C = C(J, E)$ the Banach space of continuous functions $z\colon J \to E$ with the usual norm $\|z\|_C = \max_{t \in J} \|z(t)\|$.

Let $\widetilde{B} = \{x \in C : \|x\|_C \leq b\}$. For $t \in J$ and $x \in \widetilde{B}$ put

$$\widetilde{g}(t, x) = \int_0^t g(t, s, x(s))ds.$$

Fix $\tau \in J$ and $x \in \widetilde{B}$. As the set $J \times x(J)$ is compact, from the continuity of g it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|g(t,s,x(s)) - g(\tau,s,x(s))\| < \varepsilon \quad \text{for} \quad t,s \in J \quad \text{with} \quad |t-\tau| < \delta.$$

In view of the inequality

$$\|\widetilde{g}(t,x) - \widetilde{g}(\tau,x)\| \leq m_2|t-\tau| + \int_0^\tau \|g(t,s,x(s)) - g(\tau,s,x(s))\|\,ds,$$

this implies the continuity of the function $t \to \widetilde{g}(t,x)$. On the other hand, the Lebesgue dominated convergence theorem proves that for each fixed $t \in J$ the function $x \to \widetilde{g}(t,x)$ is continuous on $\widetilde{B}$. Moreover,

$$\|\widetilde{g}(t,x)\| \leq m_2 t \quad \text{for} \quad t \in J \quad \text{and} \quad x \in \widetilde{B}.$$

Let α be the Kuratowski measure of noncompactness in E (cf. [1]).

The main result of the paper is the following

Theorem. *Let $w : I\!R_+ \mapsto I\!R_+$ be a continuous nondecreasing function such that $w(0) = 0$, $w(r) > 0$ for $r > 0$ and*

$$\int\limits_{0+} \frac{dr}{\sqrt[m]{r^{m-1}w(r)}} = \infty.$$

If

$$\alpha\left(f(t,X)\right) \leq w\left(\alpha(X)\right) \quad \text{for} \quad t \in J \quad \text{and} \quad X \subset B, \tag{4}$$

and the set $g(D^2 \times B)$ is relatively compact in E, then there exists at least one solution of (1)–(2) *defined on J.*

Proof. The problem (1)–(2) is equivalent to the integral equation

$$x(t) = p(t) + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}[f(s,x(s)) + \widetilde{g}(s,x)]ds \quad (t \in J),$$

where $p(t) = \sum_{j=1}^{m-1} \eta_j \frac{t^j}{j!}$. We define the mapping F by

$$F(x)(t) = p(t) + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}[f(s,x(s)) + \widetilde{g}(s,x)]ds \ (t \in J, x \in \widetilde{B}).$$

Owing to (3), it is known (cf. [5]) that F is a continuous mapping $\widetilde{B} \mapsto \widetilde{B}$ and the set $F(\widetilde{B})$ is equicontinuous. By the Mazur lemma the set $W = \bigcup_{0\le\lambda\le d} \lambda\overline{\mathrm{conv}}g(D^2 \times B)$ is relatively compact. Since $\{(t-s)^{m-1}\widetilde{g}(s,x) : x \in \widetilde{B}\} \subset (t-s)^{m-1}W$, we have $\alpha(\{(t-s)^{m-1}\widetilde{g}(s,x) : x \in \widetilde{B}\}) \le (t-s)^{m-1}\alpha(W) = 0$. Therefore, by the Heinz lemma [2]

$$\begin{aligned}(5)\qquad &\alpha\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}\widetilde{g}(s,x)ds : x \in \widetilde{B}\right\}\right)\\ &\le \frac{2}{(m-1)!}\int_0^t \alpha\left(\left\{(t-s)^{m-1}\widetilde{g}(s,x) : x \in \widetilde{B}\right\}\right) ds = 0.\end{aligned}$$

For any positive integer n put

$$v_n(t) = \begin{cases} p(t) & \text{if } 0 \le t \le \frac{d}{n} \\ p(t) + \frac{1}{(m-1)!}\displaystyle\int_0^{t-\frac{d}{n}} (t-s)^{m-1}[f(s,v_n(s)) + \widetilde{g}(s,v_n)]ds & \text{if } \frac{d}{n} \le t \le d. \end{cases}$$

Then, by (3), $v_n \in \widetilde{B}$ and

$$(6)\qquad \lim_{n\to\infty} \| v_n - F(v_n) \|_C = 0.$$

Put $V = \{v_n : n \in N\}$ and $Z(t) = \{x(t) : x \in Z\}$ for $t \in J$ and $Z \subset C$. As $V \subset \{v_n - F(v_n) : n \in N\} + F(V)$ and $V \subset \widetilde{B}$, from (6) it follows that the set V is equicontinuous and the function $t \mapsto v(t) = \alpha(V(t))$ is continuous on J. Applying now the Heinz lemma and (5), we get

$$\begin{aligned}&\alpha(F(V)(t)) =\\ &= \alpha\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}[f(s,v_n(s)) + \widetilde{g}(s,v_n)]ds : n \in N\right\}\right)\\ &\le \alpha\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}f(s,v_n(s))ds : n \in N\right\}\right)\\ &+ \alpha\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}\widetilde{g}(s,x)ds : x \in \widetilde{B}\right\}\right)\end{aligned}$$

$$
\begin{aligned}
&= \alpha\left(\left\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} f(s, v_n(s))ds : n \in N\right\}\right)\\
&\le \frac{2}{(m-1)!}\int_0^t \alpha\left(\left\{(t-s)^{m-1} f(s, v_n(s)) : n \in N\right\}\right) ds\\
&\le \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} \alpha(f(s, V(s))ds\\
&\le \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w(\alpha(V(s)))ds.
\end{aligned}
$$

On the other hand, from (6) and the inclusion

$$V(t) \subset \{v_n(t) - F(v_n)(t) : n \in N\} + F(V)(t)$$

it follows that $v(t) \le \alpha\left(F(V)(t)\right)$. Hence

$$v(t) \le \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w(v(s))ds \quad \text{for } \ t \in J.$$

Putting $h(t) = \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w(v(s))ds$, we see that $h \in C^m$, $v(t) \le h(t)$, $h^{(j)}(t) \ge 0$ for $j = 0, 1, \dots, m$, $h^{(j)}(0) = 0$ for $j = 0, 1, \dots, m-1$ and $h^{(m)}(t) = 2w(v(t)) \le 2w(h(t))$ for $t \in J$. By Theorem 1 of [6], from this we deduce that $h(t) = 0$ for $t \in J$. Thus $\alpha(V(t)) = 0$ for $t \in J$. Therefore for each $t \in J$ the set $V(t)$ is relatively compact in E, and by Ascoli's theorem the set V is relatively compact in C. Hence we can find a subsequence (v_{n_k}) of (v_n) which converges in C to a limit u. As F is continuous, from (6) we conclude that $u = F(u)$, so that u is a solution of (1)–(2).

Remark. It is known (cf. [7], Theorem 4) that under the assumptions of the Theorem the set of all solutions of (1)–(2) defined on J is a compact R_δ set in $C(J, E)$.

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Received 9 June 2009