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WEAK SOLUTIONS OF STOCHASTIC DIFFERENTIAL INCLUSIONS AND THEIR COMPACTNESS

Mariusz Michta

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Prof. Z. Szafrana 4a, 65–516 Zielona Góra, Poland

Dedicated to Prof. M. Kisielewicz on the occassion of his 70th birthday.

Abstract

In this paper, we consider weak solutions to stochastic inclusions driven by a semimartingale and a martingale problem formulated for such inclusions. Using this we analyze compactness of the set of solutions. The paper extends some earlier results known for stochastic differential inclusions driven by a diffusion process.

Keywords: semimartingale, stochastic differential inclusions, weak solutions, martingale problem, weak convergence of probability measures.

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1. Introduction

The major contributions in the field of stochastic inclusions have been connected with stochastic control problems (see e.g., [1, 2, 3, 10, 11, 12, 9, 20, 21] and references therein) and with the existence and properties of their strong solutions. In [13, 14, 15, 16] and [18] the existence and compactness property of *weak solutions* to Brownian motion driven stochastic differential inclusions were studied. In this work we present a *martingale problem* approach as a useful tool in the study of weak solutions of an inclusion driven by a continuous semimartingale, in which the multivalued integrand also depends on the driving process. We also consider the case of a stochastic inclusion driven by Levy's process. It extends the cases studied earlier in [13, 16, 18] and [17]. We recall at first main definitions and known facts needed in the paper. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $\{\mathcal{F}_t\}_{t\in[0,T]}$ is an increasing and right continuous family of sub σ-fields of $\mathcal{F}$. By $Comp()$ we denote the space of nonempty and compact subsets of the underlying space, equipped with the Hausdorff distance δ. Let $G = (G(t))_{t\in[0,T]}$ be a set-valued stochastic process with values in $Comp(\mathbb{R}^d \otimes \mathbb{R}^m)$, i.e., a family of $\mathcal{F}$-measurable set-valued mappings $G(t) : \Omega \to Comp(\mathbb{R}^m \otimes \mathbb{R}^d)$, each $t \in [0,T]$. For the notions of measurability, continuity, lower and upper continuity (l.s.c. and u.s.c) of set-valued mappings we refer to [6]. Similarly, G is $\mathcal{F}_t$-adapted, if $G(t)$ is $\mathcal{F}_t$-measurable for each $t \in [0,T]$. We call G predictable, if it is measurable with respect to predictable σ-field $\mathcal{P}(\mathcal{F}_t)$ in $[0,T] \times \Omega$. For a stochastic process R we introduce the following notation: $R^*_t = \sup_{s\le t}|R_s|$ and $R^* = \sup_{s\le T}|R_s|$. For a stopping time η, by R^η we denote the stopped process, i.e., $R^\eta_t = R_{\eta\wedge t}$. Let $S^p[0,T]$, $(p \ge 1)$ denote the space of all $\mathcal{F}_t$-adapted and cádlág processes $(R_t)_{t\le T}$, such that $||R||_{S^p[0,T]} := ||R^*||_{L^p} < \infty$, with $L^p = L^p(\Omega, R^1)$. A semimartingale $R = A + N$ is said to be a $H^p[0,T]$-semimartingale $(1 \le p \le \infty)$, if it has a finite $H^p[0,T] - norm$, defined by: $||R||_{H^p[0,T]} = \inf_{x=n+a} j_p(N,A)$, where $j_p(N,A) = ||\ [N,N]^{\frac{1}{2}}_T + \int_0^T |dA_s|\ ||_{L^p}$, $([N,N]_t)$ is a quadratic variation process of local martingale part N, and $|A_t| = \int_0^t |dA_s|$ represents the total variation on $[0,t]$ of the measure induced by the paths of the finite variation process A. Given a predictable set-valued process $G = (G_t)_{t\in[0,T]}$ and a d dimensional semimartingale R adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}, R_0 = 0$, let us denote

$$\mathbf{S}_R(G) := \{g \in \mathcal{P}(\mathcal{F}_t) : g(t) \in G(t) \text{ for each } t \in [0,T]\ a.e. \text{ and } g \text{ is } R \text{ integrable}\}.$$

For conditions of integrability with respect to semimartingales see e.g. [22].

Recall a set-valued stochastic process $G = (G_t)_{t\in[0,T]}$ is R-integrably bounded if there exists a predictable and R-integrable process m such that the Hausdorff distance $\delta(G_t, \{0\}) \le m_t$ a.s., each $t \in [0,T]$.

2. Weak solutions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a given filtered probability space. For any random element $R : \Omega \to \Theta$ with values in a measurable space Θ, we denote by P^R the measure on Θ being the distribution of R (under P). Let

(A^R, C^R, ν^R) denote the local characteristics of a semimartingale R, with respect to the fixed truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ (see e.g. [8] for details). For $H : [0,T] \times \Omega \to \mathbb{R}^m \otimes \mathbb{R}^d$ being any predictable and bounded (or locally bounded) mapping we will denote a stochastic integral $\int HdR$ as $H \cdot R$. Let $h^{\text{‘}} : \mathbb{R}^{d+m} \to \mathbb{R}^{d+m}$ be a fixed truncation function. For $y \in \mathbb{R}^d$, let Hy denote an m dimensional process with $(Hy)^i = \sum_{j\le d} H^{ij} y^j$, for $i \le m$. As in [8] let:

$$A^{R,H,i} = \begin{cases} A^{R,i} + [h^{\text{‘}i}(y, Hy) - h^i(y)] \cdot \nu^R & \text{if} \quad i \le d \\ \sum\limits_{j\le d} H^{i-d,j} \circ A^{R,j} + \Big[h^{\text{‘}i}(y, Hy) - (Hh(y))^{i-d}\Big] \cdot \nu^R & \text{if } d < i \le d+m \end{cases},$$

(1)

$$(2) \quad C^{R,H,ij} = \begin{cases} C^{R,ij} & \text{if} \quad i,j \le d \\ \sum_{k\le d} H^{i-d,k} \cdot C^{R,kj} & \text{if} \quad j \le d < i \le d+m \\ \sum_{k\le d} H^{j-d,k} \cdot C^{R,ik} & \text{if} \quad i \le d < j \le d+m \\ \sum_{k,l\le d} (H^{i-d,k} H^{j-d,l}) \cdot C^{R,kl} & \text{if} \quad d < i,j \le d+m \end{cases},$$

and let $\nu^{R,H}$ be defined by $I_G \cdot \nu^{R,H} = I_G(y, Hy) \cdot \nu^R$, for each Borel set G in $\mathbb{R}^{d+m}$.

By Propositions 5.3 and 5.6 Ch.IX [8] we have the following characterization for local characteristics of a stochastic integral.

Theorem 1. *Let H be any predictable and bounded (or locally bounded) mapping $H : [0,T] \times \Omega \to \mathbb{R}^m \otimes \mathbb{R}^d$ and let (A^R, C^R, ν^R) be a local characteristics of a d dimensional semimartingale R. Suppose (R, U) is a $d+m$ dimensional semimartingale. Then, $U = \int HdR$ if and only if (R, U) admits a local characteristics $(A^{R,H}, C^{R,H}, \nu^{R,H})$.*

Let $D([0,T], \mathbb{R}^n), (n \ge 1)$ denote the space of right continuous functions on $[0,T]$ with values in $\mathbb{R}^n$, with left limits, endowed with the Skorokhod topology. Let μ be a given probability measure on the space $(\mathbb{R}^m, \beta(\mathbb{R}^m))$. We consider the following stochastic inclusion:

$$dX_t \in F(t-, X, Z)dZ_t, \quad t \in [0,T], \qquad \text{(SDI)}$$
$$P^{X_0} = \mu,$$

where

$$F : [0,T] \times D([0,T], \mathbb{R}^m) \times D([0,T], \mathbb{R}^d) \to Comp(\mathbb{R}^m \otimes \mathbb{R}^d)$$

is a set-valued mapping, Z is a d dimensional semimartingale defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$.

To study weak solutions (or solution measures) to stochastic differential inclusion (SDI) we go to canonical path spaces. Similarly as in [7], let us introduce the following canonical path spaces:

1. The canonical space of driving processes: $D([0,T], \mathbb{R}^d)$ with $Z_t(y) = y(t)$ and $\mathcal{D}^d_T = \sigma\{Z_t : t \le T\}, \mathcal{D}^d_t = \sigma\{Z_s : s \le t\}, t \in [0,T]$.

2. The canonical space of solutions: $D([0,T], \mathbb{R}^m)$ with $X_t(x) = x(t)$, and σ-fields $\mathcal{F}^X_T = \sigma\{X_t : t \le T\}$ and $\mathcal{F}^X_t = \sigma\{X_s : s \le t\}, t \in [0,T]$.

3. The joint canonical path space: $\Omega^\sim = D([0,T], \mathbb{R}^m) \times D([0,T], \mathbb{R}^d)$ with $Y_t(x,z) = (x(t), z(t))$ and σ-fields $\mathcal{F}^\sim_T = \sigma\{Y_t : t \le T\}$ and $\mathcal{F}^\sim_t = \sigma\{Y_s : s \le t\}, t \in [0,T]$. Taking projections $\phi_1 : \Omega^\sim \to D([0,T], \mathbb{R}^m)$, with $\phi_1(x,z) = x$ and $\phi_2 : \Omega^\sim \to D([0,T], \mathbb{R}^d)$, with $\phi_2(x,z) = z$, we introduce on a measurable space $(\Omega^\sim, \mathcal{F}^\sim_T, (\mathcal{F}^\sim_t)$ the following processes $Z^\sim = Z \circ \phi_2$ and $X^\sim = X \circ \phi_1$.

Let (A^d, C^d, ν^d) and (A^m, C^m, ν^m) be processes defined on $D([0,T], \mathbb{R}^d)$ and $D([0,T], \mathbb{R}^m)$, respectively, satisfying the properties of local characteristics. Let us consider also processes $(A^m \circ \phi_1, C^m \circ \phi_1, \nu^m \circ \phi_1)$ and $(A^d \circ \phi_2, C^d \circ \phi_2, \nu^d \circ \phi_2)$ on $(\Omega^\sim, \mathcal{F}^\sim_T, (\mathcal{F}^\sim_t)$. Let Q be a probability measure on $(\Omega^\sim, \mathcal{F}^\sim_T, (\mathcal{F}^\sim_t)_{t\in[0,T]})$. We introduce probability measures: $P_1 = Q^{\phi_1}$ and $P_2 = Q^{\phi_2}$ on $(D([0,T], \mathbb{R}^m)$ and $(D([0,T], \mathbb{R}^d)$, respectively. Let $Z^\sim$ be a semimartingale under Q with the local characteristics $(A^d \circ \phi_2, C^d \circ \phi_2, \nu^d \circ \phi_2)$.

Definition 1. By a *weak solution* or *driving system* to the stochastic inclusion (SDI) we mean a filtered probability space $(\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t\in[0,T]}, P^*)$ on which there are defined:

(a) an $\mathcal{F}^*_t$-adapted, d dimensional semimartingale Z^*, with local characteristics $(A^d \circ \psi, C^d \circ \psi, \nu^d \circ \psi)$, where $\psi : \Omega^* \to D([0,T], \mathbb{R}^d), \psi(\omega^*) = Z^*(\omega^*)$ and $P^{*Z^*} = Q^{\phi_2}$,

(b) an m dimensional stochastic process X^*-called a solution process on $(\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t\in[0,T]}, P^*)$, such that: $P^{*X^*_0} = \mu$ and

$$X^*_t = X^*_0 + \int_0^t \gamma^*(s) dZ^*_s, \quad t \in [0,T],$$

for some $\mathcal{F}^{*X^*,Z^*}_t$-predictable mapping

$$\gamma^* \; : \; [0,T] \times \Omega^* \to \mathbb{R}^m \otimes \mathbb{R}^d,$$
$$\gamma^*(t,\omega^*) \in F(t, X^*(\omega^*), Z^*(\omega^*)).$$

We denote such solution by $(\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*)_{t\in[0,T]}, P^*, Z^*, X^*)$.

Remark 1. Let Q be a probability measure on $(\Omega^\sim, \mathcal{F}_T^\sim, (\mathcal{F}_t^\sim)_{t\in[0,T]})$ such that $Q^{X_0^\sim} = \mu$. Such a measure Q is called a *joint solution measure* a to stochastic inclusion (SDI), if there exists a weak solution to (SDI) $(\Omega^*, \mathcal{F}^*, (\mathcal{F}_t^*)_{t\in[0,T]}, P^*, Z^*, X^*)$ such that $Q = P^{*(X^*,Z^*)}$. Then, $P^{*X^*} = Q^{\phi_1}$ and $P^{*Z^*} = Q^{\phi_2}$. Hence, we can see that in a canonical setting both notions coincide. Indeed, similarly as in [7] one can show:

Proposition 1. *A probability measure Q on $(\Omega^\sim, \mathcal{F}_T^\sim, (\mathcal{F}_t^\sim)_{t\in[0,T]})$ is a solution measure to (SDI) if and only if $(\Omega^\sim, \mathcal{F}_T^\sim, (\mathcal{F}_t^\sim)_{t\in[0,T]}, Q, Z^\sim, X^\sim)$ is a weak solution to (SDI).*

In the case of a general driving semimartingale Z, the following existence result holds true (see [17] and [19]).

Theorem 2. *Let $F : [0,T] \times \mathbb{R}^{m+d} \to Comp(\mathbb{R}^m \otimes \mathbb{R}^d)$ be a set-valued function satisfying:*

(i) *F is integrably bounded (by some function $m(\cdot)$),*

(ii) *F is $([0,T] \times \mathbb{R}^{m+d})$-Borel measurable,*

(iii) *$F(t,\cdot)$ is lower semicontinuous for every fixed $t \in [0,T]$.*

If $F^* : [0,T] \times \Omega^\sim \to Comp(\mathbb{R}^m \otimes \mathbb{R}^d)$ is defined by $F^*(t,x,z) = \tilde{F}(t, x(t-), z(t-))$, where $\tilde{F}(t,a,b) = \int_0^t F(s,a,b)ds$, then there exists a weak solution to the stochastic differential inclusion:

$$\begin{aligned} dX_t &\in F^*(t,X,Z)dZ_t, \quad t \in [0,T] \\ P^{X_0} &= \mu. \end{aligned}$$

3. Martingale problem related to (SDI)

Below we present the formulation of the multivalued martingale problem related to the stochastic differential inclusion (SDI). The main results of this part states the equivalence between the existence of solution measures and solutions to the martingale problem. We start with a general formulation (see [8]). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I})$ be a filtered measurable space and let $\mathcal{H}$ be a sub-σ-field of $\mathcal{F}$. Suppose that μ is a given initial probability. By $\mathcal{X}$ we denote some family of cádlág and $\mathcal{F}_t$-adapted processes.

Definition 2. A probability P on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I})$ is a solution to the martingale problem related to $\mathcal{H}, \mathcal{X}$ and μ if

(i) $P|_{\mathcal{H}} = \mu$,

(ii) each process belonging to $\mathcal{X}$ is a local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, P)$.

We shall use notions and notations introduced in the Introduction, adapted to our canonical processes. Following a formulation in Definition 2, we will specify a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I})$, a sub σ-field $\mathcal{H}$, an initial distribution and a class of processes $\mathcal{X}$ as elements of a martingale problem related to our (SDI). They are listed in points (a), (b), (c) below. As in the previous section, we have a given bounded and predictable set-valued mapping $F : [0,T] \times \Omega^{\sim} \to \mathrm{Comp}(\mathbb{R}^m \otimes \mathbb{R}^d)$, an initial probability measure μ, processes (A^d, C^d, ν^d) defined on $D([0,T], \mathbb{R}^d)$, satisfying the properties of local characteristics. As mentioned in the Introduction, one can take a truncation function h as $h(y) = yI_{\{|y|\le 1\}}$. Below we use this function. Let us take:

(a) a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I})$ as a joint canonical space $(\Omega^{\sim}, \mathcal{F}_T^{\sim}, (\mathcal{F}_t^{\sim})_{t\in[0,T]})$,

(b) a sub σ-field $\mathcal{H} = \sigma(X_0^{\sim})$,

(c) a class: $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where

(i) $\mathcal{X}_1$ is a family consisting of processes:

$$f(Z_t^{\sim}) - f(Z_0^{\sim}) - \sum_{i\le d} \int_0^t \frac{\partial}{\partial x_i} f(Z_{s-}^{\sim}) dA_s^{Z^{\sim},i} - \frac{1}{2} \sum_{i,j\le d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(Z_{s-}^{\sim}) dC_s^{Z^{\sim},ij}$$

$$- \int_{[0,t]\times\mathbb{R}^d} \Big\{ f(Z_{s-}^{\sim} + y) - f(Z_{s-}^{\sim}) - \sum_{i\le d} \frac{\partial}{\partial x_i} f(Z_{s-}^{\sim}) y_i I_{\{|y|\le 1\}} \Big\} \nu^{Z^{\sim}}(ds, dy),$$

for each bounded function $f \in C^2(\mathbb{R}^d)$.

(b) $\mathcal{X}_2$ is a family consisting of processes:

$$f(R_t^{\sim}) - f(R_0^{\sim}) - \sum_{i\le d+m} \int_0^t \frac{\partial}{\partial x_i} f(R_{s-}^{\sim}) dA_s^{Z^{\sim},\gamma,i}$$

$$- \frac{1}{2} \sum_{i,j\le d+m} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(R_{s-}^{\sim}) dC_s^{Z^{\sim},\gamma,ij}$$

$$-\int\limits_{[0,t]\times\mathbb{R}^{d+m}}\Big\{f(R^{\sim}_{s-}+y)-f(R^{\sim}_{s-})-\sum_{i\le d+m}\frac{\partial}{\partial x_i}f(R^{\sim}_{s-})y_iI_{\{|y|\le 1\}}\Big\}\nu^{Z^{\sim},\gamma}(ds,dy),$$

for each bounded function $f\in C^2(\mathbb{R}^{m+d})$, where $R^{\sim}=\ (Z^{\sim},X^{\sim}-X^{\sim}_0)$, and for some measurable and bounded function $\gamma:[0,T]\times\Omega^{\sim}\to\mathbb{R}^m\otimes\mathbb{R}^d$.

The relation between weak solutions (or solution measures) and solutions to the related martingale problem for SDI is described by the following result.

Theorem 3 ([17]). *A probability measure Q on $(\Omega^{\sim},\mathcal{F}^{\sim}_T,(\mathcal{F}^{\sim}_t))$ is a joint solution measure to the stochastic inclusion (SDI) if and only if it is a solution to the related martingale problem.*

4. Weak compactness of the solution set

Let $\mathcal{M}(\Omega^{\sim})$ denote the space of all probability measures on the canonical space $(\Omega^{\sim},\mathcal{F}^{\sim}_T,(\mathcal{F}^{\sim}_t)_{t\in[0,T]})$, equipped with the topology of a weak convergence of probability measures (see [5]). By $\mathcal{R}^{loc}_Z(F,\mu)$ we denote the set of all probability measures $Q\in\mathcal{M}(\Omega^{\sim})$ such that Q is a solution to the martingale problem related to the stochastic inclusion (SDI). By Theorem 3, if $Q\in\mathcal{R}^{loc}_Z(F,\mu)$, then Q is a joint solution measure and there exists a weak solution system $(\Omega^*,\mathcal{F}^*,(\mathcal{F}^*_t)_{t\in[0,T]},P^*,Z^*,X^*)$. As noticed in Remark 1, the distribution law P^{*X^*} on $D([0,T],\mathbb{R}^m)$ equals the measure Q^{ϕ_1}. Since $\phi_1(X^{\sim},Z^{\sim})=X^{\sim}$, then $P^{*X^*}=Q^{X^{\sim}}$. Hence, there is a convenient way to study the properties of the solution set. Namely, let $\mathcal{R}^{loc}_Z(F,\mu)^1:=\{Q^{X^{\sim}}:Q\in\mathcal{R}^{loc}_Z(F,\mu)\}$. Clearly $\mathcal{R}^{loc}_Z(F,\mu)^1\subset\mathcal{M}(D([0,T],\mathbb{R}^m))$. Let (μ^k) be a tight sequence of initial distributions. The compactness of the set $\bigcup_{k\ge1}\mathcal{R}^{loc}_Z(F,\mu^k)^1$ was established in Theorem 5 of [17] in the case of continuous semimartingale satisfying the following condition:

Condition A: there exists the function $h(t)=o(t), t\to 0+$, such that

$$\sum_{1\le j,l\le d}E_P[Z^j,Z^j]_tE_P[Z^l,Z^l]_t+\sum_{j\le d}||(A^{Z^j})^t||^4_{H^2(P)}\le h(t),\ \text{for } t\in[0,T].$$

In a similar way we can show the same property for the set $\bigcup_{k\ge1}\mathcal{R}^{loc}_Z(F,\mu^k)$. Namely, the following result holds.

Theorem 4. *Let Z be a continuous semimartingale satisfying Condition A with $Z_0 = 0$. Let (μ^k) be a tight sequence of initial distributions and let $F : [0,T] \times C([0,T], IR^{m+d}) \to Comp(\mathbb{R}^m \otimes \mathbb{R}^d)$ be a measurable and bounded set-valued mapping such that the set $\bigcup_{k\geq 1} \mathcal{R}_Z^{loc}(F, \mu^k)$ is nonempty. Then, the set $\bigcup_{k\geq 1} \mathcal{R}_Z^{loc}(F, \mu^k)$ is a nonempty and relatively compact subset of $\mathcal{M}(C([0,T], \mathbb{R}^{m+d}))$.*

Proof. Using Prokhorov‘s Theorem ([5]), it is enough to show that the set $\bigcup_{k\geq 1} \mathcal{R}^{loc}(F, \mu^k)$ is tight. Let us remark first that

$$\lim_{a\to\infty} \sup_{Q\in\bigcup_{k\geq 1} \mathcal{R}^{loc}(F,\mu^k)} Q\{||X_0^{\sim}|| > a\}$$

$$\leq \lim_{a\to\infty} \sup_{k\geq 1} \mu^k\{x \in R^m : ||x|| > a\} = 0.$$

It is because the sequence (μ^k) is tight. Hence by Theorem 8.2 [5], it is enough to use the following criterion: for every $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{Q\in\bigcup_{k\geq 1} \mathcal{R}^{loc}(F,\mu^k)} Q\{w \in C([0,T], \mathbb{R}^{m+d}) : \Delta_T(\frac{1}{n}, w) > \epsilon\} = 0, \tag{3}$$

where $\Delta_T(\delta, w) = \sup\{||w(t) - w(s)|| : s,t \in [0,T], |s-t| < \delta\}$. Let us take an arbitrary measure Q from the set $\bigcup_{k\geq 1} \mathcal{R}_Z^{loc}(F, \mu^k)$. Then, there exist $k \geq 1$, and measurable and bounded (say by a constant $L > 0$) mappings $\gamma^k : [0,T] \times \Omega^{\sim} \to \mathbb{R}^m \otimes \mathbb{R}^d$, $\gamma^k(t,u) \in F(t,u) - dt \times dQ - a.e$ and $Q \in \mathcal{R}_Z^{loc}(\gamma^k, \mu^k)$. Taking functions $g : \mathbb{R}^{m+d} \to R$; $g(x) = x_i$, $i = 1, 2, \ldots, m+d$, we obtain, by the shape of the class $\mathcal{X}_2$ and Theorem 1, the following continuous Q-loc. martingales (on $(\Omega^{\sim}, \mathcal{F}_T^{\sim}, (\mathcal{F}_t^{\sim})_{t\in[0,T]})$):

$$N_t^{k,i} := \begin{cases} Z_t^{\sim,i} - A_t^{Z^{\sim},i} & \text{if} \quad 1 \leq i \leq d \\ X_t^{\sim,i-d} - X_0^{\sim,i-d} - A_t^{Z^{\sim},\gamma_-^k,i} & \text{if} \quad d < i \leq d+m \end{cases}. \tag{4}$$

Consequently, their second local characteristics are given by $\langle N^{k,i}, N^{k,j}\rangle_t = C^{Z^{\sim},\gamma_-^k,ij}$, $i,j = 1, 2, \ldots, d+m$. Let us take $N^k = (N^{k,d+1}, \ldots, N^{k,d+m})$. For $0 \leq t_0 < t_1 < T$, let us introduce the stopping time $\tau(u) = \inf\{s > 0 : ||X_{t_0+u}^{\sim}(u) - X_{t_0}^{\sim}(u)|| > \frac{\epsilon}{3}\} \wedge (t_1 - t_0)$, where $u \in \Omega^{\sim}$. Then by Theorem 44 from [22], the process $N_{t_0+t\wedge\tau}^k - N_{t_0}^k$ is a continuous Q-local martingale, for every fixed $k \geq 1$. We let $t_0 = 0$ for simplicity. Then by (4) we obtain

$$(X^{\sim} - X_0^{\sim})^{*2}_{t\wedge\tau} \leq 2(N^k)^{*2}_{t\wedge\tau} + 2(A^{Z^{\sim},\gamma^k_-})^{*2}_{t\wedge\tau},$$

and consequently

$$E_Q(X^{\sim} - X_0^{\sim})^{*4}_{\tau} \leq 4E_Q(N^k)^{*4}_{\tau} + 4E_Q(A^{Z^{\sim},\gamma^k_-})^{*4}_{\tau}. \tag{5}$$

Since

$$E_Q(N^k)^{*4}_{\tau} \leq m \sum_{d+1\leq i\leq d+m} E_Q\Big\{\sup_{s\leq\tau}(N^{k,i}_s)^4\Big\},$$

then applying Burkholder-Davis-Gundy inequality (see e.g., [22]) to continuous $Q-$ local martingales $N^{k,i}$, we get:

$$E_Q(N^k)^{*4}_{\tau} \leq C_4 m \sum_{d+1\leq i\leq d+m} E_Q\Big(C^{Z^{\sim},\gamma^k_-,ii}_{\tau}\Big)^2,$$

with some universal constant C_4. Consequently by (2):

$$E_Q(N^k)^{*4}_{\tau} \leq C_4 C(m) \sum_{d+1\leq i\leq d+m} \sum_{1\leq j,l\leq d} E_Q\Big\{\int_0^{\tau} |\gamma^{k,i-d,j}_{s-}||\gamma^{k,i-d,l}_{s-}||dC^{Z^{\sim},jl}_s|\Big\}^2,$$

with some constant $C(m)$. From the boundedness of F we have $|\gamma^{k,i,l}_t| \leq \sup_{a\in F(t,X^{\sim},Z^{\sim})} ||a|| \leq L\ dt \times dQ - a.e.$ Then applying the Kunita-Watanabe inequality (Theorem 25 Ch.II [22]) and Cauchy-Schwarz inequality to the right hand side above, we obtain:

$$E_Q(N^k)^{*4}_{\tau} \leq a(C_4, m, L) \sum_{1\leq j,l\leq d} E_Q[Z^{\sim,j}, Z^{\sim,j}]_{\tau} E_Q[Z^{\sim,l}, Z^{\sim,l}]_{\tau}, \tag{6}$$

where $a(C_4, m, L)$ is some constant not depending on $Z^{\sim}$ and τ.

Let us consider now the estimation of the term $E_Q(A^{Z^{\sim},\gamma^k_-})^{*4}_{\tau}$ appearing in (5). By Theorem 1, the semimartingale $\int \gamma^k_{s-} dZ^{\sim}_s$ admits its first local characteristics $A^{Z^{\sim},\gamma^k_-} = (A^{Z^{\sim},\gamma^k_-,i})_{i\leq d}$, with $A^{Z^{\sim},\gamma^k_-,i} = \sum_{j\leq d} \int \gamma^{k,i-d,j}_{s-} dA^{Z^{\sim},j}_s$, $i = d+1, d+2, \ldots, d+m$. Hence applying Emery's inequalities ([22]) and boundedness of F, one can verify that

$$E_Q(A^{Z^\sim,\gamma^k_-})^{*4}_\tau \le d^3 m \sum_{d+1\le i\le d+m}\sum_{j\le d}\Big\| \int_0^{\cdot\wedge\tau} \gamma^{k,i-d,j}_{s-} dA^{Z^\sim,j}_s \Big\|^4_{S^4(Q)}$$
$$\le m^2 d^3 c_4^4 L^4 \sum_{j\le d} ||(A^{Z^\sim,j})^\tau||^4_{H^2(Q)},$$

where c_4 is a universal constant. Using this inequality together with (6) we finally obtain the following estimation in (5)

$$E_Q(X^\sim - X^\sim_0)^{*4}_\tau \le D\Big\{ \sum_{1\le j,l\le d} E_Q[Z^{\sim,j}, Z^{\sim,j}]_\tau E_Q[Z^{\sim,j}, Z^{\sim,j}]_\tau +$$
$$+ \sum_{j\le d} ||(A^{Z^\sim,j})^\tau||^4_{H^2(Q)}\Big\},$$

for some constant $D := a(C_4, d, c_4, m, L)$ depending only on indicated constants. Now, restoring t_0 and setting $t_1 - t_0 := \alpha$, we obtain:

$$E_Q(X^\sim_{t_0+\cdot} - X^\sim_{t_0})^{*4}_\alpha \le Dh(\alpha),$$

where h is a function as in Condition A. By Tchebyshev's inequality we have:

$$Q\Big\{ \sup_{s\le\alpha} ||X^\sim_{t_0+s} - X^\sim_{t_0}|| > \epsilon \Big\} \le \frac{Dh(\alpha)}{\epsilon^4}. \tag{7}$$

Let $T^* = [T] + 1$. For an arbitrary $n \in N$, let us divide the interval $[0, T^*]$ by points $\{\frac{i}{n}\}, i = 0, 1, 2, \ldots, T^*n$. Then,

$$Q\Big\{ \Delta_T\Big(\frac{1}{n}, X^\sim\Big) > \epsilon \Big\} \le Q\Big\{ \bigcup_{i=0}^{T^*n-1} \Big\{ \sup_{0\le s\le\frac{1}{n}} ||X^\sim_{t_0+s} - X^\sim_{t_0}|| > \frac{\epsilon}{3} \Big\}\Big\}.$$

Hence and by (7) with $\alpha = \frac{1}{n}$, we get:

$$Q\Big\{ \Delta_T\Big(\frac{1}{n}, X^\sim\Big) > \epsilon \Big\} \le \frac{3^4 T^* D n h(\frac{1}{n})}{\epsilon^4}.$$

Hence by Condition A, we have:

$$\lim_{n\to\infty} \sup_{Q\in\bigcup_{k\geq 1}\mathcal{R}^{loc}(F,\mu^k)} Q\Big\{\Delta_T\Big(\frac{1}{n},X^{\sim}\Big)>\epsilon\Big\}=0.$$

In a similar way one obtain:

$$\lim_{n\to\infty} \sup_{Q\in\bigcup_{k\geq 1}\mathcal{R}^{loc}(F,\mu^k)} Q\Big\{\Delta_T\Big(\frac{1}{n},Z^{\sim}\Big)>\epsilon\Big\}=0,$$

which completes the proof.

Remark 2. Let us put in particular $Z_t := (t, W_t)$, where W is a $d-1$ dimensional Wiener process and $F(t,x,z) := (F(t,x), G(t,x))$, with $F : [0,T]\times C([0,T],\mathbb{R}^m) \to Comp(\mathbb{R}^m \otimes \mathbb{R}^1)$ and $G : [0,T]\times C([0,T],\mathbb{R}^m) \to Comp(\mathbb{R}^m \otimes \mathbb{R}^{d-1})$. Then the stochastic inclusion (SDI) has the form

$$\begin{aligned} dX_t &\in F(t,X)dt + G(t,X)dW_t,\\ P^{X_0} &= \mu, \end{aligned}$$

In this case one can choose $h(t) = d^2t^2 + t^4$. Thus Theorem 4 extends earlier results obtained in [13, 16] and [18].

For the case of a noncontinuous integrator we consider the stochastic inclusion driven by the Levy process L on the interval $[0,T]$. Namely, we consider the following inclusion

$$\begin{aligned} dX_t &\in F^*(t-,X,L)dL_t, \quad t\in[0,T]\\ P^{X_0} &= \mu \end{aligned}$$

with a set-valued mapping $F^* : [0,T]\times D([0,T],IR^2) \to Comp(\mathbb{R}^1)$ defined by $F^*(t,x,z) = \tilde{F}(t,x(t-),z(t-))$, where $\tilde{F}(t,a,b) = \int_0^t F(s,a,b)ds$ and $F : [0,T]\times\mathbb{R}^2 \to Comp(\mathbb{R}^1)$ are given. We assume $m = d = 1$ for simplicity. Since L is a semimartingale with independent increments then the local characteristics (A,C,ν) of the integrator are deterministic and they have the form: $A_t = bt, C_t = \sigma^2 t, \nu(dt,dx) = dtm(dx)$, where $b = E(L_1), \sigma > 0$ and $m(dx)$ is a measure on $\mathbb{R}^1\backslash\{0\}$ that integrates the function $\min(1,x^2)$ (see [8] for details). We assume also that the integrator L has a finite second moment. Then, $L_t = M_t + tEL_1$, where M is a square integrable martingale. Since the integrator is a cádlág process we cannot proceed as earlier. We shall use the Aldous Criterion of Tightness (see e.g., Theorem 4.5 Ch.VI in [8]). Let $\{Z^n\}$ be a sequence of semimartingales (defined possibly on different probability spaces $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t\in[0,T]}, P^n)$). We will use the following:

Definition 3 ([24])**.** The sequence $\{Z^n\}$ of semimartingales satisfies the uniform tightness condition (UT) if for every $q \in \mathbb{R}^+$ the family of random variables

$$\left\{ \int_0^q U_s^n dZ_s^n : U^n \in \mathcal{U}_q^n, n \in \mathbb{N} \right\}$$

is tight in $\mathbb{R}$, where $\mathcal{U}_q^n$ denotes the family of predictable processes of the form

$$U_s^n = U_0^n + \sum_{i=0}^{k} U_i^n I_{\{t_i < s \le t_{i+1}\}},$$

for $0 = t_0 < \ldots < t_k = q$ and every U_i^n being an $\mathcal{F}_{t_i}^n$ measurable random variable such that $|U_i^n| \le 1$, for every $i \in \mathbb{N} \cup \{0\},\ k, n \in \mathbb{N}$.

The main properties of uniformly tight sequences of semimartingales are presented below (see [23] for details).

Theorem 5. *Let $\{Z^n\}$ be a sequence of semimartingales satisfying (UT). Then the following statements hold true*

(i) *for every $q \in \mathbb{R}^+$ the sequences $\{\sup_{t\le q} |Z_t^n|\}$ and $\{[Z^n]_q\}$ are tight in $\mathbb{R}^1$,*

(ii) *if $\{U^n\}$ is a sequence of predictable processes such that for every $q \in \mathbb{R}^+$ the sequence $\{\sup_{t\le q} |U_t^n|\}$ is tight in $\mathbb{R}^1$, then the sequence of stochastic integrals $\{\int_0^\cdot U_s^n dZ_s^n\}$ satisfies (UT).*

Under the same notations as before the following theorem holds true.

Theorem 6. *Let L be a Levy process as above. We assume that $F : [0,T] \times \mathbb{R}^2 \to Comp(\mathbb{R}^1)$ is a set-valued function satisfying the assumptions of Theorem 2. Let (μ^k) be a tight sequence of initial distributions. Then, the set $\bigcup_{k\ge1} \mathcal{R}_L^{loc}(F^*, \mu^k)^1$ is nonempty and relatively compact in the space $\mathcal{M}(D([0,T], \mathbb{R}^1))$.*

Proof. The nonemptiness of the set $\bigcup_{k\geq 1}\mathcal{R}_L^{loc}(F^*,\mu^k)^1$ follows from Theorem 2 and Theorem 3. Let us take an arbitrary sequence of measures $\{R^n\}$:

$$\{R^n\}\subset\bigcup_{k\geq 1}\mathcal{R}_L^{loc}(F^*,\mu^k)^1.$$

Then, for every $n\geq 1$ there exist $k_n\geq 1$, the joint solution measure $Q^{k_n}\in\mathcal{R}_L^{loc}(F^*,\mu^{k_n})$ and (by Theorem 3 and Remark 1) a weak solution system $(\Omega^{k_n},\mathcal{F}^{k_n},(\mathcal{F}_t^{k_n})_{t\in[0,T]},P^{k_n},L^{k_n},X^{k_n})$ with the following properties:

(i) $Q^{k_n}=(P^{k_n})^{(X^{k_n},L^{k_n})}, R^n=(P^{k_n})^{X^{k_n}}$

(ii) L^{k_n} is an $\mathcal{F}_t^{k_n}$-adapted square integrable Levy process, with local characteristics $A_t=bt, C_t=\sigma^2 t, \nu(dt,dx)=dtm(dx)$ and X^{k_n} is a solution process on $(\Omega^{k_n},\mathcal{F}^{k_n},(\mathcal{F}_t^{k_n})_{t\in[0,T]},P^{k_n})$ such that $(P^{k_n})^{X_0^{k_n}}=\mu^{k_n}$ and

$$X_t^{k_n}=X_0^{k_n}+\int_0^t\gamma^{k_n}(s)dL_s^{k_n},\quad t\in[0,T],$$

for some $(\mathcal{F}_t^{k_n})^{X^{k_n},L^{k_n}}$-predictable and bounded (say by the constant C) process

$$\gamma^{k_n}\ :\ [0,T]\times\Omega^{k_n}\to\mathbb{R}^1,$$
$$\gamma^{k_n}(t)\in F(t,X^{k_n},L^{k_n})\ dt\otimes dP^{k_n}\text{-}a.s.$$

Since the sequence of processes $\{\gamma^{k_n}\}$ is uniformly bounded it follows that the sequence $\{\sup_{t\leq q}|\gamma_t^{k_n}|\}$ is tight in $\mathbb{R}^1$ for every $q\in\mathbb{R}^+$. For every $q\in\mathbb{R}^+$ let us consider the family $\mathcal{U}_q^n$ described in Definition 3. Then using first Khinthine's inequality and next Emery‘s inequality [22], we get the following estimation:

$$\begin{aligned}
P^{k_n}\Big\{\Big|\int_0^q U_s^n dL_s^{k_n}\Big|>K\Big\} &\leq \frac{1}{K^2}E^{k_n}\Big\{\sup_{0\leq t\leq q}\Big|\int_0^t U_s^n dL_s^{k_n}\Big|^2\Big\}\\
&\leq \frac{c_2^2}{K^2}\Big\|\int_0^\cdot U_s^n dL_s^{k_n}\Big\|^2_{H^2_{[0,q]}}\\
&\leq \frac{c_2^2}{K^2}\left(\sigma^2+\int x^2 m(dx)+qM\right)\int_0^q E^{k_n}(U_s^n)^2 ds\\
&\leq \frac{c_2^2 q}{K^2}\left(\sigma^2+\int x^2 m(dx)+qM\right),
\end{aligned}$$

where $M := E^{k_n}[(L_1^{k_n})^2] < \infty$ (we have assumed that the Levy process has the finite second moment). Hence, the sequence $\{L^{k_n}\}$ satisfies (UT). By Theorem 5 we claim that the sequence $\{\int_0^{\cdot} \gamma_s^{k_n} dL_s^{k_n}\}$ satisfies (UT) as well. Consequently, we infer the tightness of the sequence $\{\sup_{t\leq q} |\int_0^t \gamma_s^{k_n} dL_s^{k_n}|\}$ for every $q \in \mathbb{R}^+$. For $n, N \geq 1$ let $\mathcal{T}_N^n$ denote the set of $(\mathcal{F}_t^{k_n})^{X^{k_n}, L^{k_n}}$-stopping times that are bounded by N. Then, similarly as above one can show

$$\sup_{S,T\in\mathcal{T}_N^n: S\leq T\leq S+\theta} P^{k_n}\left\{|X_T^{k_n} - X_S^{k_n}| > \varepsilon\right\}$$

$$\leq \frac{1}{\varepsilon^2} \sup_{S,T\in\mathcal{T}_N^n: S\leq T\leq S+\theta} E^{k_n}\left\{\sup_{S\leq q\leq T}\left|\int_S^q \gamma_\tau^{k_n} L_\tau^{k_n}\right|^2\right\}$$

$$\leq \frac{c_2^2 C^2 \theta}{\varepsilon^2}\left(\sigma^2 + \int x^2 m(dx) + \theta M\right)$$

for every $\varepsilon, \theta > 0$ and $n \in \mathbb{N}$. Thus we have

$$\lim_{\theta\to 0+} \limsup_n \sup_{S,T\in\mathcal{T}_N^n: S\leq T\leq S+\theta} P^{k_n}\{|X_T^{k_n} - X_S^{k_n}| > \varepsilon\} = 0,$$

and by the Aldous Criterion of Tightness (see Theorem 4.5 Ch.VI in [8]) we claim the tightness of the sequence $\{X^{k_n}\}$, which implies the some property for the sequence $\{R^n\}$. Hence by Prohorov's Theorem we infer that the set $\bigcup_{k\geq 1} \mathcal{R}_L^{loc}(F^*, \mu^k)$ is relatively compact in the space $\mathcal{M}(D([0,T], \mathbb{R}^1))$ equipped with the topology of weak convergence.

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