Discussiones Mathematicae
Differential Inclusions, Control and Optimization 29 (2009) 19–42

FOURIER-LIKE METHODS FOR EQUATIONS WITH SEPARABLE VARIABLES

Danuta Przeworska-Rolewicz

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8, 00–956 Warszawa 10, P.O. Box 21, Poland

e-mail: rolewicz@impan.gov.pl

Abstract

It is well known that a power of a right invertible operator is again right invertible, as well as a polynomial in a right invertible operator under appropriate assumptions. However, a linear combination of right invertible operators (in particular, their sum and/or difference) in general is not right invertible. It will be shown how to solve equations with linear combinations of right invertible operators in commutative algebras using properties of logarithmic and antilogarithmic mappings. The used method is, in a sense, a kind of the variables separation method. We shall obtain also an analogue of the classical Fourier method for partial differential equations. Note that the results concerning the Fourier method are proved under weaker assumptions than these obtained in [6] (cf. also [7, 8, 11]).

Keywords: algebraic analysis, commutative algebra with unit, Leibniz condition, logarithmic mapping, antilogarithmic mapping, right invertible operator, sine mapping, cosine mapping, initial value problem, boundary value problem, Fourier method.

2000 Mathematics Subject Classification: 47A25, 47C05, 47N20, 35A25.

1. Preliminaries. Basic notions of Algebraic Analysis

We recall here the following notions and theorems (without proofs; cf. [7, 8]). Denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Q}$ the sets of positive integers, nonnegative integers, reals, complexes, integers and rational numbers, respectively, and by $\mathbb{F}$ any field of scalars. If $\mathbb{F}$ is a field of numbers, then is denoted by $\mathbb{F}[t]$ the set of all polynomials in t with coefficients in $\mathbb{F}$.

Let X be a linear space (in general, without any topology) over a field $\mathbb{F}$ of scalars of the characteristic zero.

- $L(X)$ is the set of all linear operators with domains and ranges in X;
- dom A is the domain of an $A \in L(X)$;
- $\ker A = \{x \in \text{dom } A : Ax = 0\}$ is the kernel of an $A \in L(X)$;
- $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$;
- $I(X)$ is the set of all invertible elements in X.

An operator $D \in L(X)$ is said to be *right invertible* if there is an operator $R \in L_0(X)$ such that $RX \subset \text{dom } D$ and $DR = I$, where I denotes the identity operator. The operator R is called a *right inverse* of D. By $R(X)$ we denote the set of all right invertible operators in $L(X)$. Let $D \in R(X)$. Let $\mathcal{R}_D \subset L_0(X)$ be the set of all right inverses for D, i.e., $DR = I$ whenever $R \in \mathcal{R}_D$. We have dom $D = RX \oplus \ker D$, independently of the choice of an $R \in \mathcal{R}_D$. Elements of $\ker D$ are said to be *constants*, since by definition, $Dz = 0$ if and only if $z \in \ker D$. The kernel of D is said to be the *space of constants*. We should point out that, in general, constants are different than scalars, since they are elements of the space X. If two right inverses commute with each other, then they are equal.

Clearly, if $\ker D \neq \{0\}$, then the operator D is right invertible, but not invertible. Here the invertibility of an operator $A \in L(X)$ means that the equation $Ax = y$ has a unique solution for every $y \in X$. An element $y \in \text{dom } D$ is said to be *a primitive* for an $x \in X$ if $y = Rx$ for an $R \in \mathcal{R}_D$. Indeed, by definition, $x = DRx = Dy$. Again, by definition, all $x \in X$ have primitives. Let

$$\mathcal{F}_D = \{F \in L_0(X) : F^2 = F; FX = \ker D \text{ and } \exists_{R \in \mathcal{R}_D}\ FR = 0\}.$$

Any $F \in \mathcal{F}_D$ is said to be an *initial* operator for D corresponding to R. One can prove that **any** projection F' onto $\ker D$ is an initial operator for D corresponding to a right inverse $R' = R - F'R$ independently of the choice of an $R \in \mathcal{R}_D$.

If two initial operators commute with each other, then they are equal. Thus this theory is essentially **noncommutative**. An operator F is initial for D if and only if there is an $R \in \mathcal{R}_D$ such that

$$F = I - RD \quad \textit{on } \text{dom } D. \tag{1.1}$$

It is enough to know one right inverse in order to determine all right inverses and all initial operators. Note that a superposition of a finite number of right invertible operators is again a right invertible operator.

The equation $Dx = y$ $(y \in X)$ has the general solution $x = Ry + z$, where $R \in \mathcal{R}_D$ is arbitrarily fixed and $z \in \ker D$ is arbitrary. However, if we take an *initial condition*: $Fx = x_0$, where $F \in \mathcal{F}_D$ and $x_0 \in \ker D$, then this equation has a unique solution $x = Rx + x_0$.

If $T \in L(X)$ belongs to the set $\Lambda(X)$ of all left invertible operators, then $\ker T = \{0\}$. If D is invertible, i.e., $D \in \mathcal{I}(X) = R(X) \cap \Lambda(X)$, then $\mathcal{F}_D = \{0\}$ and $\mathcal{R}_D = \{D^{-1}\}$.

If $P(t) \in \mathbb{F}[t]$ then all solutions of the equation $P(D)x = y$, $y \in X$, can be obtained by a decomposition of the rational function $1/P(t)$ into vulgar fractions. Write

$$v_{\mathbb{F}}A = \{\lambda \in \mathbb{F} \setminus \{0\} : I - \lambda A \ \text{ is invertible}\} \quad \text{for} \ \ A \in L(X).$$

Clearly, $\lambda \in v_{\mathbb{F}}A$ if and only if $1/\lambda$ is a regular value of A. Let $V(X)$ be the set of all *Volterra* operators, i.e.,

$$V(X) = \{A \in L_0(X) : A - \lambda I \ \text{ is invertible for all } \ \lambda \in \mathbb{F} \setminus \{0\}\}.$$

Then $A \in V(X)$ if and only if $v_{\mathbb{F}}A = \mathbb{F} \setminus \{0\}$.

If X is an algebra over $\mathbb{F}$ with a $D \in L(X)$ such that $x, y \in \text{dom } D$ implies $xy, yx \in \text{dom } D$, then we say that X is a *D-algebra* and we write $D \in \mathsf{A}(X)$. If X is a *commutative* algebra then $\mathbf{A}(X)$ is denoted by $\mathsf{A}(X)$. Let $D \in \mathbf{A}(X)$ and

$$f_D(x, y) = D(xy) - c_D[xDy + (Dx)y] \ \text{ for } \ x, y \in \text{dom } D, \tag{1.2}$$

where c_D is a scalar dependent on D only. Clearly, f_D is a bilinear (i.e., linear in each variable) form which is symmetric when X is commutative, i.e. when $D \in \mathsf{A}(X)$. This form is called a *non-Leibniz component* (cf. [7]). If $D \in \mathbf{A}(X)$ then the product rule in X can be written as follows:

$$D(xy) = c_D[xDy + (Dx)y] + f_D(x, y) \ \text{ for } \ x, y \in \text{dom } D.$$

If $D \in \mathbf{A}(X)$ and if D satisfies the *Leibniz condition*:

$$D(xy) = xDy + (Dx)y \ \text{ for } \ x, y \in \text{dom } D, \tag{1.3}$$

then X is said to be a *Leibniz algebra*. It means that in Leibniz algebras $c_D = 1$ and $f_D = 0$. The Leibniz condition implies that $xy \in \text{dom } D$ whenever $x, y \in \text{dom } D$, i.e., Leibniz algebras are D-algebras. If X is a Leibniz algebra with unit e then $e \in \ker D$, i.e., D is not left invertible. The set of commutative Leibniz D-algebras X with a $D \in R(X)$ and with unit $e \in \text{dom } D$ is denoted by $\mathbf{L}(D)$. Clearly, if $X \in \mathbf{L}(D)$ then $e \in \ker\ D$.

Non-Leibniz components for powers of $D \in \mathsf{A}(X)$ are determined by recurrence formulae (cf. [7, 8]).

Suppose that $D \in \mathsf{A}(X)$ and $\lambda \neq 0$ is an arbitrarily fixed scalar. Then $\lambda D \in \mathsf{A}(X)$ and $c_{\lambda D} = c_D,\ f_{\lambda D} = \lambda f_D$.

If $D_1, D_2 \in \mathsf{A}(X)$, the superposition $D = D_1 D_2$ exists and $D_1 D_2 \in A(X)$, then

$$(1.4) \qquad c_{D_1 D_2} = c_{D_1} c_{D_2} \ \text{ and for } \ x, y \in \text{dom } D = \text{dom } D_1 \cap D_2$$

$$f_{D_1 D_2}(x,y) = f_{D_1}(x,y) + D_1 f_{D_2}(x,y) + +c_{D_1} c_{D_2}[(D_1 x) D_2 y + (D_2 x) D_1 y].$$

For higher powers of D in Leibniz algebras, by an easy induction from Formulae (1.4) and the Leibniz condition, we obtain *the Leibniz formula*:

$$(1.5) \qquad D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} (D^k x) D^{n-k} y \ \text{ for } \ x, y \in \text{dom } D^n \quad (n \in \mathbb{N}).$$

Let $X \in \mathsf{A}(X)$. We denote by $M(X)$ the set of all multiplicative mappings (not necessarily linear) with domains and ranges in X:

$$M(X) = \{A : A(xy) = (Ax)(Ay) \ \text{ whenever } \ x, y \in \text{dom } A \subset X\}.$$

Suppose that $X \in \mathsf{A}(\mathrm{X})$ and $D \in R(X)$. An initial operator F for D is said to be *almost averaging* if $F(zx) = zFx$ whenever $z \in \ker D$, $x \in X$. Clearly, every multiplicative operator $F \in \mathcal{F}_D$ is almost averaging for $F(zx) = (Fz)(Fx) = zFx$ if $z \in \ker D$, $x \in X$, but not conversely (cf. [7]). If X is a D-algebra such that dim ker D=1 and $e \in \ker\ D$ then all initial operators for D are almost averaging.

Suppose that $D \in \mathsf{A}(X)$. Let a multifunction $\Omega : \text{dom } D \longrightarrow 2^{\text{dom } D}$ be defined as follows:

$$\Omega u = \{x \in \text{dom } D : Du = uDx\} \ \text{ for } \ u \in \text{dom } D \setminus \{0\}.$$

The equation

$$Du = uDx \text{ for } (u, x) \in \text{graph } \Omega$$

is said to be the *basic equation.* Clearly,

$$\Omega^{-1}x = \{u \in \text{dom } D : Du = uDx\} \text{ for } x \in \text{ dom D}.$$

The multifunction Ω is well-defined and dom $\Omega \supset \ker D \setminus \{0\}$.

Suppose that $(u, x) \in$ graph Ω, L is a selector of Ω and E is a selector of Ω^{-1}. By definitions, $Lu \in$ dom Ω^{-1}, $Ex \in$ dom Ω and the following equations are satisfied:

$$Du = uDLu, \qquad DEx = (Ex)Dx.$$

Any invertible selector L of Ω is said to be a *logarithmic mapping* and its inverse $E = L^{-1}$ is said to be an *antilogarithmic mapping.* By $G[\Omega]$ we denote the set of all pairs (L, E), where L is an invertible selector of Ω and $E = L^{-1}$. For any $(u, x) \in$ dom Ω and $(L, E) \in G[\Omega]$ elements Lu, Ex are said to be *logarithm* of u and *antilogarithm* of x, respectively. The multifunction Ω is examined in [8] (also for noncommutative algebras).

Clearly, by definition, for all $(L, E) \in G[\Omega]$, $(u, x) \in$ graph Ω we have

$$ELu = u, \quad LEx = x; \quad DEx = (Ex)Dx, \quad Du = uDLu. \tag{1.6}$$

A logarithm of zero is not defined. If $(L, E) \in G[\Omega]$ then $L(\ker D \setminus \{0\}) \subset \ker D$, $E(\ker D) \subset \ker D$. In particular, $E(0) \in \ker D$.

If $D \in R(X)$ then logarithms and antilogarithms are uniquely determined up to a constant. Moreover, if $F \in \mathcal{F}_D$ then $FE = EF$, $FL = LF$ (cf. [15]).

Let $D \in \mathsf{A}(X)$ and let $(L, E) \in G[\Omega]$. A logarithmic mapping L is said to be of the *exponential type* if $L(uv) = Lu + Lv$ for $u, v \in$ dom Ω. If L is of the exponential type then $E(x + y) = (Ex)(Ey)$ for $x, y \in$ dom Ω^{-1}. We have proved that a logarithmic mapping L is of the exponential type if and only if X is a *Leibniz commutative algebra* (cf. [8]). Moreover, $Le = 0$, i.e., $E(0) = e$. In Leibniz commutative algebras with $D \in R(X)$ a necessary and sufficient conditions for $u \in$ dom Ω is that $u \in I(X)$ (cf. [8]).

By $\mathbf{Lg}(D)$ we denote the class of these commutative algebras with $D \in R(X)$ and with unit $e \in$ dom Ω for which there exist invertible selectors of Ω, i.e., there exist $(L, E) \in G[\Omega]$. By $\mathsf{L}(D)$ we denote the class of these commutative Leibniz algebras with unit $e \in$ dom Ω for which there exist invertible selectors of Ω. By these definitions, $X \in \mathbf{Lg}(D)$ is a Leibniz algebra if and only if $X \in \mathsf{L}(D)$ and $D \in R(X)$. This class we shall denote by $\mathit{L}(D)$. It means that $\mathit{L}(D)$ is the class of these commutative Leibniz algebras with $D \in R(X)$ and with unit $e \in \text{dom}\ \Omega$ for which there exist invertible selectors of Ω, i.e., there exist $(L, E) \in G[\Omega]$.

In the same manner we define logarithmic and antilogarithmic mappings of *higher order*. Namely, let $n \in N$ be arbitrarily fixed. Suppose that $D \in \mathsf{A}(X)$. Let a multifunction $\Omega_n : \text{dom}\ D^n \longrightarrow 2^{\text{dom}\ D^n}$ be defined as follows:

$$\Omega_n u = \{x \in \text{dom}\ D^n : D^n u = u D^n x\} \quad \text{for} \quad u \in \text{dom}\ D^n. \tag{1.7}$$

Any invertible selector L_n of Ω_n is said to be a *logarithmic mapping of the order* n and its inverse $E_n = L_n^{-1}$ is said to be an *antilogarithmic mapping of the order* n. By $G[\Omega_n]$ we denote the set of all pairs (L_n, E_n), where L_n is an invertible selector of Ω_n and $E_n = L_n^{-1}$. For any $(u, x) \in \text{dom}\ \Omega_n$ and $(L_n, E_n) \in G[\Omega_n]$ elements $L_n u$, $E_n x$ are said to be *logarithm of the order* n of u and *antilogarithm of the order* n of x, respectively. The multifunctions Ω_n and relations between them are examined in [8]. Clearly, if $X \in \mathbf{Lg}(D)$ then $X \in \mathbf{Lg}(D^n)$ for all $n \in \mathbb{N}$.

If $\ker D = \{0\}$, then either X is not a Leibniz algebra or X has no unit (cf. [8]). Thus, by our definition, if $X \in \mathit{L}(D)$, then $\ker D \neq \{0\}$, i.e., the operator D is right invertible but not invertible.

2. Linear combinations of right invertible operators

We begin with

Proposition 2.1. *Suppose that* $n, r_1, \ldots, r_n \in \mathbb{N}$,

$$X \in \bigcap_{j=1}^{n} \mathit{L}(D_j), \tag{2.1}$$

(2.2)
$$D = \sum_{j=1}^{n} \alpha_j D_j^{r_j}, \quad \alpha_j \in X \quad (j = 1, \ldots, n), \quad \text{dom}\ D = \bigcap_{j=1}^{n} \text{dom}\ D_j^{r_j} \neq \emptyset,$$

$(L_{r_j}^{(j)}, E_{r_j}^{(j)}) \in G[\Omega_{r_j}^{(j)}]$, *where* $\Omega_{r_j}^{(j)}$ *is induced by* $D_j^{r_j}$ $(j = 1, \ldots, n)$, (2.3) $$x = \prod_{k=1}^{n} u_k, \quad \textit{where } u_k \in \ker D_k \cap I(X) \quad (k = 1, \ldots, n).$$

Then

(2.4) $$Dx = ax, \quad \textit{where } a = \sum_{j=1}^{n} \alpha_j a_j,$$

(2.5) $$a_j = D_j^{r_j} L_{r_j}^{(j)} \widetilde{u}_j \quad \textit{and} \quad \widetilde{u}_j = \prod_{k=1,\ k\neq j}^{n} u_k \quad (j = 1, \ldots, n),$$

i.e., $\widetilde{u}_j,\ u_j\widetilde{u}_j = x \in I(X)$ $(j = 1, \ldots, n)$.

Proof. Since, by our assumptions, the operators $D_1,\ldots,D_n$ satisfy the Leibniz condition and $D_1u_1 = \ldots\ D_nu_n = 0$, from the Leibniz Formula (1.5) we get

$$D_j^m x = D_j^m \Big(\prod_{k=1}^{n} u_k \Big) = \sum_{l=0}^{m} \binom{m}{l} (D_j^l u_j)(D_j^{m-l} \widetilde{u}_j) = u_j D_j^m \widetilde{u}_j$$

for $j = 1, \ldots, n$. Thus

$$\begin{aligned}
Dx &= \Big(\sum_{j=1}^{n} \alpha_j D_j^{r_j} \Big) \prod_{k=1}^{n} u_k \\
&= \sum_{j=1}^{n} \alpha_j u_j D_j^{r_j} \prod_{k=1,\ k\neq j}^{n} u_k \\
&= \Big(\prod_{k=1}^{n} u_k \Big) \sum_{j=1}^{n} \alpha_j \Big(\prod_{k=1,\ k\neq j}^{n} u_k^{-1} \Big) D_j^{r_j} \Big(\prod_{k=1,\ k\neq j}^{n} u_k \Big) \\
&= x \sum_{j=1}^{n} \alpha_j \widetilde{u}_j^{-1} D_j^{r_j} \widetilde{u}_j = x \sum_{j=1}^{n} \alpha_j D_j^{r_j} L_{r_j}^{(j)} \widetilde{u}_j = x \sum_{j=1}^{n} \alpha_j a_j = xa.
\end{aligned}$$

■

Proposition 2.2. *Suppose that all assumptions of Proposition* 2.1 *are satisfied. Then there are* $R_j \in \mathcal{R}_{D_j}$ *such that* $R_j^{r_j} a_j \in \operatorname{dom} (\Omega_{r_j}^{(j)})^{-1}$ $(j = 1, \ldots, n)$ *and*

(2.6) $$\widetilde{u}_j = E_{r_j}^{(j)} (R_j^{r_j} a_j) \qquad (j = 1, \ldots, n).$$

Proof. By our assumptions, $a_j = D_j^{r_j} L_{r_j}^{(j)} \widetilde{u}_j$ for $j = 1, \ldots, n$. Hence there are $R_j \in \mathcal{R}_{D_j}$ such that $L_{r_j}^{(j)} \widetilde{u}_j = R_j^{r_j} a_j$ (cf. [8]), i.e., $\widetilde{u}_j = E_{r_j}^{(j)} L_{r_j}^{(j)} \widetilde{u}_j = E_{r_j}^{(j)}(R_j^{r_j} a_j)$. ■

Proposition 2.3. *Suppose that all assumptions of Proposition* 2.1 *are satisfied and* $r_1 = \ldots = r_n = 1$. *Then the operator* D *defined by* (2.2) *satisfies the Leibniz condition.*

Proof. Let $x, y \in \mathrm{dom}\ D$. Clearly, $x, y \in \mathrm{dom}\ D$ whenever $x, y \in \bigcap_{j=1}^{n} \mathrm{dom}\ D_j$. Since $D_1, \ldots, D_n$ satisfy the Leibniz condition, we get

$$\begin{aligned} D(xy) &= \sum_{j=1}^{n} \alpha_j D_j(xy) = \sum_{j=1}^{n} \alpha_j (x D_j y + y D_j x) = \\ &= x \sum_{j=1}^{n} \alpha_j D_j y + y \sum_{j=1}^{n} \alpha_j D_j x = xDy + yDx. \end{aligned}$$

■

Proposition 2.4. *Suppose that all assumptions of Proposition* 2.1 *are satisfied. Let*

$$U_n = \left\{ \prod_{k=1}^{n} u_k : u_k \in \ker D_k \cap I(X) \quad (k = 1, \ldots, n) \right\} \quad (n \in \mathbb{N}). \tag{2.7}$$

Then selectors L *of the multifunction* Ω *induced by* D *satisfy the equality* $DLx = a$ *for* $x \in U_n$.

Proof. By Equation (2.4), we have $Dx = ax$, where $x \in I(X)$. Thus, by definition, $DLx = x^{-1}Dx = a$ for any selector L of Ω. ■

Propositions 2.3 and 2.4 imply

Corollary 2.1. *Suppose that all assumptions of Proposition* 2.1 *are satisfied. If* $r_1 = \ldots = r_n = 1$ *and* $U_n \in \mathbf{Lg}(D)$, *then* $U_n \in \mathsf{L}(D)$.

Proposition 2.5. *Suppose that all assumptions of Proposition* 2.1 *are satisfied and* $a = 0$. *Then there are* $R_j \in \mathcal{R}_{D_j}$ *such that* $R_j^{r_j} a_j \in \mathrm{dom}\ (\Omega_{r_j}^{(j)})^{-1}$ $(j = 1, \ldots, n)$ *and*

$$x = \frac{1}{n} \sum_{j=1}^{n} u_j \widetilde{u}_j = \frac{1}{n} \sum_{j=1}^{n} u_j E_{r_j}^{(j)}(R_j^{r_j} a_j) \in \ker D, \tag{2.8}$$

where $u_j \in \ker D_j \cap I(X)$ $(j = 1, \dots, n)$.

Proof. By our assumptions and Proposition 2.1, $Dx = ax = 0$ and $x = u_j \widetilde{u}_j$ $(j = 1, \dots, n)$. Hence $x = \frac{1}{n}\sum_{j=1}^{n} u_j \widetilde{u}_j$. This, and Proposition 2.2 together imply (2.8). ■

Proposition 2.6. *Suppose that all assumptions of Proposition* 2.1 *are satisfied and* $a \in I(X)$. *Then the equation*

$$Dx = y, \quad y \in X \tag{2.9}$$

has a solution

$$x = y\Big(\sum_{j=1}^{n} \alpha_j D_j^{r_j} L_{r_j}^{(j)} \widetilde{u}_j\Big)^{-1}. \tag{2.10}$$

Proof. By our assumptions, $ax = Dx = y$. Since $a \in I(X)$, we get $x = a^{-1}y$. Propositions 2.1 and 2.2 together imply that

$$x = a^{-1}y = y\Big(\sum_{j=1}^{n} \alpha_j a_j\Big)^{-1} = y\Big(\sum_{j=1}^{n} \alpha_j D_j^{r_j} L_{r_j}^{(j)} \widetilde{u}_j\Big)^{-1}.$$

■

Corollary 2.2. Suppose that all assumptions of Proposition 2.1 are satisfied, $r_1 = \ldots = r_n = 1$ and $a \in I(X)$. Then the equation (2.9) has a solution

$$x = y\Big(\sum_{j=1}^{n} \alpha_j D_j \sum_{k=1,\ k\neq j}^{n} L_1^{(j)} u_k\Big)^{-1}. \tag{2.11}$$

Proof. Proposition 2.6 and the Leibniz condition together imply that

$$x = y\Big(\sum_{j=1}^{n} \alpha_j D_j L_1^{(j)} \prod_{k=1,\ k\neq j}^{n} u_k\Big)^{-1} = y\Big(\sum_{j=1}^{n} \alpha_j D_j \sum_{k=1,\ k\neq j}^{n} L_1^{(j)} u_k\Big)^{-1}.$$

■

Some more generalized approaches to problems with linear combinations of right invertible operators of order one on vectors fields and magnifolds have been given by Virsik [17] and Multarzyński [3, 4].

3. Trigonometric elements and mappings

We shall show now an approach to the trigonometric identity in Leibniz D-algebras with unit e (but not necessarily with logarithms). Clearly, without additional assumptions we cannot expect too much.

Proposition 3.1. *Suppose that $X \in \mathbf{L}(D)$, $x \in \operatorname{dom} D^2$ and x, Dx are not zero divisors. If $x^2 + (Dx)^2 = e$, then*

$$\alpha x + \beta Dx \in \ker\ (D^2 + I) \quad \textit{whenever } \alpha, \beta \in \mathbb{F}. \tag{3.1}$$

Proof. Let $y = -Dx$. Then $Dy = -D^2x$ and

$$0 = De = D[x^2 + (Dx)^2] = 2xDx + 2(Dx)D^2x = 2(Dx)(x + D^2x) = 2y(x - Dy).$$

Since $y = -Dx$ is not a zero divisor, we have $x - 2y = 0$. Hence $Dy = x$ and $y = -Dx = -D^2y$, which implies $y \in \ker\ (D^2 + I)$. On the other hand, $x = Dy = -D^2x$, which implies $x \in \ker\ (D^2 + I)$. ■

Proposition 3.2. *Suppose that all assumptions of Proposition* 3.1 *are satisfied. If Condition* (3.1) *holds for x and Dx and $u = x^2 + (Dx)^2$, then $u \in \ker D$.*

Proof. Let $u = x^2 + (Dx)^2$. Then $Du = 2xDx + 2(Dx)D^2x = 2(Dx)(x + Dx) = 2(Dx)(D^2 + I)x = 0$, which implies $u \in \ker D$. ■

Corollary 3.1. *Suppose that all assumptions of Proposition* 3.1 *are satisfied, Condition* (3.1) *holds for x and Dx, $F \in \mathcal{F}_D \cap M(X)$, $Fx = e$, $FDx = 0$ and $u = x^2 + (Dx)^2$. Then $u = e$, i.e., $x^2 + (Dx) = e$.*

Proof. Since F is a multiplicative initial operator and $Fx = e$, $FDx = 0$, we find $u = F[x^2 + (Dx)^2] = (Fx)^2 + (FDx)^2 = e^2 + 0 = e$. ■

Proposition 3.3. *Suppose that all assumptions of Proposition* 3.1 *and Condition* (3.1) *are satisfied, $F \in \mathcal{F}_D \cap M(X)$ and $Fx = e$. Then $FDx = 0$.*

Proof. By our assumptions, $e = Fe = F[x^2 + (Dx)^2] = (Fx)^2 + (FDx)^2 = e + (FDx)^2$, which implies $(FDx)^2 = 0$. Hence $FDx = 0$. ■

Proposition 3.4. *Suppose that all assumptions of Proposition* 3.1 *are satisfied. If* $x_\pm \in \ker\ (D \pm iI)$ *and* $x = \frac{1}{2}(x_+ + x_-)$, $y = \frac{1}{2i}(x_+ - x_-)$, *then*

(i) $x, y \in \ker\ (D^2 + I)$, $Dx = -y$, $Dy = x$ *and* $\frac{1}{2}(x \pm y) \in \ker\ (D^2 \mp iI)$;

(ii) $x^2 + y^2 = x_+ x_- \in \ker\ D$.

Proof. Points (i) is proved by checking. In order to prove (ii), observe that, by the Leibniz condition and our assumptions,

$$D(x_+x_-) = x_+Dx_- + x_-Dx_+ = ix_+x_- - ix_+x_- = 0. \qquad \blacksquare$$

Observe that $x_\pm$ are eigenvectors of the operator D corresponding to the eigenvalues $\mp i$, respectively.

Here and in the sequel we assume that $\mathbb{F}$ is an algebraically closed field of scalars. For instance, $\mathbb{F} = \mathbb{C}$. The following results are slightly stronger (with some proofs slightly simpler than in [7]):

Definition 3.1. Let X be a linear space over $\mathbb{F}$. If $\lambda \in \mathbb{F}$ is an eigenvalue of an operator $D \in R(X)$, then every eigenvector x_λ corresponding to λ is said to be an *exponential element* (shortly: an *exponential*). $\square$

This means that x_λ is an exponential if and only if $x_\lambda \neq 0$ and $x_\lambda \in \ker(D - \lambda I)$.

Proposition 3.5. *Let* X *be a linear space* (*over* $\mathbb{F}$). *Suppose that* $D \in R(X)$. *If* $0 \neq x_\lambda \in \ker(I - \lambda R)$ *for an* $R \in \mathcal{R}_D$ *and a* $\lambda \in \mathbb{F}$, *then* $x_\lambda \in \ker(D - \lambda I)$, *i.e.,* x_λ *is an exponential.*

Proof. By our assumption, $(D - \lambda I)x_\lambda = (D - \lambda DR)x_\lambda = D(I - \lambda R)x_\lambda = 0$. $\blacksquare$

By an easy induction we get

Proposition 3.6. *Suppose that* X *is a linear space* (*over* $\mathbb{F}$), $D \in R(X)$ *and* $\{\lambda_n\} \subset \mathbb{F}$ *is a sequence of eigenvalues such that* $\lambda_i \neq \lambda_j$ *for* $i \neq j$. *Then for an arbitrary* $n \in \mathbb{N}$ *the exponentials* $x_{\lambda_1}, \ldots, x_{\lambda_n}$ *are linearly independent.*

Proposition 3.7. *Suppose that* X *is a linear space* (*over* $\mathbb{F}$), $D \in R(X)$, F *is an initial operator for* D *corresponding to an* $R \in \mathcal{R}_D$ *and* x_λ *is an exponential. Then* x_λ *is an eigenvector for* R *corresponding to the eigenvalue* $1/\lambda$ *if and only if* $Fx_\lambda = 0$, *i.e., if* R *is not a Volterra operator.*

Proof. *Sufficiency.* Since $Dx_\lambda = \lambda x_\lambda$ and $Fx_\lambda = 0$, we get $x_\lambda = x_\lambda - Fx_\lambda = (I-F)x_\lambda = RDx_\lambda = \lambda Rx_\lambda$. Hence $x_\lambda \in \ker(I-\lambda R)$. Since $x_\lambda \neq 0$, we conclude that x_λ is an eigenvector for R corresponding to $1/\lambda$.

Necessity. Suppose that $1/\lambda$ is an eigenvalue of R and the corresponding eigenvector x_λ is an exponential. Then $Fx_\lambda = (I-RD)x_\lambda = (I-\lambda R)x_\lambda = -\lambda(R-\frac{1}{\lambda}I)x_\lambda = 0$. ■

Theorem 3.1. *Suppose that X is a linear space (over $\mathbb{F}$), $D \in R(X)$, $\ker D \neq \{0\}$, $R \in \mathcal{R}_D$ and $\lambda \in v_{\mathbb{F}}R$. Then*

(i) *λ is an eigenvalue of D and the corresponding exponential is*

$$x_\lambda = e_\lambda(z), \quad \text{where } e_\lambda = (I-\lambda R)^{-1},\ z \in \ker D; \tag{3.2}$$

whenever $e_\lambda = (I-\lambda R)^{-1}$ exists, is said to be an exponential operator;

(ii) *the dimension of the eigenspace X_λ corresponding to the eigenvalue λ is equal to the dimension of the space of constants, i.e.,* $\dim X_\lambda = \dim\ker D \neq 0$*;*

(iii) *if $\lambda \neq 0$, then there exist non-trivial exponentials: $e_\lambda(z) \neq 0$;*

(iv) *exponentials are uniquely determined by their initial values, i.e., if F is an initial operator for D corresponding to R, then $F[e_\lambda(z)] = z$;*

(v) *if R is a Volterra operator, then every $\lambda \in \mathbb{F}$ is an eigenvalue of D, i.e., for every $\lambda \in \mathbb{F}$ there exist exponentials.*

Proof. (i) By definition, $(I-\lambda R)e_\lambda(z) = (I-\lambda R)(I-\lambda R)^{-1}z = z$, where $z \in \ker D$. Thus $e_\lambda(z) = z + \lambda Re_\lambda(z)$, which implies $De_\lambda(z) = Dz + \lambda DRe_\lambda(z) = \lambda e_\lambda(z)$.

(ii) Since by our assumptions, the operator $e_\lambda = I - \lambda R$ is invertible, dim X_λ=dim $\{e_\lambda(z) : z \in \ker D\}$=dim $\{(I-\lambda R)^{-1}z : \ker D\}$=dim $\ker D \neq 0$.

(iii) If $\lambda \neq 0$ and $e_\lambda(z) = (I-\lambda R)^{-1}z = 0$ then $z = (I-\lambda R)e_\lambda(z) = 0$, This contradicts our assumption that $\ker D \neq \{0\}$.

(iv) By definitions and (i), we have $Fe_\lambda(z) = (I-RD)e_\lambda(z) = (I-\lambda R)$ $e_\lambda(z) = z$.

(v) If $R \in V(X)$ then $v_{\mathbb{F}}R = \mathbb{F} \setminus \{0\}$. Clearly, for $\lambda = 0$ the operator $I - \lambda R$ is also invertible. Hence, by (i), every scalar λ is an eigenvalue of D. ■

Definition 3.2. Let $\mathbb{F} = \mathbb{C}$. Suppose that X is a linear space (over $\mathbb{C}$, $D \in R(X)$, $\ker D \neq \{0\}$ and $R \in \mathcal{R}_D \cap V(X)$. Then the operators

$$(3.3)\qquad c_\lambda = \frac{1}{2}(e_{\lambda i} + e_{-\lambda i}), \quad s_\lambda = \frac{1}{2i}(e_{\lambda i} - e_{-\lambda i}) \qquad (\lambda \in R)$$

are said to be *cosine* and *sine operators*, respectively (or: *trigonometric operators*). Elements $c_\lambda(z)$, $s_\lambda(z)$, where $z \in \ker D$, are said to be *cosine* and *sine elements*, respectively (or: *trigonometric elements*). □

Theorem 3.2. *Suppose that all assumptions of Definition* 3.2 *are satisfied. Then*

$$(3.4)\qquad c_\lambda = (I + \lambda^2 R^2)^{-1}, \quad s_\lambda = \lambda R(I + \lambda^2 R^2)^{-1} \qquad (\lambda \in \mathbb{R})$$

$$(3.5)\qquad Dc_\lambda = -\lambda s_\lambda, \quad Ds_\lambda = \lambda c_\lambda \qquad (\lambda \in \mathbb{R})$$

$$(3.6)\qquad c_0(z) = z, \quad s_0(z) = 0, \quad Fs_\lambda(z) = 0 \quad for\ z \in \ker D,\ \lambda \in \mathbb{R}.$$

Moreover, whenever $z \in \ker D$, $\lambda \in \mathbb{R}$, *the element* $c_\lambda(z)$ *is even with respect to* λ *and the element* s_λ *is odd with respect to* λ.

Proof. By the first Formula of (3.4), for $\lambda \in \mathbb{R}$ we get

$$\begin{aligned} c_\lambda &= \frac{1}{2}\left[(I - \lambda iR)^{-1} + (I + \lambda iR)^{-1}\right] \\ &= \frac{1}{2}(I - \lambda iR)^{-1}(I + \lambda iR)^{-1}(I + \lambda iR + I - \lambda iR) \\ &= \frac{1}{2}(I + \lambda^2 R^2)^{-1} 2I = (I + \lambda^2 R^2)^{-1}. \end{aligned}$$

A similar proof for s_λ. By definitions, if $\lambda \in \mathbb{R}$, then

$$\begin{aligned} Dc_\lambda &= \frac{1}{2}D(e_{\lambda i} + e_{-\lambda i}) = \frac{1}{2}(\lambda i e_{\lambda i} + \lambda i e_{-\lambda i}) \\ &= \frac{1}{2}\lambda i(e_{\lambda i} + e_{-\lambda i}) = -\frac{\lambda}{2i}(e_{\lambda i} + e_{-\lambda i}) = -\lambda s_\lambda. \end{aligned}$$

Since $DR = I$, we have $Ds_\lambda = \lambda DR(I + \lambda^2 R^2)^{-1} = \lambda(I + \lambda^2 R^2)^{-1} = \lambda c_\lambda$.

Let $z \in \ker D$. Let $\lambda = 0$. Then $c_0(z) = z$, $s_0(z) = 0$. Since $FR = 0$, for every $\lambda \in R$ we have $Fs_\lambda(z) = \lambda FR(I + \lambda^2 R^2)^{-1} = 0$. Let $z \in \ker D$. Then

$$c_{-\lambda}(z) = [I + (-\lambda)^2 R^2)^{-1}](z) = (I + \lambda^2 R^2)^{-1} z = c_\lambda(z);$$

$$s_{-\lambda}(z) = -\lambda R[I + (-\lambda)^2 R^2)^{-1}](z) = -\lambda R(I + \lambda^2 R^2)^{-1} z = -s_\lambda(z). \quad \blacksquare$$

Consider now trigonometric elements in algebras. It is easy to verify

Proposition 3.6. *Suppose that* $D \in \mathsf{A}(X) \cap R(X)$, $\ker D \neq \{0\}$ *and* $R \in \mathcal{R}_D \cap V(X)$. *Then*

$$[c_\lambda(z)]^2 + [s_\lambda(z)]^2 = e_{\lambda i}(z) e_{-\lambda i}(z) \quad \textit{for all } z \in \ker D,\ \lambda \in \mathbb{R}. \tag{3.6}$$

$$D[e_{\lambda i}(z) e_{-\lambda i}(z)] = c_D z + f_D(e_{\lambda i}(z), e_{-\lambda i z}) \quad \textit{for all } z \in \ker D,\ \lambda \in \mathbb{R}. \tag{3.7}$$

Corollary 3.2. *Suppose that* X *is a Leibniz* D*-algebra,* $\ker D \neq \{0\}$ *and* $R \in \mathcal{R}_D \cap V(X)$. *Then*

$$D[e_{\lambda i}(z) e_{-\lambda i}(z)] = z \quad \textit{for all } z \in \ker D,\ \lambda \in \mathbb{R}, \tag{3.8}$$

Proof. Since X is a Leibniz D-algebra, we have c_D=1 and $f_D = 0$. Hence Formulae (3.6) and (3.7) imply (3.8). $\blacksquare$

An immediate consequence of Corollary 3.2 is

Corollary 3.3. *Suppose that* X *is a Leibniz* D*-algebra,* $\ker D \neq \{0\}$ *and* $R \in \mathcal{R}_D \cap V(X)$. *Then the Trigonometric Identity holds, i.e.,*

$$[c_\lambda(z)]^2 + [s_\lambda(z)]^2 = z \quad \textit{for all } z \in \ker D,\ \lambda \in \mathbb{R}. \tag{3.9}$$

Proposition 3.7 (cf. [8])**.** *Suppose that* $X \in \mathbf{Lg}(D)$, $\lambda g = Re \in \operatorname{dom} \Omega^{-1}$ *for every* $R \in \mathcal{R}_D$ *and* $\lambda \in v_{\mathbb{F}} R$. *Then there are* $(L, E) \in G[\Omega]$ *such that*

$$E(\lambda g) = (I - \lambda R)^{-1} z = e_\lambda z \in \ker(D - \lambda I) \quad \textit{for all } z \in \ker D.$$

Proof. Let $R \in \mathcal{R}_D$ be fixed. Elements of the form $u = e_\lambda z = (I - \lambda R)^{-1} z$ are well-defined for all $z \in \ker D$ and $(D - \lambda I)u = D(I - \lambda R)u = Dz = 0$.

Moreover, $Du = \lambda u = u\lambda e = u\lambda DRe = uD(\lambda g)$, which implies that $\lambda g \in \operatorname{dom} \Omega^{-1}$ and there are $(L, E) \in G[\Omega]$ such that $e_\lambda = u = E(\lambda g)$. ■

Definition 3.3 (cf. [8]). Suppose that $\mathbb{F} = \mathbb{C}$, $X \in \mathbf{Lg}(D)$ and $\mathbf{E}_1 = \operatorname{dom} \Omega^{-1}$ is *symmetric*, i.e., $-x \in \mathbf{E}_1$ whenever $x \in \mathbf{E}_1$. Let $(L, E) \in G[\Omega]$. Write

$$(3.10)\quad Cx = \frac{1}{2}[E(ix) + E(-ix)], \quad Sx = \frac{1}{2i}[E(ix) - E(-ix)] \ \text{ for } \ ix \in \mathbf{E}_1.$$

The mappings C and S are said to be *cosine* and *sine* mappings or *trigonometric* mappings. Elements Cx and Sx are said to be *cosine* and *sine* elements or *trigonometric* elements. □

Clearly, trigonometric mappings and elements have such properties as the classical cosine and sine functions. Namely, we have (proofs can be found in [8]):

Proposition 3.8 (cf. [8]). *Suppose that all assumptions of Definition* 3.3 *are satisfied. Let* $(L, E) \in G[\Omega]$. *Then trigonometric mappings* C *and* S *are well-defined for all* $ix \in \mathbf{E}_1$ *and have the following properties:*

(i) *The de Moivre formulae hold:*

$$E(ix) = Cx + iSx, \quad E(-ix) = Cx - iSx \quad for\ ix \in \mathbf{E}_1.$$

In particular, if X *is a commutative Leibniz algebra then*

$$(3.11)\qquad (Cx + iSx)^n = C(nx) + iS(nx) \quad for\ ix \in \mathbf{E}_1\ and\ n \in \mathbb{N};$$

(ii) C *and* S *are even and odd functions of their argument, respectively, i.e.,* $C(-x) = Cx$, $S(-x) = -Sx$ *for* $ix \in \mathbf{E}_1$ *and* $C(0) = z \in \ker\ D \setminus \{0\}$, $S(0) = 0$. *In particular, for all* $ix \in \mathbf{E}_1$

$$(3.12)\qquad (Cx)^2 + (Sx)^2 = \frac{1}{2}[E(ix)E(-ix) + E(ix)E(ix)].$$

(iii) *The mappings* C', S' *defined as follows:* $C'x = C(x+z)$, $S'x = S(x+z)$ *for* $ix \in E_0$, $z \in \ker D$ *also satisfy assertions* (i)–(ii).

(iv) *For all* $ix \in \operatorname{dom} \Omega^{-1}$

$$(Cx)^2 + (Sx)^2 = E(ix)(E(-ix); \tag{3.13}$$

$$DCx = -(Sx)Dx, \quad DSx = (Cx)Dx. \tag{3.14}$$

Corollary 3.4 (cf. [8])**.** *Suppose that all assumptions of Proposition* 3.8 *are satisfied and X is a Leibniz D-algebra with unit e. Then the Trigonometric Identity holds, i.e.,*

$$(Cx)^2 + (Sx)^2 = e \quad \textit{whenever } ix \in \mathbf{E}_1. \tag{3.15}$$

The following question arises: Do non-Leibniz algebras with the Trigonometric Identity (3.15) exist? The answer to this question is negative, i.e., non-Leibniz algebras with the Trigonometric Identity (3.15) do not exist (cf. [11]). In other words: The Leibniz condition is necessary and sufficient for the Trigonometric Identity to hold.

In order to apply trigonometric mappings, we shall make use of the following condition:

$[\mathbf{C}]_\mathbf{n}$ $\quad \mathbb{F} = \mathbb{C}$, $n \in \mathbb{N}$ *is arbitrarily fixed,* $X \in \mathbf{Lg}(D^n)$, $\Omega_1 = \Omega$ *and* dom Ω_n^{-1} *is* symmetric, *i.e.,* $-x \in \text{dom } \Omega_n$ *whenever* $x \in \text{dom } \Omega_n$.

Suppose now that Condition $[\mathbf{C}]_\mathbf{2}$ holds and $X \in L(D)$. Suppose that $\lambda \in \mathbb{C}$, $R \in \mathcal{R}_D$, $g = Re$ and $\lambda ig \in \text{dom } \Omega_1$. If $(L_1, E_1) \in G[\Omega_1]$ and $(L_2, E_2) \in G[\Omega_2]$ then

$$\begin{aligned} \ker(D^2 + \lambda^2 I) &= \{z_1 E(\lambda ig) + z_2 E(-\lambda ig) : z_1, z_2 \in \ker D\} \\ &= \{zC(\lambda g) + \widetilde{z} S(\lambda g) : z, \widetilde{z} \in \ker D\} \\ &= \{(z'' g + z') E_2(\frac{\lambda^2 g^2}{2}) : z', z'' \in \ker D\}. \end{aligned} \tag{3.16}$$

The assumption that $\lambda i, -\lambda i \in v_{\mathbb{C}} R$ ensures that $\lambda ig, -\lambda ig \in \text{dom } \Omega_1^{-1}$. In this case, $-\lambda^2 \in v_{\mathbb{C}} R^2$.

4. Fourier-like problems for right invertible operators

We will apply properties of trigonomeric mappings and elements in order to find non-trivial solutions of some homogeneous initial and boundary value problems for difference of two right invertible operators (of the first and second order).

Proposition 4.1. *Let* $X \in \mathbf{L}(D_i)$, $\ker\ D_i \neq \{0\}$, $i = 1, 2$, $\ker\ D_1 \cap\ \ker\ D_2 = \mathbb{C}e = \{\lambda e\}_{\lambda\in\mathbb{C}}$. *Suppose that* $x = uv$, *where* $u \in \ker\ D_2$, $v \in \ker\ D_1$. *Then*

$$(4.1)\qquad (D_2 - D_1^2)x = u(D_2 + \lambda^2 I)v - v(D_1^2 + \lambda^2 I)v \quad for\ all\ \lambda \in \mathbb{C}\ \{0\}.$$

Proof. By our assumptions, $D_2u = 0$, $D_1v = 0$ and both operators D_1, D_2 satisfy the Leibniz condition. This and Leibniz Formula (1.5) together imply that

$$(D_2 - D_1^2)x = (D_2 - D_1^2)(uv) = D_2(uv) - D_1^2(uv)$$

$$= uD_2v + vD_2u - uD_1^2v - 2(D_1u)(D_1v) - vD_1^2u = uD_2v - vD_1^2u$$

$$= u(D_2v + \lambda^2 v) - \lambda^2 uv - vD_1u = u(D_2 + \lambda^2 I)v - v(D_1^2 + \lambda^2 I)u. \quad \blacksquare$$

Proposition 4.1 immediately implies

Corollary 4.1. *Let* $X \in \mathbf{L}(D_i)$, $\ker\ D_i \neq \{0\}$, $i = 1, 2$, $\ker\ D_1 \cap\ \ker\ D_2 = \mathbb{C}e = \{\lambda e\}_{\lambda\in\mathbb{C}}$. *Suppose that* $x = uv$, *where* $u \in \ker\ D_2$, $v \in \ker\ D_1$. *Then* $(D_2 - D_1^2)x = 0$ *if and only if* $u(D_2 + \lambda^2 I)v - v(D_1^2 + \lambda^2 I)v = 0$ *for all* $\lambda \in \mathbb{C}\ \{0\}$.

Corollary 4.2. *Let* $X \in \mathbf{L}(D_i)$, $\ker\ D_i \neq \{0\}$, $i = 1, 2$, $\ker\ D_1 \cap\ \ker\ D_2 = \mathbb{C}e = \{\lambda e\}_{\lambda\in\mathbb{C}}$ *and* $x = uv$, *where* $u \in \ker\ D_2$, $v \in \ker\ D_1$. *Then* $(D_2 - D_1^2)x = 0$ *if and only if* $u \in I(\ker\ D_2)$, $v \in I(\ker\ D_1)$ *and there is a* $\lambda \in \mathbb{C} \setminus \{0\}$ *such that*

$$(4.2)\qquad u^{-1}D_1^2u = vD_2v = -\lambda^2 e.$$

Proof. Equalities (4.2) hold if and only if $(D_1^2+\lambda^2 I)u = 0$, $(D_2+\lambda^2 I)v = 0$. This, and Corollary 4.1 together imply that $(D_2 - D_1^2)x = 0$ if and only if Equalities (4.2) hold. $\blacksquare$

Theorem 4.1. *Suppose that* $X \in \mathbf{L}(D_i)$, ker $D_i \neq \{0\}$, $i = 1, 2$, ker $D_1 \cap$ ker $D_2 = \mathbb{C}e = \{\lambda e\}_{\lambda\in\mathbb{C}}$ *and almost averaging* $F_0, F_1 \in \mathcal{F}_{D_1}$, $F_2 \in \mathcal{F}_{D_2}$ *correspond to* $R_0, R_1 \in \mathcal{R}_{D_1}$, $R_2 \in \mathcal{R}_{D_2}$, *respectively. Suppose, moreover, that* $x = uv$, *where* $u \in I(\text{ker } D_2)$, $v \in I(\text{ker } D_1)$ *and there is a* $\lambda \in v_{\mathbb{C}}R_0$ *such that Equalities* (4.2) *hold and a* u *such that* F_1u*=0. Then the homogeneous initial value problem*

$$(D_2 - D_1^2)x = 0, \tag{4.3}$$

with the homogeneous boundary condition

$$F_0x = 0, \quad F_1x = 0 \tag{4.4}$$

and with the homogeneous initial condition

$$F_2x = 0 \tag{4.5}$$

is ill posed, since it has a non-trivial solution $x = uv$, *where* v *is an eigenvector of* R_2 *corresponding to the eigenvalue* -λ^2.

Proof. By Corollary 4.3, elements u, v are invertible by our assumption, hence they are not zero divisors and $x = uv$ is a non-trivial solution of the equation $(D_2 - D_1^2)x = 0$. Since $\lambda \in v_{\mathbb{C}}R_0$, Equalities (4.2) imply that $v \in$ ker $(D_2 + \lambda^2 I) =$ ker $D_2(I + \lambda^2 I)R_2$, i.e., $v = -\lambda^{-2}R_2$, $u \in$ ker $(D_1^2 + \lambda^2 I)$. Since $F_0R_0 = 0$, we have $F_0u = 0$. Since $u \in$ ker D_2, $v \in$ ker D_1 and initial operators F_0, F_1, F_2 are almost averaging, we find $F_0x = F_0(uv) = vF_0u = 0$, $F_1(uv) = vF_1u = 0$, $F_2x = F_2(uv) = uF_2v = u(-\lambda{-}2)F_2R_2v = 0$ (for $F_2R_2 = 0$). ∎

Theorems 4.1 and 3.2 together imply

Corollary 4.3. *Suppose that* $X \in \mathbf{L}(D_i)$, ker $D_i \neq \{0\}$, $i = 1, 2$, ker $D_1 \cap$ ker $D_2 = \mathbb{C}e = \{\lambda e\}_{\lambda\in\mathbb{C}}$ *and almost averaging* $F_0, F_1 \in \mathcal{F}_{D_1}$, $F_2 \in \mathcal{F}_{D_2}$ *correspond to* $R_0, R_1 \in \mathcal{R}_{D_1}$, $R_2 \in \mathcal{R}_{D_2}$, *respectively. Suppose, moreover, that* $x = uv$, *where* $u \in I(\text{ker } D_2)$, $v \in I(\text{ker } D_1)$ *and there are a* $\lambda \in v_{\mathbb{C}}R_0$ *such that Equalities* (4.2) *hold and a* $z_0 \in$ ker D_1 *such that* $F_1u = F_1s_\lambda(z_0) = 0$. *Then the initial value problem* (4.3)–(4.5) *is ill posed and its non-trivial solution is* $x = uv$, *where* $u = s_\lambda(z_0)$, $s_\lambda = \lambda R_0(\lambda^2 I + R_0^2)$, $F_1u = 0$ *for a* $z_0 \in$ ker D_1, v *is an eigenvector of* R_2 *corresponding to the eigenvalue*-λ^2.

Theorem 4.2. *Suppose that* $X \in \mathrm{L}(D_1) \cap \mathrm{L}(D_2)$, *Condition* $[\mathbf{C}]_\mathbf{2}$ *is satisfied with respect to the multifunction* $\Omega_1^{(1)}$ *induced by* D_1, $(L_1^{(1)}, E_1^{(1)}) \in G[\Omega_1^{(1)}]$, $S^{(1)}$ *is a sine mapping induced by* $E_1^{(1)}$, F_0, F_1, F_2 *are almost averaging initial operators corresponding to* $R_0, R_1 \in \mathcal{R}_{D_1}$, $R_2 \in \mathcal{R}_{D_2}$, *respectively,* $g_1 = R_0 e$, *there exists a* λ *such that* $i\lambda \in v_{\mathbb{C}} R_0$, $i\lambda g_1 \in \mathrm{dom}\ (\Omega_1^{(1)})^{-1}$, $S^{(1)}(\lambda g_1) \in \ker D_2$ *and* $F_1 S^{(1)}(\lambda g_1) = 0$. *Then the initial value problem* (4.3)–(4.5) *is ill-posed and its non-trivial solution is* $x = uv$, *where* $u = S^{(1)}(\lambda g_1) \in \ker D_2$, $v \in \ker D_1$ *is an eigenvector of* R_2 *corresponding to the eigenvalue* $-\lambda^2$, *i.e.,* $0 \neq v \in \ker(I + \lambda^2 R_2)$.

Proof. Let $x = uv$. Then, by our assumptions, $(D_2 + \lambda^2 I)v = D_2(I + \lambda^2 R_2)v = 0$. Since both operators D_1 and D_2 satisfy the Leibniz condition and $u \in \ker D_2$, we can apply Corollary 4.1 in a similar way, as before. Since F_0, F_1, F_2 are almost averaging and $F_0 E_1^{(j)}(\pm i\lambda g_1) = e$ (cf. [8]), we find

$$F_0 x = F_0(uv) = vF_0 u = vF_0 S^{(1)}(\lambda g_1) = v\frac{1}{2i}F_0\left[E^{(1)}(i\lambda g_1) - E^{(1)}(-i\lambda g_1)\right]$$

$$= \frac{1}{2i}v\left[F_0 E^{(1)}(i\lambda g_1) - F_0 E^{(1)}(-i\lambda g_1)\right] = \frac{1}{2i}v(e - e) = 0;$$

$$F_1 x = F_1(uv) = vF_1 u = vF_1 S^{(1)}(\lambda g_1) = 0;$$

$$F_2 x = F_2(uv) = uF_2 v = uF_2(-\lambda^2 R_2 v) = -\lambda^2 u F_2 R_2 v = 0. \qquad \blacksquare$$

Theorem 4.3. *Suppose that* $X \in \mathrm{L}(D_1) \cap \mathrm{L}(D_2)$, *Condition* $[\mathbf{C}]_\mathbf{2}$ *is satisfied with respect to the multifunction* $\Omega_1^{(1)}$ *induced by* D_1, $(L_1^{(1)}, E_1^{(1)}) \in G[\Omega_1^{(1)}]$, $C^{(1)}$ *is a cosine mapping induced by* $E_1^{(1)}$, F_0, F_1, F_2 *are almost averaging initial operators corresponding to* $R_0, R_1 \in \mathcal{R}_{D_1}$, $R_2 \in \mathcal{R}_{D_2}$, *respectively,* $g_1 = R_0 e$, *there exist a* λ *such that* $i\lambda \in v_{\mathbb{C}} R_0$, $i\lambda g_1 \in \mathrm{dom}\ (\Omega_1^{(1)})^{-1}$, $C^{(1)}(\lambda g_1) \in \ker D_2$ *and* $F_0 C^{(1)}(\lambda g_1) = 0$. *Then the initial value problem* (4.3), (4.4),

$$F_1 D x = 0 \tag{4.6}$$

is ill-posed and its non-trivial solution is $x = uv$, *where* $u = C^{(1)}(\lambda g_1) \in \ker D_2$, v *is an eigenvector of* R_2 *corresponding to the eigenvalue* $-\lambda^2$, *i.e.,* $0 \neq v \in \ker(I + \lambda^2 R_2)$.

Proof. Let $x = uv$. Then, by our assumptions, $u \in \ker D_2$, $DC^{(1)}(\lambda g_1) = -\lambda S^{(1)}(\lambda g_1)$. Thus, in a similar manner as in the proof of Theorem 4.1, we prove that $(D_2 - D_1^2)x = 0$, $F_0 x = 0$, $F_2 x = 0$. Condition $F_1 x = 0$ follows from the fact that (as before) F_1 is almost averaging, hence $F_1 x = vF_1 u = vF_1 C^{(1)}(\lambda g_1) = \frac{1}{\lambda} vF_1 DS^{(1)}(\lambda) = 0$. ∎

Corollary 4.4. *Suppose that all assumptions of Theorem* 4.2 *are satisfied and* $F_1 = F_0$, *hence also* $R_1 = R_0$. *Then equation* (4.3) *has a non-trivial solution* $x = uv$, *where* $u = C^{(1)}(\lambda g_1) \in \ker D_2$, v *is an eigenvector of* R_2 *corresponding to the eigenvalue* $-\lambda^2$. *This solution satisfies the homogeneous initial conditions*

$$F_0 x = 0, \quad F_0 Dx = 0, \quad F_2 x = 0. \tag{4.7}$$

Hence the problem (4.3), (4.7) *is ill-posed.*

Theorem 4.4. *Suppose that* $X \in L(D_1) \cap L(D_2)$. *Condition* $[\mathbf{C}]_\mathbf{2}$ *is satisfied with respect to the multifunction* $\Omega_1^{(2)}$ *induced by* D_2, $(L_1^{(2)}, E_1^{(2)}) \in G[\Omega_1^{(2)}]$, $S^{(2)}$, $C^{(2)}$ *are sine and cosine mappings induced by* $E_1^{(2)}$, F_0, F_1, F_2 *are almost averaging initial operators corresponding to* $R_0, R_1 \in \mathcal{R}_{D_1}$, $R_2 \in \mathcal{R}_{D_2}$, *respectively,* $g_2 = R_2 e$, *there exist a* λ *such that* $i\lambda \in v_{\mathbb{C}} R_2$, $-\lambda^2 g_2 \in$ dom $(\Omega_1^{(2)})^{-1}$, $z_0, z_1 \in \ker D_1$, $z_2 \in \ker D_2 \cap I(X)$ *and*

$$u = z_0 S^{(2)}(\lambda g_2) + z_1 C^{(2)}(\lambda g_2) \in I(X). \tag{4.8}$$

If $v = z_2 E_1^{(2)}(-\lambda^2 g_2) \in \ker D_1$ *then* $x = uv \in I(X)$ *is a non-trivial solution of Equation* (4.3).

Proof. By the Leibniz condition, $v = z_2 E_1^{(2)}(-\lambda^2 g_2) \in I(X)$. Then

$$(D_2 + \lambda^2 I)v = D_2[z_2 E_1^{(2)}(-\lambda^2 g_2) + \lambda^2 v]$$

$$= -\lambda^2 z_2 E_1^{(2)}(-\lambda g_2) D_2 R_2 e + \lambda^2 v = -\lambda^2 v + \lambda^2 v = 0.$$

By Formulae (3.16), $u \in \ker(D_2^2 + \lambda^2 I)$. In a similar manner, as in the proof of Corollary 4.1, we get

$$(D_1 - D_2^2)x = (D_1 - D_2^2)(uv) = uD_2 v - vD_1^2 u = u(-\lambda^2 v) - u(-\lambda^2 v) = 0.$$

∎

Theorem 4.5. *Suppose that* $X \in L(D_1) \cap L(D_2)$*, Condition* $[\mathbf{C}]_{\mathbf{2}}$ *is satisfied with respect to the multifunctions* $\Omega_1^{(i)}$ *induced by* D_i*,* $(L_1^{(i)}, E_1^{(i)}) \in G[\Omega_1^{(i)}]$*,* $S^{(i)}$ *are sine mappings induced by* $E_1^{(i)}$ $(i = 1, 2)$*,* F_0, F_1, F_2, F_3 *are almost averaging initial operators corresponding to* $R_0, R_1 \in \mathcal{R}_{D_1}$*,* $R_2, R_3 \in \mathcal{R}_{D_2}$*, respectively,* $g_1 = R_0 e$*,* $g_2 = R_2 e$*, there exist a* λ *such that* $i\lambda \in v_{\mathbb{C}} R_0 \cap v_{\mathbb{C}} R_2$*,* $i\lambda g_i \in \mathrm{dom}\ (\Omega_1^{(i)})^{-1}$*,* $S^{(i)}(\lambda g_i) \in \ker D_j$ $(j \neq i;\ i, j = 1, 2)$ *and* $F_1 S^{(1)}(\lambda g_1) = 0$*,* $F_3 S^{(2)} S(\lambda g_2) = 0$*. Then the equation*

$$(D_1^2 - D_2^2)x = 0 \tag{4.9}$$

has a non-trivial solution $x = uv$*, where* $u = S^{(1)}(\lambda g_1) \in \ker D_2$*,* $v = S^{(2)}(\lambda g_2) \in \ker D_1$*. This solution satisfies the homogeneous boundary conditions*

$$F_0 x = 0, \quad F_1 x = 0, \quad F_2 x = 0, \quad F_3 x = 0. \tag{4.10}$$

Hence the problem (4.9)–(4.10) *is ill-posed.*

Proof. Let $x = uv$. By definition, $D_1^2 u = -\lambda^2 u$, $D_2^2 v = -\lambda^2 v$. Hence, in a similar way, as in the proof of Corollary 4.1, we find

$$(D_1^2 - D_2^2)x = (D_2^1 - D_2^2)(uv) = vD_1^2 u - uD_2^2 v = -\lambda^2 uv + \lambda^2 uv = 0.$$

By our assumptions, $F_1 u = 0$, $F_3 v = 0$ and $u \in \ker D_2$, $v \in \ker D_1$. Since F_1 and F_3 are almost averaging, we get $F_1 x = F_1(uv) = vF_1 u = 0$, $F_3 x = F_3(uv) = uF_3 v = 0$. As in the proof of Theorem 4.1, we find $F_0 x = F_0(uv) = vF_0 u = vF_0 S^{(1)}(\lambda g_1) = 0$. Similarly, $F_2 x = F_2(uv) = uF_2 v = F_2 S^{(2)}(\lambda g_2) = 0$. ∎

Theorem 4.6. *Suppose that* $X \in L(D_1) \cap L(D_2)$*, Condition* $[\mathbf{C}]_{\mathbf{2}}$ *is satisfied with respect to the multifunction* $\Omega_1^{(1)}$ *induced by* D_1*,* $(L_1^{(1)}, E_1^{(1)}) \in G[\Omega_1^{(1)}]$*,* $S^{(1)}$ *is a sine mapping induced by* $E_1^{(1)}$*,* F_0, F_1, F_2, F_3 *are almost averaging initial operators corresponding to* $R_0, R_1 \in \mathcal{R}_{D_1}$*,* $R_2, R_3 \in \mathcal{R}_{D_2}$*, respectively,* $g_1 = R_0 e$*, there exists a* λ *such that* $i\lambda \in v_{\mathbb{C}} R_0$*,* $i\lambda g_1 \in \mathrm{dom}\ (\Omega_1^{(1)})^{-1}$*,* $u = S^{(1)}(\lambda g_1) \in \ker D_2$*,* $F_1 S^{(1)}(\lambda g_1) = 0$ *and* $0 \neq v \in \ker D_1$ *is an eigenvector of the operator* $R_2 R_3$ *corresponding to the eigenvalue* $-\lambda^2$*. Then Equation* (4.9) *has a non-trivial solution* $x = uv$ *which satisfies the homogeneous* mixed *boundary conditions*

$$F_0 x = 0, \quad F_1 x = 0, \quad F_2 x = 0, \quad F_3 D_2 x = 0. \tag{4.11}$$

Hence the problem (4.9), (4.11) *is ill-posed.*

Proof. Following the proofs of Theorems 4.1 and 4.2, we prove that $x = uv$ and $F_0x = F_1x = 0$. By our assumptions, $D_2u = 0$, $F_2R_2 = 0$, $F_3R_3 = 0$, $v = -\lambda^2 R_2R_3$, hence $D_2v = -\lambda^2 R_3v$. Since F_2 and F_3 are almost averaging, we get

$$F_2x = F_2(uv) = uF_2v = -\lambda^2 uF_2R_2R_3v = 0,$$

$$F_3D_2x = F_3D_2(uv) = F_3(uD_2v + vD_2u)$$

$$= F_3(uD_2v) = uF_3D_2v = -\lambda^2 uF_3R_3v = 0. \qquad \blacksquare$$

Corollary 4.5. *Suppose that all assumptions of Theorem* 4.5 *are satisfied and* $F_3 = F_2$*, hence* $R_3 = R_2$*. Then Equation* (4.9) *has a non-trivial solution* $x = uv$*, where* $0 \neq v \in \ker D_1$ *is an eigenvector of the operator* R_2^2 *corresponding to the eigenvalue* $-\lambda^2$*. This solution satisfies the homogeneous boundary conditions*

$$F_0x = 0, \quad F_1x = 0 \tag{4.12}$$

and the homogeneous initial conditions

$$F_2x = 0, \quad F_2D_2x = 0. \tag{4.13}$$

Hence the problem (4.9), (4.12), (4.13) *is ill-posed.*

We should point out that we do not assume any right inverse under consideration to be a Volterra operator.

Under appropriate assumptions the *Sylvester inertia law* holds in algebras with logarithms (cf. [14]). We therefore can say that Equation (4.3) is *parabolic-like* and Equation (4.9) is *hyperbolic-like* whenever X is an algebra with logarithms. Indeed, these equations have forms $(D_2 - D_1^2)x = 0$, $(D_2^2 - D_2^1)x = 0$ of the classical canonic parabolic and hyperbolic equations, respectively (cf. [5]).

Clearly, a linear combination of solutions x_{λ_n} of any problem considered above corresponding to the eigenvalues λ_n, is again a solution of that problem. Even more. Consider Equation (4.3). If we are given $\lambda_j \in v_{\mathbb{C}}R_2$ such that $\lambda_j \neq \lambda_k$ for $j \neq k$ $(j, k = 1, \ldots, n;\ n \in \mathbb{N})$ then the corresponding eigenvectors v_{λ_j} $(j = 1, \ldots, n)$ are linearly independent and a linear combination

$$x = \sum_{j=1}^{n} \alpha_j x_{\lambda_j} = \sum_{j=1}^{n} \alpha_j u_{\lambda_j} v_{\lambda_j},$$

where $\alpha_j \in \mathbb{C}$, $x_\lambda = u_{\lambda_j} v_{\lambda_j}$, $u_{\lambda_j} \in \ker\ (D_1^2 + \lambda_j^2)$ $(j = 1, \ldots, n)$, is again a solution of Equation (4.3). A similar conclusion can be obtained for Equation (4.9).

Throughout the paper we have assumed several times that $F_1 u$=0, where u was a sine element. However, under appropriate assumptions (X is a complete linear space over $\mathbb{C}$, D is closed) in a complex extension of X, exponentials, sine and cosine elements are $2\pi e$-periodic:

$$E[i(x + 2\pi e)] = E(ix),\ C(x + 2\pi e) = Cx,\ S(x + 2\pi e) = Sx,$$

whenever these elements are well-defined (cf. [8], Chapter 9). If it is the case, we conclude that

$$F_1 u = F_1 S(\lambda g) = F_1 S(\lambda g + 2\pi e) = F_1 S(\lambda g) = S(0) = 0,$$

whenever $g = Re$, $R \in \mathcal{R}_D$, $\lambda \in v_{\mathbb{C}} R$.

References

[1] A. Favini, *Sum of operators' method in abstract equations*, in: Partial Differential Equations, Progr. Nonlinear Differential Equations Appl. **22** (Burkhäuser, Boston, 1996) 132–146.

[2] V. Lovass-Nagy and D.L. Powers, *On under- and over-determined initial value problems*, Int. J. Control **19** (1974), 653–656.

[3] P. Multarzyński, *On some right invertible operators in differential spaces*, Demonstratio Math. **37** (4) (2004), 905–920.

[4] P. Multarzyński, *On divided difference operators in function algebras*, Demonstratio Math. **41** (2) (2008), 273–289.

[5] W. Pogorzelski, Integral Equations and Their Applications, 1-st Polish ed. Vol. I-1953, Vol. II-1958, Vol. III-1960; PWN-Polish Scientific Publishers, Warszawa; English ed. Pergamon Press and PWN-Polish Scientific Publishers, Oxford-Warszawa, 1966.

[6] D. Przeworska-Rolewicz, *Remarks on boundary value problems and Fourier method for right invertible operators*, Math. Nachrichten **72** (1976), 109–117.

[7] D. Przeworska-Rolewicz, Algebraic Analysis, D. Reidel and PWN-Polish Scientific Publishers, Dordrecht-Warszawa, 1988.

[8] D. Przeworska-Rolewicz, Logarithms and Antilogarithms. An Algebraic Analysis Approach. With Appendix by Z. Binderman, Kluwer Academic Publishers, Dordrecht, 1998.

[9] D. Przeworska-Rolewicz, *Linear combinations of right invertible operators in commutative algebras with logarithms*, Demonstratio Math. **31** (4) (1998), 887–898.

[10] D. Przeworska-Rolewicz, *Postmodern Logarithmo-technia*, Computers and Mathematics with Applications **41** (2001), 1143–1154.

[11] D. Przeworska-Rolewicz, Non-Leibniz algebras with logarithms do not have the trigonometric identity, in: Algebraic Analysis and Related Topics, Proc. Intern. Conf. Warszawa, September 21-25, 1999. Banach Center Publications, 53. Inst. of Math., Polish Acad. of Sci., Warszawa, 2000, 177–189.

[12] D. Przeworska-Rolewicz, *Algebraic Analysis in structures with Kaplansky-Jacobson property*, Studia Math. **168** (2) (2005), 165–186.

[13] D. Przeworska-Rolewicz, *Some summations formulae in commutative Leibniz algebras with logarithms*, Control and Cybernetics **36** (3) (2007), 841–857.

[14] D. Przeworska-Rolewicz, *Sylvester inertia law in commutative Leibniz algebras with logarithms*, Demonstratio Math. **40** (2007), 659–669.

[15] D. Przeworska-Rolewicz, Nonlinear separable equations in linear spaces and commutative Leibniz algebras. Preprint 691, Institute of Mathematics, Polish Academy of Sciences, Warszawa, September 2008; http//www.impan.pl/Preprints/p691; Annales Polon. Math. (to appear).

[16] D. Przeworska-Rolewicz, Fourier-like methods for equations with separable variables. Preprint 693, Institute of Mathematics, Polish Academy of Sciences, Warszawa, October 2008; http//www.impan.pl/Preprints/p693.

[17] G. Virsik, *Right inverses of vector fields*, J. Austral. Math. Soc. (Series A), **58** (1995), 411–420.

Received 3 March 2009