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VECTOR AND OPERATOR VALUED MEASURES AS CONTROLS FOR INFINITE DIMENSIONAL SYSTEMS: OPTIMAL CONTROL

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Abstract

In this paper we consider a general class of systems determined by operator valued measures which are assumed to be countably additive in the strong operator topology. This replaces our previous assumption of countable additivity in the uniform operator topology by the weaker assumption. Under the relaxed assumption plus an additional assumption requiring the existence of a dominating measure, we prove some results on existence of solutions and their regularity properties both for linear and semilinear systems. Also presented are results on continuous dependence of solutions on operator and vector valued measures, and other parameters determining the system which are then used to prove some results on control theory including existence and necessary conditions of optimality. Here the operator valued measures are treated as structural controls. The paper is concluded with some examples from classical and quantum mechanics and a remark on future direction.

Keywords: evolution equations, Banach spaces, operator valued measures, strong operator topology, existence of solutions, optimal control.

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1. Introduction

In a recent paper [2] the author studied systems governed by operator valued measures under the assumption that they are countably additive in the uniform operator topology. The objective of this paper is to relax this assumption by use of countable additivity in the strong operator topology in place of countable additivity in the uniform operator topology. This is a far reaching generalization since it also admits operators which have bounded semivariation rather than bounded variation. These results are expected to be useful in structural control theory as developed in [1]. Here we have dealt with the question of existence of optimal operator valued measures and vectors and also presented some necessary conditions of optimality.

Closely related to the problems studied here is the identification (inverse) problem where one is required to find the best operator valued function (not measure) that minimizes the identification criterion. For such problems the reader may like to refer to the monograph [5].

2. PRELIMINARIES

Function Spaces: Let $I \equiv [0,T], T < \infty$, be an interval and let Σ denote the sigma algebra of subsets of the set I and suppose that E is a Banach space. Let $B(I,E)$ denote the space of bounded Σ measurable functions on I with values in E. Furnished with the sup norm topology, this is a Banach space. We are also interested in the Banach space $L_1(I,E)$ of all Lebesgue-Bochner integrable functions on I with values in E.

Let χ_σ denote the characteristic function of the set $\sigma \in \Sigma$. A function $f \in B(I,E)$ is said to be a simple function if there exists a finite integer n and a finite family of pairwise disjoint members $\{\sigma_i\}_{i=1}^n$ of Σ satisfying $\bigcup_{i=1}^n \sigma_i = I$ and elements $\{e_i\} \in E$ so that f has the representation

$$f(t) = \sum_{i=1}^{n} \chi_{\sigma_i}(t) e_i.$$

We may denote the class of simple functions by $\mathcal{S}(I,E)$ and note that it is a dense (sup norm topology) subspace of $B(I,E)$.

Vector Measures: Let $\mathcal{M}_c(\Sigma, F)$ denote the space of bounded countably additive vector measures defined on the sigma algebra Σ with values in the Banach space F. For each $\mu \in \mathcal{M}_c(\Sigma, F)$, we write

$$|\mu| \equiv |\mu|(I) \equiv \sup_{\pi} \left\{ \sum_{\sigma \in \pi} \| \mu(\sigma) \|_F \right\}$$

where the supremum is taken over all partitions π of the interval I into a finite number of disjoint members of Σ. Furnished with this norm topology (total variation norm), $\mathcal{M}_c(\Sigma, F)$ is a Banach space. For any $\sigma \in \Sigma$, define the variation of μ on σ by

$$V(\mu)(\sigma) \equiv V(\mu, \sigma) \equiv |\mu|(\sigma).$$

Since μ is countably additive and bounded, this defines a countably additive bounded positive measure on Σ see [8, Proposition 9, p. 3]. In case $F = R$, the real line, we have the space of real valued signed measures. We denote this simply by $\mathcal{M}_c(\Sigma)$ in place of $\mathcal{M}_c(\Sigma, R)$. Clearly, for $\nu \in \mathcal{M}_c(\Sigma)$, $V(\nu)$ is also a countably additive bounded positive measure.

Operator Valued Measures: Let E and F be any pair of Banach spaces and $\mathcal{L}(E, F)$ the space of bounded linear operators from E to F. Suppose $\mathcal{L}(E, F)$ is furnished with the uniform operator topology. We may denote this by $\mathcal{L}_u(E, F) \equiv (\mathcal{L}(E, F), \tau_u)$. This is a Banach space. Similarly, we let $\mathcal{L}_s(E, F) \equiv (\mathcal{L}(E, F), \tau_s)$ denote the space of bounded linear operators furnished with the strong operator topology. This is a locally convex sequentially complete linear topological vector space. This follows from the uniform boundedness principle and Banach-Steinhaus theorem.

A set function Φ mapping Σ to $\mathcal{L}(E, F)$ is said to be an operator valued measure if for each $\sigma \in \Sigma$, $\Phi(\sigma) \in \mathcal{L}(E, F)$ and $\Phi(\emptyset) = 0$ the zero operator. We may denote by $\mathcal{M}_c(\Sigma, \mathcal{L}_u(E, F))$ the space of countably additive (in the sense of uniform operator topology) operator valued measures having bounded total variation. That is $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_u(E, F))$ if for every disjoint family $\{\sigma_i\} \subset \Sigma$ we have

$$M\Big(\bigcup \sigma_i\Big) = \sum M(\sigma_i),$$

in the sense that

$$\lim_{n\to\infty} \Big\| M\Big(\bigcup \sigma_i\Big) - \sum_{i=1}^{n} M(\sigma_i) \Big\|_{\mathcal{L}(E,F)} = 0$$

and it has bounded variation, that is $\overline{M}(\sigma) < \infty$, for every $\sigma \in \Sigma$ where

$$\overline{M}(\sigma) \equiv \sup_{\pi} \Big\{ \sum_{\Delta \in \pi} \| M(\sigma \cap \Delta) \|_{\mathcal{L}(E,F)} \Big\}$$

with the supremum taken over all finite partitions π of I by disjoint members of Σ. If $\overline{M}(I) < \infty$, the measure is said to have bounded (total) variation. It is not difficult to verify that this defines a norm. Since $\mathcal{L}(E,F)$ is a Banach space with respect to the uniform operator topology, $\mathcal{M}_c(\Sigma, \mathcal{L}_u(E,F))$ is also a Banach space with respect to the total variation norm. It is known that [see 8] if M has bounded variation, then the measure induced by the variation $\overline{M}(\cdot)$ is countably additive if and only if $M(\cdot)$ is countably additive. Clearly, in this situation $\nu_M(\cdot) \equiv \overline{M}(\cdot)$ is a countably additive bounded positive measure.

For operator valued measures this is a rather strong topology. For broader applicability we need a weaker topology. This involves the strong operator topology τ_s and what is known as the semi variation as defined below. Let $\mathcal{M}_c(\Sigma, \mathcal{L}_s(E,F))$ denote the space of $\mathcal{L}(E,F)$ valued measures which are countably additive with respect to the strong operator topology in the sense that the vector measure

$$\sigma \longrightarrow M(\sigma)e$$

is countably additive on Σ for each $e \in E$. The semivariation of $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E,F))$ on a set $\sigma \subset I, \sigma \in \Sigma$, is given by the following expression:

$$\hat{M}(\sigma) \equiv \sup\left\{ \Big\| \sum_{i=1}^{n} M(\sigma \cap \sigma_i)e_i \Big\|_F, \sigma_i \cap \sigma_j = \emptyset, \sigma_i \in \Sigma, e_i \in B_1(E), n \in N \right\},$$

where $B_1(E)$ denotes the closed unit ball in E centered at the origin. Clearly, it follows from the definitions that

$$\hat{M}(K) \leq \overline{M}(K), \ \forall \ K \in \Sigma. \tag{1}$$

It is not difficult to construct examples of operator valued measures which have finite semivariation but infinite variation. A simple example is given by the operator determined by the tensor product $M(\cdot) \equiv \mu(\cdot) \otimes e^*$ where μ is an F-valued vector measure having finite semi variation but infinite total variation and $e^* \in E^*$. Since $M(\cdot)e = \mu(\cdot)(e^*, e)_{E^*,E}$, it is clear that $M(\sigma) \in \mathcal{L}(E,F)$, for all $\sigma \in \Sigma$ and that it has finite semi variation.

Note that $\hat{M}(\cdot)$ can also be computed as follows

$$\hat{M}(\sigma) = \sup\left\{ \Big\| \int_\sigma M(ds)f(s) \Big\|_F : f \in \mathcal{S}(I,E), \| \ f \ \|_{B(I,E)} \leq 1 \right\}, \sigma \in \Sigma.$$

Clearly, it follows from this [9, Theorem 14] that

$$(2) \qquad \Big\| \int_\sigma M(ds)f(s) \Big\|_F \leq \hat{M}(\sigma) \parallel f \parallel_{B(\sigma,E)} \quad \forall\, \sigma \in \Sigma.$$

Define the semivariation of M on I by

$$\hat{M}(I) = \sup\{\hat{M}(\sigma),\, \sigma \in \Sigma\}.$$

Thus if the semivariation is finite, each $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E,F))$ determines a bounded linear operator L from $B(I,E)$ to F given by

$$Lf \equiv \int_I M(ds)f(s).$$

However, the converse is false. According to the generalized Riesz representation theorem [16, Theorem 2.2, Theorem 2.6, Corollary 2.6.1, Theorem 2.7], the representing measure m_L corresponding to a bounded linear operator L from $C(I,E)$ to F is a finitely additive measure in the strong operator topology with values $m_L(\sigma) \in \mathcal{L}(E, F^{**}), \sigma \in \Sigma$. Here we are not interested in the converse. Readers interested in the converse may find many interesting results in [16, 13].

Recall that a set function $\eta : \Sigma \longrightarrow [0,\infty]$ is called a submeasure if (i): $\eta(\emptyset) = 0$, (ii): it is monotone, that is, for every $G_1, G_2 \in \Sigma$ and $G_1 \subset G_2$, $\eta(G_1) \leq \eta(G_2)$, and (iii): it is subadditive (superadditive) if for every pair of disjoint sets $\sigma_1, \sigma_2 \in \Sigma$, $\eta(\sigma_1 \cup \sigma_2) \leq \eta(\sigma_1) + \eta(\sigma_2)$ $(\eta(\sigma_1 \cup \sigma_2) \geq \eta(\sigma_1) + \eta(\sigma_2))$.

It is easy to verify that the set function $\hat{M}(\cdot)$ induced by the semivariation of the operator valued measure M is a nonnegative finitely subadditive extended real valued set function mapping Σ to $[0,\infty]$. In fact, the following stronger result is known and its proof can be found in [2].

Lemma 2.1. *The set function $\hat{M}(\cdot)$ induced by the semivariation of any*

$$M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E,F))$$

is a monotone countably subadditive nonnegative extended real valued set-function. In contrast, the set function $\overline{M}(\cdot)$ induced by the variation of any $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_u(E,F))$ is a countably additive nonnegative extended real valued measure.

Proof. See [2].

In the sequel we need the celebrated Vitali-Hahn-Saks-Nikodym theorem [11, Theorem IV.10.6, p. 321], [10, Lemma 1.3, p. 247] and [8, Theorem 1.4.8, p. 23]. For easy reference we quote it here.

Lemma 2.2 (Vitali-Hahn-Saks-Nikodym). *Let $\{m_n\}$ denote a sequence of vector valued set functions mapping Σ into E, and suppose that, for each $\Gamma \in \Sigma$, $\lim_{n\to\infty} m_n(\Gamma) = m(\Gamma)$ exists. If m_n is countably additive for each n then so is m; the sequence $\{m_n\}$ is uniformly countably additive and converges uniformly on Σ to m.*

For $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E,F))$ having bounded semivariation on I, and any $f \in B(I,E)$, the measure γ defined by

$$\gamma(\cdot) \equiv \int_{(\cdot)} M(ds)f(s)$$

is a vector valued set function $\gamma : \Sigma \longrightarrow F$. Since M is countably additive in the strong operator topology having bounded semivariation on I and $f \in B(I,E)$, the measure γ is a well defined countably additive F valued bounded vector measure.

The following result shows that the set function determined by the semivariation of a strongly countably additive operator valued measure admits an extended real valued positive countably additive dominating measure. It follows from a fundamental result of the classical measure theory given in Munroe [14, Theorem 11.2, p. 87].

Lemma 2.3. *For every $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E,F))$ having bounded semivariation, there exists a sigma algebra Ξ of subsets of the set I and a countably additive positive measure μ_M on Ξ having bounded variation on I such that $\hat{M}(\sigma) \le \mu_M(\sigma)$ for all $\sigma \in \Xi$.*

Proof. Let $\mathcal{C}$ denote any sequential covering class of subsets of the set I and $\hat{M}$ the set function determined by the semivariation of the operator valued measure $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E,F))$ with $\hat{M}(\emptyset) = 0$. For each $D \subset I$, define

$$\mu_M^*(D) \equiv \inf\left\{ \sum_{i=1}^{\infty} \hat{M}(D_i) : D_i \in \mathcal{C}, \bigcup D_i \supset D \right\}.$$

If for a given D no such covering exists we set $\mu_M^*(D) = \infty$. By Lemma 2.1, $\hat{M}$ is a monotone, nonnegative, extended real valued, countably subadditive set function on Σ with $\hat{M}(\emptyset) = 0$. Hence μ_M^* is a well defined nonnegative extended real valued countably subadditive set function defined on all subsets D of I with $\mu_M^*(\emptyset)$. It is easy to verify that μ_M^* is an outer measure (see Munroe [14, p. 85]). Let Ξ denote the class of all μ_M^* measurable sets from I. Then it follows from Munroe [14, Theorem 11.2, p. 87], that Ξ is a completely additive class and that there exists a countably additive measure μ_M on Ξ such that $\mu_M^*(\Gamma) = \mu_M(\Gamma)$ for all $\Gamma \in \Xi$. In other words, the restriction of $\mu_M^*(\cdot)$ on Ξ is a countably additive measure. Recall that the union of a countable sequence of sets can be described by the union of a countable sequence of disjoint sets. Thus without loss of any generality we may assume the sequence $\{D_i\}$ to be disjoint. Since $\hat{M}$ is monotone and countably subadditive we have

$$\hat{M}(D) \leq \hat{M}\Big(\bigcup D_i\Big) \leq \sum_{i=1}^{\infty} \hat{M}(D_i)$$

for any (disjoint) family $\{D_i\} \in \mathcal{C}$ covering the set D. Taking infimum on the righthand side of the above expression over the covering class, we obtain $\hat{M}(D) \leq \mu_M^*(D)$ for all $D \subset I$. Hence, restricted to Ξ, we have $\hat{M}(\sigma) \leq \mu_M^*(\sigma) = \mu_M(\sigma)$. Since M has finite semivariation, the outer measure μ_M^* can be constructed so as to have finite variation and hence μ_M has finite variation. This completes the proof. ■

Now we are prepared to undertake the study of dynamic systems and their control.

3. Existence of solutions

In this section, we consider the question of existence of solutions and their regularity properties.

3.1. Linear system

Consider the linear system

(3) $\quad dx = Axdt + M(dt)x(t-),\ x(0) = x(0+) = \xi,\ t \in I \equiv [0, T],$

on a Banach space E where A is the generator of a C_0 semigroup on E and M is an operator valued measure. Throughout the rest of the paper we assume that E is a reflexive Banach space. Under fairly general assumptions on the operator valued measure M, we prove that the Cauchy problem has a unique solution.

Theorem 3.1. *Consider the system* (3) *and suppose A is the infinitesimal generator of a C_0 semigroup $S(t), t \geq 0$, on the Banach space E. Let $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ be an operator valued measure countably additive in the strong operator topology having bounded semivariation on I and there exists a countably additive bounded positive measure μ_M having bounded variation on I such that $\hat{M}(\sigma) \leq \mu_M(\sigma)$ for all $\sigma \in \Sigma$. Then for each $\xi \in E$, the evolution equation* (3) *has a unique mild solution $x \in B(I, E)$.*

Proof. Using the variation of constants formula we have

$$x(t) = S(t)\xi + \int_0^t S(t-s)M(ds)x(s-) \equiv (Gx)(t),\ t \in I. \tag{4}$$

We prove the existence by showing that the operator G as defined above has a unique fixed point in $B(I, E)$. First, note that G maps $B(I, E)$ to $B(I, E)$. Indeed, since I is a bounded interval, there exists a finite positive number $\mathfrak{S}$ such that

$$\sup\{\| S(t) \|_{\mathcal{L}(E)},\ t \in I\} \leq \mathfrak{S}.$$

Consider the linear operator L determined by the expression

$$(Lf)(t) \equiv \int_0^t S(t-r)M(dr)f(r),\ t \in I. \tag{5}$$

Since the operator valued measure M is assumed to have bounded semivariation, and the semigroup is bounded by $\mathfrak{S}$ on the interval I, it is easy to verify that for $f \in B(I, E)$, $Lf \in B(I, E)$. Thus for any fixed but arbitrary $t \in I$, $(Lf)(t) \in E$ and, as a consequence of the Hahn-Banach theorem, there exists an $e_0^* \in B_1(E^*)$, possibly dependent on t, such that

$$\| (Lf)(t) \|_E = \langle e_0^*, (Lf)(t) \rangle_{E^*,E} = \int_0^t \langle S^*(t-r)e_0^*, \nu_f(dr) \rangle \tag{6}$$

where

$$\nu_f(\sigma) \equiv \int_\sigma M(ds)f(s),\ \sigma \in \Sigma. \tag{7}$$

Since M has finite semivariation and $f \in B(I, E)$, it is clear that the measure ν_f is a countably additive bounded E-valued vector measure and it follows from the above identity and the inequality (2) that

$$\| \nu_f(\sigma) \|_E \leq \hat{M}(\sigma) \| f \|_\sigma, \forall \sigma \in \Sigma \tag{8}$$

where $\| f \|_\sigma \equiv \sup\{\| f(s) \|_E, s \in \sigma\}$. In general, the adjoint semigroup is only w^*-continuous. However, since E is assumed to be a reflexive Banach space it is strongly continuous. Thus the function $g_t(s) \equiv S^*(t-s)e_0^*, s \in I_t \equiv [0,t]$ is continuous and it follows from the boundedness of the semigroup on I that it is an element of $C(I_t, E^*)$ for all $t \in I$. For convenience of notation, let us denote the expression (6) by

$$Z_t \equiv \| (Lf)(t) \|_E = \int_{I_t} \langle g_t(r), \nu_f(dr) \rangle_{E^*,E}. \tag{9}$$

Let $\{\sigma_i, i = 0, 1, \ldots, n-1\}$ be any partition of the interval I_t by disjoint members of Σ, in particular $\sigma_i \equiv (t_i, t_{i+1}]$, $t_0 = 0$ and $t_n = t$. Since $g_t(\cdot)$ is an E^*-valued continuous function defined on I_t, the integral (9) denoted by Z_t can be approximated by the sum

$$Z_t^n \equiv \sum_{i=0}^{n-1} \langle g_t(t_i), \nu(\sigma_i) \rangle_{E^*,E}$$

and one can easily verify that $Z_t^n \to Z_t$ as $n \to \infty$. Instead of the lower sum one could also use the upper sum

$$\tilde{Z}_t^n \equiv \sum_{i=0}^{n-1} \langle g_t(t_{i+1}), \nu(\sigma_i) \rangle_{E^*,E}$$

to arrive at the same conclusion. Since $g_t(\cdot)$ is bounded by $\mathfrak{S}$ it follows from this that

$$Z_t^n \leq \sum_{i=0}^{n-1} \| g_t(t_i) \|_{E^*} \| \nu(\sigma_i) \|_E \leq \mathfrak{S} \sum_{i=0}^{n-1} \| \nu(\sigma_i) \|_E . \tag{10}$$

Using the inequality (8) into (10) and our assumption on the existence of the dominating measure μ_M, that is $\hat{M}(\sigma) \leq \mu_M(\sigma)$ for all $\sigma \in \Sigma$, we obtain

$$Z_t^n \leq \mathfrak{S} \sum_{i=0}^{n-1} \| \nu(\sigma_i) \|_E \leq \mathfrak{S} \sum_{i=0}^{n-1} \| f \|_{\sigma_i} \hat{M}(\sigma_i) \leq \mathfrak{S} \sum_{i=0}^{n-1} \| f \|_{\sigma_i} \mu_M(\sigma_i). \tag{11}$$

In view of this inequality it follows from (9) that for each $t \in I$

$$\| (Lf)(t) \|_E \leq \mathfrak{S} \int_0^t \| f(s) \| \mu_M(ds), \; t \in I. \tag{12}$$

Thus, for any $f \in B(I, E)$, it follows from the definition of the operator G and the above inequality that

$$\| (Gf)(t) \| \leq \mathfrak{S} \| \xi \| + \mathfrak{S} \int_0^t \| f(s-) \|_E \, \mu_M(ds).$$

Since $t \in I$ is arbitrary and μ_M has bounded total variation on I, it follows from this inequality that

$$\sup \{ \| (Gf)(t) \|_E, t \in I \} \; \leq \; \mathfrak{S} \| \xi \| + \mathfrak{S} \| f \|_{B(I,E)} \, \mu_M(I) < \infty.$$

For $x, y \in B(I, E)$ with $x(0) = y(0) = \xi$, define

$$\rho_t(x, y) \equiv \sup \{ \| x(s) - y(s) \|_E, 0 \leq s \leq t \}, \tag{13}$$

and $\rho(x, y) \equiv \rho_T(x, y)$. Clearly, it follows from the definition of the operator G and the inequality (12) that

$$\| (Gx)(t) - (Gy)(t) \| \leq \; \mathfrak{S} \int_0^t \| x(s-) - y(s-) \|_E \, \mu_M(ds), t \in I.$$

Using the definition of $\rho_t, t \in I$, as given by (13) and noting that it is a nondecreasing (bounded measurable) function of t on I, it follows from this inequality that

$$\rho_t(Gx, Gy) \leq \mathfrak{S} \int_0^t \rho_s(x, y) \mu_M(ds). \tag{14}$$

Define the function V_M by $V_M(t) \equiv \mu_M((0, t+]), t \in I$. Since μ_M is a nonnegative measure having bounded variation on bounded sets, V_M is a nonnegative nondecreasing right continuous function of bounded variation on I. Thus the inequality (14) can be written as

$$\rho_t(Gx, Gy) \leq \mathfrak{S} \int_0^t \rho_s(x, y) \; dV_M(ds) \tag{15}$$

which after one iteration gives

$$\rho_t(G^2x, G^2y) \leq \mathfrak{S} \int_0^t \rho_s(Gx, Gy)\ dV_M(ds). \tag{16}$$

Substituting equation (15) into equation (16) and repeating this process n times we arrive at the following inequality

$$\rho_t(G^n x, G^n y) \leq \big((\mathfrak{S} V_M(t))^n/n!\big)\rho_t(x, y), t \in I, \tag{17}$$

which, in turn, leads to the following inequality

$$\rho(G^n x, G^n y) \leq \Big(\mathfrak{S}^n (V_M(T))^n/n!\Big)\rho(x, y). \tag{18}$$

From this last expression it is clear that for n sufficiently large G^n is a contraction and hence by the Banach fixed point theorem G^n has a unique fixed point in $B(I, E)$ which is also the unique fixed point of G itself. This proves the existence of a unique mild solution of the evolution equation (3). ■

Remark 3.2. As a corollary of the above result we can conclude that to each pair $\{A, M(\cdot)\}$ satisfying the hypothesis of the above theorem there corresponds a unique strongly measurable (strong operator topology) evolution operator $U(t, s), 0 \leq s < t < \infty$ so that the mild solution of

$$dx = Axdt + M(dt)x(t-), x(s+) = \xi, t > s$$

is given by $x(t) \equiv U(t, s)\xi$. If M has no atom at the point $\{s\}$, we have $x(s+) = x(s) = x(s-)$. On the other hand, if it has an atom at this point, we have $\xi = x(s+) = (I + M(\{s\}))x(s-)$.

Remark 3.3. If the operator valued measure $M(\cdot)$ has a countable set of atoms, the solution x is piecewise continuous and if it is free of atoms, the solution is continuous.

Remark 3.4. The assumption that the semivariation $\hat{M}$ is dominated by a countably additive bounded positive measure is not too restrictive. For example, $\hat{M}(\sigma) \leq \overline{M}(\sigma), \forall \sigma \in \Sigma$, and $\overline{M}(\cdot)$ is a countably additive positive measure (possibly unbounded, that is, $\overline{M}(I) = \infty$ while $\hat{M}(I) < \infty$).

This shows that the set

$$\mathcal{M} \equiv \Big\{ M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E)) : \ \exists \ \mu_M \in \mathcal{M}_c^+(\Sigma) \text{ satisfying } \hat{M}(\sigma) \leq \mu_M(\sigma) \leq \overline{M}(\sigma) \ \forall \ \sigma \in \Sigma \Big\} \neq \emptyset.$$

The validity of the above assumption is also justified by Lemma 2.3.

3.2. Nonlinear system

Here we consider a class of nonlinear systems. Let E and F be any pair of Banach spaces and consider the system

$$dx = Axdt + M(dt)x + f(t,x)dt + g(t,x)\nu(dt), x(0) = x_0 \tag{19}$$

on the Banach space E where A is the generator of a C_0-semigroup in E and $M(\cdot)$ is an operator valued measure, $f : I \times E \longrightarrow E$ and ν is an F valued vector measure and $g : I \times E \longrightarrow \mathcal{L}(F,E)$.

Theorem 3.5. *Suppose A generates a C_0-semigroup $S(t)$, $t \geq 0$ on E and $M \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ having bounded semivariation and admitting a dominating measure $\mu_M \in \mathcal{M}_c^+(\Sigma)$. Let F be another Banach space and $\nu \in \mathcal{M}_c(\Sigma, F)$ a countably additive F-valued measure having bounded total variation and the nonlinear operators $\{f, g\}$ are Borel measurable in both the variables on $I \times E$ with $f : I \times E \longrightarrow E$ and $g : I \times E \longrightarrow \mathcal{L}(F,E)$ satisfying the following conditions: there exist $K \in L_1^+(I)$ and $L \in L_1^+(I, |\nu|)$ so that*

$$\begin{aligned} &\| f(t,x) \|_E \leq K(t)[1+ \| x \|_E], \\ &\| g(t,x) \|_{\mathcal{L}(F,E)} \leq L(t)[1+ \| x \|_E] \ \forall \ x \in E, \end{aligned} \tag{20}$$

and for every positive number $r < \infty$, there exist $K_r \in L_1^+(I)$ and $L_r \in L_1^+(I, |\nu|)$ such that

$$\| f(t,x) - f(t,y) \|_E \leq \ K_r(t)[\, \| x - y \|_E \,], \ \forall \ x, y \in B_r(E) \tag{21}$$

$$\| g(t,x) - g(t,y) \|_{\mathcal{L}(F,E)} \leq \ L_r(t)[\, \| x - y \|_E \,] \ \forall \ x, y \in B_r(E). \tag{22}$$

Then for each initial state $x_0 \in E$, the system (19) *has a unique mild solution $x \in B(I, E)$.*

Proof. Again the proof is based on the Banach fixed point theorem. We present only a brief outline. Using the growth assumption (20) one can establish an a-priori bound. Indeed, if $x \in B(I,E)$ is any solution of equation (19) then it must satisfy the following integral equation

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-r)M(dr)x(r-) \\ &\quad + \int_0^t S(t-s)f(s,x(s))ds + \int_0^t S(t-r)g(r.x(r-))\nu(dr). \end{aligned} \tag{23}$$

This is equivalent to the fixed point problem $x = Gx$ in the Banach space $B(I,E)$ where G denotes the integral operator determined by the right-hand expression of equation (23). Using this expression and the dominating measure μ_M, one can easily deduce that

$$\| x \|_{B(I,E)} \leq C \exp\{\mathfrak{S}\tilde{\mu}_M(I)\},$$

where

$$C \equiv \mathfrak{S}\Big(\| x_0 \| + \int_I K(s)ds + \int_I L(s)|\nu|(ds)\Big)$$

and

$$\tilde{\mu}_M(\sigma) \equiv \mu_M(\sigma) + \int_\sigma K(\theta)d\theta + \int_\sigma L(\theta)|\nu|(d\theta),\ \sigma \in \Sigma.$$

Since the vector measure ν has bounded variation, and $K \in L_1^+(I)$, $L \in L_1^+(I,|\nu|)$ and, by assumption, μ_M is countably additive with bounded variation, it follows from the preceding expression that $\tilde{\mu}_M$ is a countably additive bounded positive measure with $\tilde{\mu}_M(I) < \infty$. Hence there exists a finite positive number r so that

$$\| x \|_{B(I,E)} \leq C \exp\{\mathfrak{S}\tilde{\mu}_M(I)\} \leq r$$

and so if x is any solution, $x(t) \in B_r(E)$ for all $t \in I$. Using this a-priori bound and the local Lipschitz properties (21)–(22) one can easily prove, as in the linear case, that for a sufficiently large integer n the operator G^n is a contraction on $B(I,E)$. Thus by the Banach fixed point theorem, G^n and hence G has a unique fixed point $x \in B(I,E)$. This ends the proof. ■

3.3. Continuous dependence of solutions

In the study of structural control theory it is essential to establish continuous dependence of solutions: $M \longrightarrow x(M)$. Since continuity is dependent on the topology, it is important to introduce an appropriate topology on $\mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ which is able to cover a range of applications. We consider sequential convergence and denote the topology by τ_v.

Definition 3.6. A sequence $M_n \in M_c(\Sigma, \mathcal{L}_s(E))$ is said to converge in the τ_v topology to $M_0 \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$, denoted by $M_n \xrightarrow{\tau_v} M_0$, if for every $f \in B(I, E)$ and every $K \in \Sigma$

$$\int_K M_n(ds) f(s) \xrightarrow{s} \int_K M_0(ds) f(s) \text{ in } E.$$

Now we consider the question of continuous dependence.

Theorem 3.7. *Consider the linear system* (3) *with A being the infinitesimal generator of a C_0 semigroup $S(t)$, $t \geq 0$, in E and $\xi \in E$. Then the solution map $M \longrightarrow x(M)$ is continuous with respect to the τ_v topology on $\mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ and sup-norm topology on $B(I, E)$.*

Proof. Let $\{x_n, x_o\} \in B(I, E)$ denote the solutions of equation (3) corresponding to $\{M_n, M_o\} \subset \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ respectively and suppose $M_n \xrightarrow{\tau_v} M_o$. Computing the difference we have

$$x_n(t) - x_o(t) = e_n(t) + \int_0^t S(t-r) M_n(dr)\{x_n(r-) - x_o(r-)\},\ t \in I,$$

where

$$e_n(t) \equiv \int_0^t S(t-r)\big(M_n(dr) - M_o(dr)\big) x_o(r-).$$

Define

$$\Psi_n(t) \equiv \sup\{\| x_n(s) - x_o(s) \|_E,\ 0 \leq s \leq t\}$$

and

$$\hat{e}_n(T) \equiv \sup\{\| e_n(t) \|_E,\ t \in I\}.$$

Using the above expressions one can easily verify that

$$\Psi_n(t) \le \hat{e}_n(T) + \mathfrak{S} \int_0^t \Psi_n(r) \hat{M}_n(dr)$$

where $\hat{M}_n(\cdot)$ is the submeasure induced by the semivariation of M_n. By virtue of Lemma 2.3, there exists a countably additive bounded positive μ_{M_n} dominating $\hat{M}_n$. Thus the above inequality can be replaced by the following inequality

$$\Psi_n(t) \le \hat{e}_n(T) + \mathfrak{S} \int_0^t \Psi_n(r) \mu_{M_n}(dr). \tag{24}$$

Using generalized Gronwall inequality [15, Lemma 5] it follows from this that

$$\Psi_n(t) \le \hat{e}_n(T) \exp\{\mathfrak{S} \mu_{M_n}(I)\},\ t \in I.$$

Since $M_n \xrightarrow{\tau_v} M_o$ and $\mathcal{L}_s(E)$ is a sequentially complete locally convex topological vector space (thanks to the uniform boundedness principle and Banach-Steinhaus theorem) $\sup_n \hat{M}_n(I) < \infty$. Hence there exists a finite positive number β such that $\mu_{M_n}(I) \le \beta$ for all $n \in N$ and therefore,

$$\Psi_n(t) \le (\exp\{\mathfrak{S}\beta\}) \hat{e}_n(T),\ \forall\, t \in I.$$

Define the family of measures

$$\gamma_n(\sigma) \equiv \int_\sigma M_n(dr) x_o(r-), \qquad \gamma_o(\sigma) \equiv \int_\sigma M_o(dr) x_o(r-),\ \sigma \in \Sigma.$$

Since $x_o \in B(I, E)$, and M_n is countably additive in the strong operator topology, $\{\gamma_n\}$ is a sequence of countably additive E valued vector measures. Thus it follows from τ_v convergence of M_n to M_o that for every $K \in \Sigma$, $\gamma_n(K) \xrightarrow{s} \gamma_o(K)$ as $n \to \infty$. Then by the Vitali-Hahn-Sacks-Nikodym theorem [Lemma 2.2], γ_o is countably additive and γ_n converges to γ_o uniformly on Σ, that is,

$$\lim_{n\to\infty} \big\{\sup\{\| \gamma_n(\sigma) - \gamma_o(\sigma) \|_E \colon \sigma \in \Sigma\}\big\} = 0.$$

Then it follows from the boundedness and strong continuity of the semigroup $S(t), t \in I$, that

$$e_n(t) \equiv \int_0^t S(t-r)\{\gamma_n(dr) - \gamma_o(dr)\} \stackrel{s}{\longrightarrow} 0, \text{ in } E$$

uniformly in t on I. Thus we have $\hat{e}_n(T) \to 0$ as $n \to \infty$ and hence $\lim_{n\to\infty} \Psi_n(t) = 0$ uniformly in $t \in I$. In other words, $x_n \equiv x(M_n) \to x(M_o) \equiv x_o$ in $B(I,E)$. This completes the proof. ∎

We can prove a similar continuous dependence of solutions for the nonlinear system (19).

Theorem 3.8. *Consider the nonlinear system* (19) *and suppose the assumptions of Theorem* 3.5 *hold. Then the solution map* $M \longrightarrow x(M)$ *is continuous with respect to the* τ_v *topology on* $\mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ *and sup-norm topology on* $B(I,E)$.

Proof. As the proof is quite similar to that of Theorem 3.7, we present only a brief outline. Let $M_n \stackrel{\tau_v}{\longrightarrow} M_o$ and let $\{x_n, x_o\} \subset B(I,E)$ denote the solutions of (19) corresponding to $\{M_n, M_o\}$ respectively. Since a τ_v convergent sequence $\{M_n\} \subset \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ is bounded having bounded semivariation with the limit $M_o \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$, it follows from the growth assumption (20) that the sequence of solutions $x_n \in B(I,E)$, corresponding to $\{M_n\}$, is bounded and hence there exists a finite positive number r such that $x_n(t), x_o(t) \in B_r(E)$ for all $t \in I$. Based on this fact one can use the local Lipschitz properties (21)–(22) and the definitions for $\Psi_n(t)$ and $\hat{e}_n(T)$, as in the linear case (Theorem 3.7), to arrive at the following inequality

$$\Psi_n(t) \le \hat{e}_n(T) + \mathfrak{S} \int_0^t \Psi_n(\tau)\tilde{\mu}_n(d\tau),\ t \in I, \tag{25}$$

where

$$\tilde{\mu}_n(\sigma) \equiv \mu_{M_n}(\sigma) + \int_\sigma K_r(\theta)d\theta + \int_\sigma L_r(\theta)|\nu|(d\theta),\ \sigma \in \Sigma,$$

and μ_{M_n} is as defined in Theorem 3.7. This inequality is similar to the inequality (24) as in Theorem 3.7. Since the measure μ_{M_n} and the measure $|\nu|(\cdot)$, induced by the variation of the countably additive measure ν, are countably additive having bounded total variation, we have $\tilde{\mu}_n$ a countably additive (positive) measure having bounded total variation. From here on, following similar steps as in Theorem 3.7, we arrive at the conclusion. ∎

Remark 3.9. In fact, under some additional assumptions such as weak convergence of vector measures, we can prove continuous dependence of solutions of the system (19) with respect to all the parameters $\{M, \nu\}$.

Remark 3.10. Preceding results are based on the assumption that E is a reflexive Banach space. In particular, this assumption was used to prove the basic Theorem 3.1 by exploiting the fact that under this assumption the adjoint semigroup is also strongly continuous. This allows one to approximate the integral (9) by Rieman-Stiltjes sums. However, this is not essential if one assumes that E^* is separable. In that case, $s \to g_t(s) \equiv S^*(t-s)e_0^*, s \in I_t$, is strongly measurable by the Pettis measurability theorem and hence it follows from the boundedness of the semigroup on I that $g_t \in B(I_t, E^*)$. This is all that is necessary to derive the inequality (12).

4. Optimal control

We consider two classes of control problems. The first consists of controls which are vector measures $\nu \in \mathcal{U}_{ad} \subset \mathcal{M}_c(\Sigma, F)$. This type of control problems have been studied by the author in [7] without involving structural controls. The other class consists of structural controls where the admissible controls are operator valued measures $M \in \mathcal{B}_{ad} \subset \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$. This class of control problems have been studied also recently by the author in [2] under the assumption that the operator valued measures are countably additive with respect to the uniform operator topology having bounded variation. Here we relax this assumption to include operator valued measures which are only countably additive with respect to the strong operator topology having finite semivariation.

4.1. Vector measures as controls

We consider a control problem for the nonlinear system (19) with the vector measure ν being the control. Let F be a Banach space and $\mathcal{M}_c(\Sigma, F)$ denote the space of F-valued countably additive bounded vector measures on $\Sigma \equiv \sigma(I)$ having bounded variations on I. Let $\mathcal{U}_{ad} \subset \mathcal{M}_c(\Sigma, F)$ denote the class of admissible controls. The objective is to find a control from the admissible class that minimizes the cost functional given by

$$J(\nu) \equiv \int_I \ell(t, x(t))dt + \Phi(|\nu|) \tag{26}$$

where $x \in B(I,E)$ is the (mild) solution of equation (19) corresponding to the control $\nu \in \mathcal{U}_{ad}$, and ℓ and Φ are suitable functions to be defined shortly. Our objective is to present sufficient conditions that guarantee the existence of optimal control.

Theorem 4.1. *Consider the control system* (19) *with the operators* $\{A, M(\cdot), f, g\}$ *satisfying the basic assumptions of Theorem* 3.5. *Further, suppose* g *is uniformly Lipschitz with respect to the second argument and for each* $y \in B(I,E)$ *and* $t \in I$, $G_y(t) \equiv g(t, y(t))$ *is a compact operator valued function with values in* $\mathcal{L}(F,E)$ *with both* F *and its dual* F^* *satisfying the Radon-Nikodym property. Suppose* $\ell : I \times E \longrightarrow R$ *is measurable in the first argument and lower semicontinuous in the second and there exists a* $c \in R$ *such that* $\ell(t,\xi) \geq c$ *for all* $(t,\xi) \in I \times E$; *and* $\Phi : [0,\infty] \longrightarrow [0,\infty]$ *is an extended real valued nondecreasing continuous function. Then, if* $\mathcal{U}_{ad}$ *is a weakly compact subset of* $\mathcal{M}_c(\Sigma, F)$, *there exists an optimal control.*

Proof. Suppose $\mathcal{U}_{ad} \subset \mathcal{M}_c(\Sigma, F)$ is weakly compact and let $\{\nu_n\} \subset \mathcal{U}_{ad}$ be a minimizing sequence and $\{x_n\} \subset B(I,E)$ the corresponding sequence of solutions of the evolution equation (19) for $\nu = \nu_n$. Clearly, by definition,

$$\lim_{n \to \infty} J(\nu_n) = \inf \{J(\nu) : \nu \in \mathcal{U}_{ad}\} \equiv m.$$

Since $\ell \geq c$ and Φ is nonnegative it is clear that $m > -\infty$. Thus the infimum exists and hence the problem is to show that the infimum is attained on $\mathcal{U}_{ad}$. Since $\mathcal{U}_{ad}$ is weakly compact, there exists a subsequence of the given sequence, relabeled as the original sequence, and an element $\nu_o \in \mathcal{U}_{ad}$, such that

$$\nu_n \xrightarrow{w} \nu_o \quad \text{in} \quad \mathcal{M}_c(\Sigma, F). \tag{27}$$

We prove that $\nu \longrightarrow J(\nu)$ is weakly lower semicontinuous, that is,

$$J(\nu_o) \leq \liminf_{n \to \infty} J(\nu_n),$$

whenever $\nu_n \xrightarrow{w} \nu_o$. Let x_o denote the (mild) solution of equation (19) corresponding to the control measure ν_o. First, we show that $x_n \xrightarrow{s} x_o$ in $B(I,E)$. Since $\mathcal{U}_{ad}$ is weakly compact, it follows from the Bartle-Dunford-Schwartz theorem [8, Theorem 5, p. 105] that there exists a countably additive bounded nonnegative measure μ such that

$$\lim_{\mu(\sigma)\to 0} \nu(\sigma) = 0 \ \text{ uniformly in } \quad \nu \in \mathcal{U}_{ad}.$$

From this result and the assumption that $\{F, F^*\}$ satisfy the Radon-Nikodym property, we conclude that every $\nu \in \mathcal{U}_{ad}$ has a Radon-Nikodym derivative with respect to the measure μ and it is given by $d\nu = f d\mu$ for some $f \in L_1(\mu, F)$. We have used $L_1(\mu, F)$ to denote the class of Lebesgue-Bochner μ integrable F valued functions. Thus there exists an isometric isomorphism Υ of $\mathcal{U}_{ad}$ onto a subspace of $L_1(\mu, F)$. Since compactness is preserved under isomorphism, $\Upsilon(\mathcal{U}_{ad})$ is a weakly compact subset of $L_1(\mu, F)$. Using the expression (23) corresponding to ν_n and ν_o and subtracting one from the other we have

$$\begin{aligned}[x_n(t) - x_o(t)] &= \int_0^t S(t-r)M(dr)[x_n(r-) - x_o(r-)] \\ &+ \int_0^t S(t-s)[f(s, x_n(s)) - f(s, x_o(s)]ds \\ &+ \int_0^t S(t-r)[g(r, x_n(r-)) - g(s, x_o(r-)]\nu_n(dr) \\ &+ \int_0^t S(t-r)g(s, x_o(r-)[\nu_n(dr) - \nu_o(dr)]. \end{aligned} \tag{28}$$

Again, defining $\Psi_n(t) \equiv \sup\{\| x_n(s) - x_o(s) \|_E, 0 \le s \le t\}$ and the measures α_n by

$$\alpha_n(\sigma) \equiv \mu_M(\sigma) + \int_\sigma K_r(s)ds + L|\nu_n|(\sigma),\ \sigma \in \Sigma \tag{29}$$

and the function

$$e_n(t) \equiv \int_0^t S(t-s)G_{x_o}(s)(\nu_n - \nu_o)(ds),\ t \in I, \tag{30}$$

where $G_{x_o}(t) \equiv g(t, x_o(t))$, it follows from the above expression that

$$\Psi_n(t) \le \hat{e}_n(t) + \int_0^t \mathfrak{S}\Psi_n(r)\alpha_n(dr),\ t \in I$$

where $\hat{e}_n(t) \equiv \sup\{|e_n(s)|_E, 0 \le s \le t\}$. Hence by virtue of the generalized Gronwall inequality [15, Lemma 5] we obtain

$$\Psi_n(t) \le \hat{e}_n(T) \exp\left\{\mathfrak{S}\alpha_n(I)\right\}. \tag{31}$$

In view of the isomorphism mentioned above, for every pair $\{\nu_n, \nu_o\} \in \mathcal{U}_{ad}$, there exists a pair $\{h_n, h_o\} \in L_1(\mu, F)$ so that $h_n \stackrel{w}{\longrightarrow} h_o$ in $L_1(\mu, F)$ whenever $\nu_n \stackrel{w}{\longrightarrow} \nu_0$ in $\mathcal{M}_c(\Sigma, F)$. Thus the expression (30) is equivalent to the following expression

$$e_n(t) \equiv \int_0^t S(t-s) G_{x_o}(s)(h_n(s) - h_o(s))\mu(ds),\ t \in I. \tag{32}$$

Since $x_o \in B(I, E)$, by hypothesis $G_{x_o}(s), s \in I$, is a compact operator valued function with values in $\mathcal{L}(F, E)$. Clearly, for any $e^* \in B_1(E^*)$, we have

$$\langle\, e^*, e_n(t)\,\rangle_{E^*,E} = \int_I \langle\, \chi_{[0,t]}(s) G^*_{x_o}(s) S^*(t-s)e^*, h_n(s) - h_o(s)\,\rangle_{F^*,F}\, \mu(ds)$$

where χ_σ denotes the characteristic function of the set σ. Since the multiplication of a compact operator by a bounded operator is compact and the adjoint of a compact operator is compact, it follows from the above expression that $< e^*, e_n(t) > \longrightarrow 0$ uniformly with respect to $e^* \in B_1(E^*)$ and $t \in I$. The reader can easily verify this by assuming the contrary and noting that this contradicts weak convergence of h_n to h_o in $L_1(\mu, F)$. Hence

$$e_n(t) \stackrel{s}{\longrightarrow} 0 \ \text{ in } \ E$$

uniformly on I. Since $\{\nu_n\} \subset \mathcal{U}_{ad}$ and $\mathcal{U}_{ad}$ is weakly compact, $\sup\{|\nu_n|(I), n \in \mathcal{N}\} < \infty$, where $\mathcal{N}$ denotes the set of nonnegative integers. Thus it follows from the inequality (29) that there exists a finite positive number b such that $\sup\{|\alpha_n|(I),\, n \in \mathcal{N}\} \le b < \infty$ and hence it follows from (31) and the uniform convergence of e_n to zero that $\Psi_n(t) \longrightarrow 0$ uniformly on I. In other words, $x_n(t) \stackrel{s}{\longrightarrow} x_o(t)$ in E uniformly on I. Now it follows from the lower semicontinuity of ℓ in the second argument that

$$\ell(t, x_o(t)) \le \liminf_{n\to\infty} \ell(t, x_n(t)) \ \text{ for } \ a.a \ \ t \in I.$$

Hence by use of Fatou's Lemma one can verify that

$$\int_I \ell(t, x_o(t))dt \leq \liminf_{n\to\infty} \int_I \ell(t, x_n(t))dt. \tag{33}$$

Considering the second part of the objective functional (26), and recalling that a norm is a weakly lower semicontinuous functional, we have

$$|\nu_o| \leq \liminf_{n\to\infty} |\nu_n|.$$

Since Φ is a nonnegative, monotone, nondecreasing continuous function, it follows from the above inequality that

$$\Phi(|\nu_o|) \leq \liminf_{n\to\infty} \Phi(|\nu_n|). \tag{34}$$

Combining (33) and (34) we have

$$J(\nu_0) \leq \liminf_{n\to\infty} J(\nu_n). \tag{35}$$

Hence

$$J(\nu_o) \leq \liminf_{n\to\infty} J(\nu_n) \leq \lim_{n\to\infty} J(\nu_n) = m.$$

Since $\nu_o \in \mathcal{U}_{ad}$ and m is the infimum of J on it, $m \leq J(\nu_o)$. From these facts we conclude that ν_o is an optimal control proving existence. ■

Remark 4.2. A special case of the system (19), where the control appears linearly, is given by

$$dx = Axdt + M(dt)x(t-) + f(t,x)dt + \Gamma(t)\nu(dt), x(0) = x_0$$

where $\Gamma(t) \in \mathcal{L}(F, E)$ is a compact operator valued function. Clearly, Theorem 4.1 holds for this case.

4.2. Structural control

Here we consider a structural control problem for the system (19) where ν is fixed and the operator valued measure $M(\cdot)$ is the control or the decision variable. Let $\mathcal{B}_{ad} \subset \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ denote the admissible set and suppose it is furnished with the τ_v topology introduced in Definition 3.6. The problem is to find a $M_o \in \mathcal{B}_{ad}$ that imparts a minimum to the cost functional given by

$$J(M) \equiv \int_I \ell(t,x)dt + \Phi(M), \tag{36}$$

where $x \in B(I,E)$ is the solution corresponding to M. The existence of an optimal control is proved in the following theorem.

Theorem 4.3. *Consider the control system* (19) *with the parameters* $\{A, M, f, g, \nu\}$ *satisfying the basic assumptions of Theorem* 3.5. *Suppose the admissible set* $\mathcal{B}_{ad}$ *is sequentially compact in the* τ_v *topology, the cost integrand* $\ell : I \times E \longrightarrow R$ *is measurable in the first argument and lower semicontinuous in the second and there exists a* $c \in R$ *such that* $\ell(t,\xi) \geq c$ *for all* $(t,\xi) \in I \times E$; *and* $\Phi : \mathcal{M}_c(\Sigma, \mathcal{L}_s(E)) \longrightarrow [0,\infty]$ *is an extended real valued function lower semicontinuous in the* τ_v *topology. Then there exists an optimal structural control minimizing the cost functional* (36).

Proof. Since ℓ is bounded below and Φ is nonnegative, $J(M) > -\infty$ and hence the infimum exists, that is, $Inf\{J(M), M \in \mathcal{B}_{ad}\} \equiv m > -\infty$. We show that the infimum is attained in $\mathcal{B}_{ad}$. Let $\{B_n\} \subset \mathcal{B}_{ad}$ be a minimizing sequence, that is,

$$\lim_{n\to\infty} J(M_n) = m.$$

By virtue of τ_v compactness of the set $\mathcal{B}_{ad}$, there exists a subsequence of the sequence $\{M_n\}$, relabeled as the original sequence, and an element $M_o \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ such that

$$M_n \xrightarrow{\tau_v} M_o.$$

Let $\{x_n\}$ and x_o denote the solutions of the system (19) corresponding to the sequence $\{M_n\}$ and M_o respectively. Then it follows from the continuous dependence result given by Lemma 3.8 that, along a subsequence if necessary,

$$x_n \xrightarrow{s} x_o \text{ in } B(I,E).$$

Thus $x_n(t) \xrightarrow{s} x_o(t)$ in E for each $t \in I$ and hence by the lower semicontinuity of ℓ in the second argument, we have

$$\ell(t, x_o(t)) \leq \liminf_{n\to\infty} \ell(t, x_n(t) \text{ for } a.a \ \ t \in I. \tag{37}$$

Since Φ is lower semicontinuous in the τ_v topology, we also have

$$\Phi(M_o) \leq \liminf_{n\to\infty} \Phi(M_n). \tag{38}$$

Using (36),(37) and (38) and Fatou's Lemma one can easily verify that

$$J(M_o) \leq \liminf_{n\to\infty} J(M_n). \tag{39}$$

Since $\mathcal{B}_{ad}$ is τ_v compact, it is τ_v closed and hence $M_o \in \mathcal{B}_{ad}$ and consequently $m \leq J(M_o)$. Thus it follows from (39) that

$$m \leq J(M_o) \leq \liminf_{n\to\infty} J(M_n) \leq \lim_{n\to\infty} J(M_n) = m.$$

This proves that the infimum is attained on $\mathcal{B}_{ad}$. In other words, an optimal structural control exists. ∎

Remark 4.4. An example of a functional Φ that satisfies the hypothesis of the above theorem is given by any cylindrical function of the form,

$$\Phi(M) \equiv q(M(\varphi_1), M(\varphi_2), \ldots, M(\varphi_r)) \tag{40}$$

where

$$q : E^r \to [0, \infty]$$

is an extended real valued lower semicontinuous function defined on the Cartesian product of r copies of the Banach space E,

$$M(\varphi_k) \equiv \int_I M(ds)\varphi_k(s), k = 1, 2, \ldots, r$$

and $\varphi_k \in B(I, E)$ for each k.

5. NECESSARY CONDITIONS OF OPTIMALITY

Here we present two sets of necessary conditions of optimality for control problems; one with controls which are vector measures, and the other with controls which are operator valued measures (structural control).

5.1. Vector measure as control

We present necessary conditions of optimality for the control problem (26) subject to the dynamics described by the evolution equation (19). For this we need some additional regularity for the parameters $\{f, g, \ell\}$.

$\underline{\mathbf{H}(\mathbf{f},\mathbf{g},\ell)}$: The parameters $\{f, g, \ell\}$ are Frechet differentiable in the second argument and, for each $r > 0$, there exists a positive constant b_r such that the Frechet derivatives $\{f_x, g_x, \ell_x\}$ satisfy the following bounds

$$\| f_x(t,\xi) \|_{\mathcal{L}(E)} \leq b_r \quad \forall \quad (t,\xi) \in I \times B_r(E) \tag{41}$$

$$\| g_x(t,\xi)e \|_{\mathcal{L}(F,E)} \leq b_r |e|_E \quad \forall \quad (t,\xi) \in I \times B_r(E), e \in E \tag{42}$$

$$\| \ell_x(t,\xi) \|_{E^*} \leq b_r \quad \forall \quad (t,\xi) \in I \times B_r(E). \tag{43}$$

Note that under the above assumptions we have $g_x(t,\xi) \in \mathcal{L}(E, \mathcal{L}(F,E))$ and $g_x^*(t,\xi) \in \mathcal{L}(E^*, \mathcal{L}(F,E^*))$ for each $(t,\xi) \in I \times B_r(E)$. For convenience of notation we shall denote the space $\mathcal{M}_c(\Sigma, F)$ by X and its (topological) dual by X^*. Since X, furnished with the total variation norm, is a Banach space its (topological) dual X^* is well defined. Let

$$D(\nu) \equiv \{\zeta \in X^* : \langle \zeta, \nu \rangle_{X^*,X} = |\nu|\}$$

denote the normalized duality map.

Now we can state the following necessary conditions of optimality.

Theorem 5.1. *Consider the system* (19) *with the objective functional* (26) *and suppose that* $\{A, M, f, g, \Phi, \mathcal{U}_{ad}\}$ *satisfy the assumptions of Theorem* 4.1. *Further, suppose the parameters* $\{f, g, \ell\}$ *also satisfy the hypothesis* $\mathbf{H}(\mathbf{f},\mathbf{g},\ell)$ *as stated above and* $\Phi(|\nu|) = |\nu|$, *(the variation norm) and* $\mathcal{U}_{ad}$ *is convex. Then for the pair* $\{\nu^o, x^o\} \in \mathcal{U}_{ad} \times B(I,E)$ *to be optimal it is necessary that there exists a* $\psi \in B(I,E^*)$ *such that the triple* $\{\nu^o, x^o, \psi\}$ *satisfy the following inequality*

$$\int_I \langle g^*(t, x^o(t))\psi(t), (\nu - \nu^o)(dt) \rangle_{F^*,F} + \langle \zeta, \nu - \nu^o \rangle_{X^*,X} \geq 0 \tag{44}$$

for all $\nu \in \mathcal{U}_{ad},\ \zeta \in D(\nu^o)$, *the evolution equation*

$$\begin{aligned} dx^o =\ & Ax^o dt + M(dt)x^o(t-) + f(t, x^o(t))dt \\ & + g(t, x^o(t-))\nu^o(dt), \quad x^o(0) = x_0, \end{aligned} \tag{45}$$

and the adjoint evolution

$$\begin{aligned} -d\psi =\ & A^*\psi dt + M^*(dt)\psi(t+) + f_x^*(t, x^o(t))\psi(t)dt \\ & + g_x^*(t, x^o(t-))(\psi(t+))\nu^o(dt) + \ell_x(t, x^o(t))dt, \quad \psi(T) = 0. \end{aligned} \tag{46}$$

Proof. The proof is quite similar to that of [1, Theorem 5.2]. We simply note that under the assumptions (41)–(43), the adjoint equation (46) has a unique (mild) solution in $B(I, E^*)$. ■

Remark 5.2. To the knowledge of the author, characterization of the dual X^* of the space X does not seem to have been addressed in the literature. However, it is easily verified that the elements of $C(I, F^*), B(I, F^*)$ induce continuous linear functionals on X through the map $f \longrightarrow L_f$ determined by

$$L_f(\mu) \equiv \int_I \langle f(t), \mu(dt) \rangle_{F^*,F}.$$

Thus under this correspondence we have the embedding

$$\{C(I, F^*) \subset B(I, F^*)\} \subset X^* \equiv (\mathcal{M}_c(\Sigma, F))^*.$$

Further, using the Hahn-Banach theorem one can easily verify that $B(I, F^*)$ is a total subspace of $X^* \equiv (\mathcal{M}_c(\Sigma, F))^*$ in the sense that $L_f(\mu) = 0$ for all $f \in B(I, F^*)$ implies $\mu = 0$. Thus we have the locally convex linear topological space $\big(\mathcal{M}_c(\Sigma, F), \mathcal{T}_{B(I,F^*)}\big)$ where $\mathcal{T}_{B(I,F^*)}$ denotes the $B(I, F^*)$ topology of $\mathcal{M}_c(\Sigma, F)$. The dual of this locally convex topological space is of course the space $B(I, F^*)$ itself.

In view of the above remark, the necessary condition (44) can be reformulated as follows:

$$\int_I \langle g^*(t, x^o(t))\psi(t) + \zeta(t), (\nu - \nu^o)(dt) \rangle_{F^*,F} \geq 0, \tag{47}$$

for all $\nu \in \mathcal{U}_{ad}, \zeta \in D(\nu^o) \cap B(I, E^*)$.

Remark 5.3. Given that ℓ satisfies the assumption (43) or more generally, $\ell_x(\cdot, x^o(\cdot)) \in L_1(I, E^*)$, a simple cost functional that is Gateaux differentiable is given by

$$J(\nu) \equiv \int_I \ell(t,x)dt + \Psi(\nu) \tag{48}$$

where

$$\Psi(\nu) = (1/2)\sum_{i=1}^{m}\left(\int_I \langle\, \eta_i(t), \nu(dt)\,\rangle_{F^*,F}\right)^2$$

with $\{\eta_i\}$ being any finite set of linearly independent elements of $B(I, F^*)$. In this case, the necessary condition (47) remains intact with the subdifferential $D(\nu^o)$ replaced by the Gateaux gradient

$$\zeta(t) = D\Psi(\nu^o) = \sum\left(\int_I \langle\, \eta_i(t), \nu^o(dt)\,\rangle\right)\eta_i(t), t \in I$$

which is a singleton and an element of $B(I, F^*)$.

5.2. Structural control

For simplicity of presentation, here we consider the problem of structural optimization of a simplified version of the system (19) given by

$$dx = Axdt + M(dt)x(t-) + f(t,x)dt + \Gamma(t)\nu(dt), x(0) = x_0. \tag{49}$$

Consider the system (49) with the objective functional (36) and let $\mathcal{B}_{ad} \subset \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ denote the class of admissible structural controls. The problem is to find a control $M_o \in \mathcal{B}_{ad}$ that minimizes the functional $J(M)$ given by (36). This is too general a problem; we need some additional structure. Let $p \in C^1(R^m)$ and let Φ be given by the cylindrical function,

$$\Phi(M) \equiv p\big((1/2)(e_i^*, Me_i)^2, \ldots, (1/2)(e_m^*, Me_m)^2\big) \tag{50}$$

where

$$(e_i^*, Me_i) \equiv \int_I \langle\, e_i^*(t), M(dt)e_i(t)\,\rangle_{E^*,E}$$

with $\{e_i \in B(I,E)\}$ and $\{e_i^* \in B(I,E^*)\}$ being any given set of elements from the spaces indicated. This functional is Gateaux differentiable and its differential in the direction $M - M_o$ is given by

$$\begin{aligned} d\Phi(M_o, M - M_o) &= \sum_{i=1}^{m} \partial_i p^0 \ (e_i^*, M_o e_i)(e_i^*, (M - M_o)e_i) \\ &\equiv \sum_{i=1}^{m} c_i(M_o)(e_i^*, (M - M_o)e_i) \end{aligned}$$

where

$$\partial_i p^0 \equiv \partial_i p\big((1/2)(e_1^*, M_o e_1)^2, \ldots, (1/2)(e_m^*, M_o e_m)^2\big)$$

and $\partial_i p$ denotes the partial derivative of p with respect to the i-th variable.

Theorem 5.4. *Consider the system* (49) *with the cost functional* (36) *where* Φ *is given by* (50). *Suppose* $\{A, f, \ell, \nu\}$ *satisfy the assumptions of Theorem* 5.1, $\Gamma \in L_1(|\nu|, \mathcal{L}(F,E))$ *and that the admissible set* $\mathcal{B}_{ad}$ *is a convex subset of* $\mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$. *Then for the pair* $\{M_o, x^o\}$ *to be optimal it is necessary that there exists a* $\psi \in B(I, E^*)$ *so that the triple* $\{M_o, x^o, \psi\}$ *satisfy the following inequality,*

$$(51)\quad \int_I \langle\, \psi(s), (M - M_o)(ds)x^o(s-)\,\rangle + d\Phi(M_o, M - M_o) \geq 0, \ \forall \ M \in \mathcal{B}_{ad},$$

the evolution equation

$$(52)\quad dx^o = Ax^o dt + M_o(dt)x^o(t-) + f(t, x^o(t))dt + \Gamma(t)\nu(dt), \ x^o(0) = x_0,$$

and the adjoint evolution

$$(53)\quad \begin{aligned} -\, d\psi(t) &= A^*\psi(t)dt + M_o^*(dt)\psi(t+) + f_x^*(t, x^o(t))\psi(t)dt \\ &\quad + \ell_x(t, x^o(t))dt, \psi(T) = 0. \end{aligned}$$

Proof. We present a brief outline of the proof. Under the given assumptions, it follows from Theorem 3.5 that for each $M \in \mathcal{B}_{ad}$ the system (49) has a unique mild solution. Let $M_o \in \mathcal{B}_{ad}$ be the optimal strategy and $M \in \mathcal{B}_{ad}$ an arbitrary element. Clearly, for any $\varepsilon \in [0,1]$, $M_\varepsilon \equiv M_o + \varepsilon(M - M_o) \in \mathcal{B}_{ad}$.

Letting $x^o \in B(I,E)$ and $x^\varepsilon \in B(I,E)$ denote the (mild) solutions of the state equation (49) corresponding to the choices M_o and M_ε respectively, it follows from the optimality of the pair $\{M_o, x^o\}$ that

$$\begin{aligned} & dJ(M_o, M - M_o) \\ & = \int_I \langle \ell_x(t, x^o(t)), y(t) \rangle_{E^*,E} dt + d\Phi(M_o, M - M_o) \geq 0 \ \forall\, M \in \mathcal{B}_{ad}, \end{aligned} \tag{54}$$

where y is the mild solution of the evolution equation

$$\begin{aligned} dy = {} & Aydt + M_o(dt)y(t-) + f_x(t, x^o(t))y(t)dt \\ & + (M - M_o)(dt)x^o(t-),\ y(0) = 0. \end{aligned} \tag{55}$$

Note that under the assumptions of Theorem 5.1, this equation has a unique mild solution $y \in B(I,E)$. Under the assumption $\underline{\mathbf{H}(\mathbf{f}, \mathbf{g}, \ell)}$, $\ell_x(\cdot, x^o(\cdot)) \in L_1(I, E^*)$ and thus the functional

$$y \longrightarrow \int_I \langle \ell_x(t, x^o(t)), y(t) \rangle_{E^*,E} dt$$

is a continuous linear functional on $B(I,E)$. Since $x^o \in B(I,E)$ and $(M - M^o) \in \mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ having bounded semivariation, it is clear that

$$\gamma(\sigma) \equiv \int_\sigma (M - M_o)(ds)x^o(s-),\ \sigma \in \Sigma$$

is a countably additive bounded vector measure with values in E. Thus it follows from continuous dependence of solutions of the evolution equation (55) with respect to the τ_v topology on $\mathcal{M}_c(\Sigma, \mathcal{L}_s(E))$ and supnorm topology on $B(I,E)$ that $(M - M_o)x^o \longrightarrow y$ is continuous and linear. Thus the composition map

$$(M - M_o)x^o \longrightarrow y \longrightarrow \int_I \langle \ell_x(t, x^o(t)),\ y(t) \rangle dt$$

is a continuous linear functional on $\mathcal{M}_c(\Sigma, E)$ and by duality there exists a $\psi \in (\mathcal{M}_c(\Sigma, E))^*$ such that

$$\int_I \langle \ell_x(t, x^o(t)), y(t) \rangle_{E^*,E} dt = \int_I \langle \psi(t), (M - M_o)(dt)x^o(t-) \rangle_{E^*,E}. \tag{56}$$

From this identity we obtain the necessary condition (51). Using this and the variational equation (55) and setting $\psi(T)=0$, one can formally verify that ψ is a (mild) solution of the evolution equation

$$\begin{aligned} -d\psi \; &= \; A^*\psi dt + f_x^*(t,x^o(t))\psi dt + M_o^*(dt)\psi(t+) \\ &\quad + \ell_x(t,x^o(t))dt, \psi(T)=0. \end{aligned} \tag{57}$$

For rigorous justification of this step, it is necessary to use the Yosida approximation $I_n \equiv nR(n,A)$ of the identity operator on E where $R(\lambda,A)$ denotes the resolvent of the operator A for $\lambda \in \rho(A)$, the resolvent set of A. Clearly, the domain of I_n is all of E and the range is $D(A) \subset E$. For convenience of notation set $F_o(t) \equiv f_x(t,x^o(t))$ and $\ell_o(t) \equiv \ell_x(t,x^o(t))$. Using the Yosida regularization, expression (56) is approximated by the following identity

$$\begin{aligned} &\int_I \langle\, I_n\ell_x(t,x^o(t)), y^n(t)\,\rangle_{E^*,E}dt \\ &= \int_I \langle\, \psi^n(t), I_n(M-M_o)(dt)x^o(t-)\,\rangle_{E^*,E} \end{aligned} \tag{58}$$

where $\{y^n,\psi^n\}$ are the solutions of the following evolution equations,

$$\begin{aligned} dy = \; &Aydt + I_nM_o(dt)y(t-) + I_nF_o(t)y(t)dt \\ &+ I_n(M-M_o)(dt)x^o(t-),\; y(0)=0 \end{aligned} \tag{59}$$

$$\begin{aligned} -d\psi = \; &A^*\psi dt + (I_nF_o(t))^*\psi dt + (I_nM_o)^*(dt)\psi(t+) \\ &+ I_n^*\ell_o(t))dt,\; \psi(T)=0, \end{aligned} \tag{60}$$

approximating the mild solutions of the evolution equations (55) and (57) respectively. This step guarantees that $y^n(t) \in D(A)$ and $\psi^n(t) \in D(A^*)$ for all $t \in I$. By use of the generalized Gronwall inequality and Lebesgue dominated convergence theorem one can verify that $y^n \xrightarrow{s} y$ in $B(I,E)$ and $\psi^n \xrightarrow{s} \psi$ in $B(I,E^*)$. This justifies rigorously the derivation of the adjoint evolution equation (53). Now by our existence theorems, one can conclude that this equation has a unique mild solution $\psi \in B(I,E^*)$ which, in general, is a proper subspace of the dual of the space of vector measures $\mathcal{M}_c(\Sigma,E)$. Thus we have all the necessary conditions of optimality as stated. This completes the brief outline of our proof. ■

Remark 5.5. It is interesting to observe that in the process of proof of the above result, it was found that there exists an adjoint process ψ in the dual space $(\mathcal{M}_c(\Sigma, E))^*$ which then finds itself in the space $B(I, E^*)$, a subspace of the dual.

6. Two examples

For illustration we present two examples from the classical and quantum mechanics.

E1: (Structural Dynamics) A large family of structural dynamics can be described by the following second order evolution equation,

$$\begin{aligned} & dy_t + Aydt + D(dt)y_t + K(dt)y = h(t, Ly, y_t), \\ & y(0) = x_0, y_t(0) = x_1, \ t \geq 0, \end{aligned} \tag{61}$$

where $y(t) \equiv \{y(t,\xi), \xi \in \Omega\}$ represents the spatial configuration of the body at time t (such as displacement of a membrane, a beam, road-bed of a suspension bridge etc from the rest state). In general, the operator A is a positive selfadjoint unbounded operator in a Hilbert space $H = L_2(\Omega)$ with domain and range $D(A), R(A) \subset H$. For example, it may represent a plate or a beam operator with required boundary conditions. The operators D and K are respectively the damping and stiffness operators. In general [2], these are operator valued measures, $D : \Sigma \longrightarrow \mathcal{L}(H)$ and $K : \Sigma \longrightarrow \mathcal{L}(D(\sqrt{A}), H)$ where $\sqrt{A}$ denotes the positive square root of the operator A. Typically, the operator $L = \sqrt{A}$ but it can be any member of the family $\{A^\alpha, 0 \leq \alpha \leq (1/2)\}$. For such structural problems it is natural to choose the energy space $E \equiv D(\sqrt{A}) \times H$ as the state space with Hilbertian structure induced by the scalar product

$$(x, z)_E \equiv (\sqrt{A}x_1, \sqrt{A}z_1)_H + (x_2, z_2)_H \ x, z \in E.$$

Using E for the state space and defining the state vector $x \equiv (x_1, x_2) = (y, y_t)$ we can write equation (61) as a first order differential equation on the Hilbert space E given by

$$dx = \mathcal{A}xdt + M(dt)x + f(t,x)dt, x(0) = x_0 \equiv (y(0), y_t(0)), \ t \geq 0, \tag{62}$$

where

$$\mathcal{A} \equiv \begin{pmatrix} 0 & I_H \\ -A & 0 \end{pmatrix}, M(\sigma) \equiv \begin{pmatrix} 0 & 0 \\ -K(\sigma) & -D(\sigma) \end{pmatrix}, \sigma \in \Sigma$$

with the domain $D(\mathcal{A}) = D(A) \times D(\sqrt{A})$ and range in E and the operator f is given by

$$f(t,x) \equiv \begin{pmatrix} 0 \\ h(t, \sqrt{A}x_1, x_2) \end{pmatrix}.$$

Under fairly general assumptions one can verify that $\mathcal{A}$ is skewadjoint and so $i\mathcal{A}$ is selfadjoint. Thus it follows from the well known Stones theorem [4, Theorem 3.1.4, p. 71] that $\mathcal{A}$ generates a C_0-unitary group $U(t), t \in R$, on the Hilbert space E. Further, it follows from the assumptions on the operator valued measures K and D that $M : \Sigma \longrightarrow \mathcal{L}(E)$ is a bounded operator valued measure. We assume that h is C^1 in the last two arguments and that there exists $\alpha \in L_1^+(I)$ so that

$$|h(t,u,v)|_H \le \alpha(t)\big(1 + |u|_H + |v|_H\big) \ \forall \ u,v \in H$$

and that it is locally Lipschitz on $H \times H$. Under this assumption, $f : I \times E \longrightarrow E$ satisfies the linear growth condition and it is also locally Lipschitz on E. The system (62) is a special case of the system (19) and so under the above assumptions it follows from Theorem 3.5 that it has a unique mild solution given by the solution of the integral equation,

$$\begin{aligned} x(t) = U(t)x_0 + \int_0^t U(t-s)M(ds)x(s-) \\ + \int_0^t U(t-s)f(s,x(s))ds, t \in I. \end{aligned} \tag{63}$$

Smart materials such as piezoelectric and magnetostrictive composites are used in modern technology to build active actuators and sensors. They provide active damping and stiffness for high precision machines or devices thereby suppressing vibration and preventing potential structural damage. Short bursts of currents can change the flexural (mechanical) properties of these materials providing structural damping and stiffness as and when required. In this sense, we may consider the operator valued measure as a control providing the necessary damping and stiffness. For vibration suppression, an appropriate cost function may be given by

$$J(M) \equiv (1/2)\int_I (Qx(t),x(t))_E dt + (1/2)\sum_{i=1}^{m}\left(\left\| \int_I M(dt)f_i(t))\right\|_E\right)^2 \tag{64}$$

where $Q \in \mathcal{L}(E)$ is a positive, symmetric operator in the Hilbert space E. In case Q is taken as the identity operator, the scalar product,

$$(Qx,x) = \| x(t) \|_E^2 = |\sqrt{A}x_1(t)|_H^2 + |x_2(t)|_H^2,$$

gives the sum of (elastic) potential and kinetic energies. The elements $\{f_i\} \subset B(I,E))$ of the second term of the cost functional can be chosen by the designer to meet specific requirements. For example, if we choose

$$f_i(t) = \chi_{\sigma_i}(t)u_i, u_i \in \partial B_1(E) \equiv \{u \in E : \| u \|_E = 1\},$$

where $\{\sigma_i\}$ are disjoint members of Σ, we have

$$\Phi(M) \equiv (1/2)\sum_{i=1}^{m}\left(\int_I \| M(dt)f_i(t) \|_E\right)^2 = (1/2)\sum_{i=1}^{m} \| M(\sigma_i)u_i \|_E^2 .$$

Though this is neither a measure of total variation nor a measure of semivariation, it represents (is proportional to) the intensity of energy used to effect structural changes for vibration control. Thus it makes sense to minimize the cost functional given by (64) subject to the dynamics (62). In this case the necessary conditions of optimality given by Theorem 5.4 reduce to

$$\begin{aligned}&\int_I \langle\, \psi(s), (M - M_o)(ds)x^o(s-)\,\rangle_E \\ &+ \sum_{i=1}^{m}\langle\, M_o(\sigma_i)u_i, (M - M_o)(\sigma_i)u_i\,\rangle_E \geq 0, \forall\ M \in \mathcal{B}_{ad},\end{aligned} \tag{65}$$

$$dx^o = \mathcal{A}x^o dt + M_o(dt)x^o(t-) + f(t,x^o(t))dt, \quad x^o(0) = x_0, \tag{66}$$

$$\begin{aligned}-d\psi(t) &= \mathcal{A}^*\psi(t)dt + M_o^*(dt)\psi(t+) + f_x^*(t,x^o(t))\psi(t)dt \\ &\quad + Qx^o(t)dt, \psi(T) = 0.\end{aligned} \tag{67}$$

Using these necessary conditions one can develop a numerical algorithm for computing the extremals containing the optimal policies.

E2: (Molecular Dynamics) Study of molecular dynamics and their control [17] has become an interesting topic in recent years because of the enormous prospect of developing smart materials with incredible structural properties (for example nanotechnology leading to smart materials). The impact of such development on future engineering materials with application to biotechnology and medicine is unprecedented. Changes in the molecular structure can be effected through control of the potential of the unperturbed Hamiltonian of the associated Schrödinger equation. This is done by laser induced electric field perturbing the original (unperturbed) Hamiltonian of the Schrödinger equation.

A simplified laser driven Schrödinger equation is described by

$$i(\partial\phi/\partial t) = H_o\phi + v(t)\chi_\Gamma\phi, \phi(0,\xi) = \phi_0(\xi), (t,\xi) \in I \times \Omega, \tag{68}$$

where H_o denotes the Hamiltonian of the unperturbed system which includes the operators corresponding to kinetic and (Coulomb) potential energies. In writing this equation we have used atomic units, that is $m_e = 1$, $e = 1$, $\hbar = 1$, $(1/4\pi\epsilon_0) = 1$ where $m_e, e, \hbar, \epsilon_0$ respectively denote the electron mass, the elementary charge, the reduced Planck constant and the dielectric permittivity of vacuum. A differential operator, again denoted by H_o, associated with this Hamiltonian may be defined so as to absorb the required boundary conditions (Dirichlet or Neumann in case Ω is a bounded domain). In general, the operator so obtained is selfadjoint or has a selfadjoint extension (denoted by H_o again) generating the unitary group $S(t) \equiv e^{-iH_o t}, t \in R$. The characteristic function χ_Γ, $\Gamma \subset \Omega$, is used to localize the controlled electric field $v(t)$ induced by the laser beam. Since the laser pulses are shot at the molecular system locally with pulse duration being of the order of 10^{-18} to 10^{-15} seconds and intensities of the order of $10^{12} watts/cm^2$, it is reasonable to consider them as Dirac measures spatially localized. Thus, in general, the input $v(t)\chi_\Gamma(\xi), t \geq 0, \xi \in \Omega$ can be replaced by a general vector measure, for example an $L_\infty(\Omega)$ valued set function $\sigma \ni \Sigma \longrightarrow V_0(\sigma) \equiv V_0(\sigma,\xi), \xi \in \Omega$, vanishing outside $\Gamma \subset \Omega$. Define the operator valued measure $\sigma \ni \Sigma \longrightarrow V_0(\sigma) \in \mathcal{L}(L_2)$ by

$$(V_0(\sigma)\phi)(\xi) \equiv V_0(\sigma,\xi)\phi(\xi), \ \xi \in \Omega.$$

Using this operator,we may write equation (68) as

$$i(d\phi) = H_o\phi dt + V_0(dt)\phi, \phi(0) = \phi_0(\cdot), \, t \in I. \tag{69}$$

As usual, this equation is defined on a complex Hilbert space. Writing the real and imaginary parts of the wave function ϕ ($\phi = \phi_1 + i\phi_2$) as a vector (ϕ_1, ϕ_2) and denoting it by x, we can model this as a differential equation on the real Hilbert space $H \equiv L_2(\Omega) \times L_2(\Omega)$ as follows

$$dx = \mathcal{A}xdt + M(dt)x, x(0) = x_0 \equiv (\phi_{0,1}, \phi_{0,2}) \tag{70}$$

where

$$\mathcal{A} \equiv \begin{pmatrix} 0 & H_o \\ -H_o & 0 \end{pmatrix}, \, M(\sigma) \equiv \begin{pmatrix} 0 & V_0(\sigma) \\ -V_0(\sigma) & 0 \end{pmatrix}, \, \sigma \in \Sigma$$

with the domain $D(\mathcal{A}) = D(H_o) \times D(H_o)$ and range $R(\mathcal{A})$ in H. Clearly, it follows from the assumption on the operator valued measure V_0 that $M : \Sigma \longrightarrow \mathcal{L}(H)$ is also an operator valued measure. Since H_o is assumed to be selfadjoint, it is easy to verify that $i\mathcal{A}$ is selfadjoint. Thus again it follows from the Stones theorem that $\mathcal{A}$ generates a unitary group $\{U(t), \, t \in R\}$ on H. If, as an $L_\infty(\Omega)$-valued vector measure, V_0 is countably additive having bounded variation, then the operator valued measure M is countably additive in the uniform operator topology having bounded variation. If, on the other hand, V_o, considered as an operator valued measure with values from $\mathcal{L}(L_2(\Omega))$, is countably additive in the strong operator topology having finite semivariation, then so is M with values from $\mathcal{L}(H)$. In either case, the (mild) solution of equation (70) is given by the solution of the integral equation

$$x(t) = U(t)x_0 + \int_0^t U(t-s)M(ds)x(s-), \, t \in I. \tag{71}$$

For this problem one may demand the control to force the system to stay close to a desired molecular state trajectory $x_d \in B(I, H)$. In that case, the quadratic form (Qx, x) in the cost functional (64) is replaced by $(Q(x - x_d), (x - x_d))$, and accordingly, in the adjoint equation (67), Qx is replaced by $Q(x - x_d)$. Thus the necessary conditions of optimality given by (65)–(67) also hold for this system provided f is set equal to zero. Another possibility is to demand the control to force the system to reach(as close as possible) a desirable molecular state $x_d \in H$ at the terminal time T. This may be formulated with the cost functional given by

$$J(M) \equiv (1/2)(Q[x(T) - x_d], [x(T) - x_d]) + \Phi(M),$$

where x is the solution of equation (71) corresponding to M. The associated adjoint equation (67) is given by

$$(72) \qquad -d\psi(t) = \mathcal{A}^*\psi(t)dt + M_o^*(dt)\psi(t+), \; \psi(T) = -Q(x(T) - x_d).$$

This result can be easily extended to a certain class of nonlinear Scrödinger equations of the form

$$(73) \qquad i(d\phi) = H_o\phi dt + f(\phi)dt + V_0(dt)\phi, \phi(0) = \phi_0(\cdot), \; t \in I$$

where

$$f(\varphi) \equiv \lambda|\varphi|^{p-1}\varphi$$

with $\lambda > 0$ and $p \geq 1$. It is known that in the case of dissipative nonlinearities one can use the Lyapunov technique relying on energy estimates and prove global existence results. For example, given that $H_0 = -\Delta$ (the Laplacian), it is easy to verify that in the absence of the external force ($V_0 = 0$), the system is conservative giving a-priori bounds in terms of the (Hilbert and Sobolev) bounds of initial data. Using these bounds and certain standard techniques involving the fixed point theory, one can prove the existence of solutions for the nonlinear problem. In this case, the abstract model on the real Hilbert space H has the form

$$(74) \qquad dx = \mathcal{A}xdt + F(x)dt + M(dt)x, \; x(0) = x_0,$$

where $F(x) = (f_2(x), -f_1(x))$ with f_1 and f_2 denoting the real and imaginary parts of f, respectively.

Questions of weak and exact controllability for linear and semilinear problems using vector measures as controls have been treated in [6]. It would be interesting to extend such results to the Quantum system described above with operator valued measures as controls. This remains an open problem.

Remark 6.1. For other interesting examples see [1, 2, 7]. Currently, we are also working on similar problems with dynamics determined by strongly nonlinear parabolic operator valued measures. Semilinear problems were considered in [18].

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