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FIXED POINT THEORY FOR MULTIVALUED MAPS IN FRÉCHET SPACES VIA DEGREE AND INDEX THEORY

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Abstract

New fixed point results are presented for multivalued maps defined on subsets of a Fréchet space E. The proof relies on the notion of a pseudo open set, degree and index theory, and on viewing E as the projective limit of a sequence of Banach spaces.

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1. Introduction

This paper presents applicable fixed point theorems for multivalued maps defined between Fréchet spaces. Our results in particular will apply to Kakutani, R_δ and more generally J maps. Our theory is based on degree and index theory in Banach spaces and on viewing a Fréchet space as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in N}$ (here $N = \{1, 2, \ldots\}$). The usual results in the literature in the non-normable situation are rarely of interest from an application viewpoint (this point seems to be overlooked by many authors) since the set constructed using degree is usually open and bounded and so has an empty interior.

For the remainder of this section, we present some definitions and some known facts. Let (X, d) be a metric space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty]$ defined by (here $A \in \Omega_X$)

$$\alpha(A) = \inf\{r > 0 : \ A \subseteq \cup_{i=1}^{n} A_i \ \text{ and } \ diam\,(A_i) \le r\}.$$

Let S be a nonempty subset of X. For each $x \in X$, define $d(x, S) = \inf_{y\in S}\ d(x, y)$. We say a set is countably bounded if it is countable and bounded. Now suppose $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then $G : S \to 2^X$ is

(i) countably k-set contractive (here $k \ge 0$) if $G(S)$ is bounded and $\alpha(G(W)) \le k\,\alpha(W)$ for all countably bounded sets W of S,

(ii) countably condensing if $G(S)$ is bounded, G is countably 1-set contractive and $\alpha(G(W)) < \alpha(W)$ for all countably bounded sets W of S with $\alpha(W) \ne 0$,

(iii) hemicompact if each sequence $\{x_n\}_{n\in N}$ in S has a convergent subsequence whenever $d(x_n, G\,(x_n)) \to 0$ as $n \to \infty$.

We now recall a result from the literature [1].

Theorem 1.1. *Let (Y, d) be a metric space, D a nonempty, complete subset of Y, and $G : D \to 2^Y$ a countably condensing map. Then G is hemicompact.*

Let A be a compact subset of a metric space X. A is called ∞-proximally connected in X if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any $n = 1, 2, \ldots$ and any map $g : \partial \triangle^n \to N_\delta(A)$ there exists a map $g' : \triangle^n \to N_\epsilon(A)$

such that $g(x) = g'(x)$ for $x \in \partial \triangle^n$; here $\triangle^n$ is the n-dimensional standard simplex and $N_\epsilon(A) = \{x \in X : \ dist\,(x, A) < \epsilon\}$. Let X and Y be two metric spaces and $F : X \to 2^Y$. We say $F \in J(X, Y)$ if F is upper semicontinuous with nonempty, compact, ∞-proximally connected values; see [5] for examples of J maps. If Z is another metric space and $F \in J(X, Y)$ with $r : Z \to X$ continuous, then it is well known [5] that $F \circ r \in J(Z, Y)$. In this paper, we will also discuss a special subclass of J maps, namely the Kututani maps. Let $F : X \to CK(Y)$; here $CK(Y)$ denotes the family of nonempty compact convex subsets of Y. We say $F : X \to CK(Y)$ is Kakutani if F is upper semicontinuous.

Let Ω be a bounded open subset of a Banach space E and assume $T : \overline{\Omega} \to 2^E$ is a Kakutani countably condensing map with $0 \notin (I - T)(\partial\Omega)$. Then [7, Chapter 2 and 3, 8, 9] guarantees that $\deg(I - T, \Omega, 0)$ is well defined and has the usual properties.

Next let Ω be an open subset of a Banach space E and assume $T \in J(\overline{\Omega}, E)$ is a compact map with $0 \notin (I - T)(\partial\Omega)$. Then [3, pp. 4868] guarantees that $\deg(I - T, \Omega, 0)$ is well defined and has the usual properties. It is possible to extend the degree for countably condensing J maps (see [2]). Let E be a Banach space and Ω an open bounded subset of E. Also let $T \in J(\overline{\Omega}, E)$ be a countably condensing map with $0 \notin (I - T)(\partial\Omega)$. Let

$$A_1 = \overline{co}\,(T\,(\overline{\Omega})), \quad A_n = \overline{co}\,(T\,(\overline{\Omega} \cap A_{n-1}))$$

for $n = 2, 3, \ldots$ and

$$A_\infty = \cap_{n=1}^{\infty} A_n.$$

<u>Fix</u> a retraction $R : E \to A_\infty$. If $\Omega \cap A_\infty = \emptyset$, we let the degree of $I - T$ on Ω with respect to 0, denoted $\deg(I - T, \Omega, 0)$, be zero. If $\Omega \cap A_\infty \neq \emptyset$ we let

$$\deg(I - T, U, \Omega, 0) = \deg(I - T \circ R, R^{-1}(\Omega), 0)$$

where the right hand side is the Andres, Gabor, Gorniewicz degree.

Let C be a closed convex subset of a Banach space E and U an open bounded subset of E. Assume $T : \overline{W} \to 2^C$ is a Kakutani countably condensing map with $x \notin T\,x$ for $x \in \partial W$; here $W = U \cap C$ and in this situation $\overline{W}$ (respectively ∂W) denotes the closure of W in C (respectively the boundary of W in C). Then [2, 4, 8] guarantee that $ind(T, C, W)$ is well defined and has the usual properties.

It is possible to extend the index for countably condensing J maps (see [2]). Let C be a closed convex subset of a Banach space E and U an open bounded subset of E. Assume $T \in J(\overline{W}, C)$ is a countably condensing map with $x \notin T\,x$ for $x \in \partial W$ where $W = U \cap C$. Let

$$A_1 = \overline{co}\,(T\,(\overline{W})), \quad A_n = \overline{co}\,(T\,(\overline{W} \cap A_{n-1}))$$

for $n = 2, 3, \ldots$ and

$$A_\infty = \cap_{n=1}^{\infty} A_n.$$

Fix a retraction $R : E \to A_\infty$. If $W \cap A_\infty = \emptyset$, we let $ind(T, C, W) = 0$. If $W \cap A_\infty \neq \emptyset$ we let

$$ind(T, C, W) = \deg(I - T \circ R, R^{-1}(U), 0)$$

where the right hand side is the Andres, Gabor, Gorniewicz degree (see [3]).

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : \ x_\alpha = \pi_{\alpha,\beta}(x_\beta) \ \forall\, \alpha,\, \beta \in I,\, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [6 pp. 439] $\cap_{\alpha \in I} E_\alpha$.)

2. Fixed point theory in Fréchet spaces.

Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \ldots \quad \text{for every} \quad x \in E. \tag{2.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x - y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume the following condition holds:

$$(2.3) \quad \begin{cases} \text{for each } \ n \in N, \ \ \text{there exists a Banach space } \ (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } \ j_n : \mathbf{E}_n \to E_n. \end{cases}$$

Remark 2.1. (i) For convenience the norm on E_n is denoted by $|\cdot|_n$.
(ii) Usually in applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.
(iii) Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$), then $x \in E$. However, if $x \in E_n$ then x is not necessarily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally, we assume

$$(2.4) \qquad E_1 \supseteq E_2 \supseteq \dots \ \text{ and for each } \ n \in N, \ |x|_n \leq |x|_{n+1} \ \forall \ x \in E_{n+1}.$$

Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [6]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\,\mu_{n,m}\,j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\,\mu_n(X)$, and we let $\overline{X_n}$ and ∂X_n denote respectively the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$pseudo - int\,(X) = \{x \in X : \ j_n\,\mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } \ n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$.

If U is a pseudo-open bounded subset of E, then for each $n \in N$ we have that U_n is open and bounded.

To see that U_n is open first notice $U_n \subseteq \overline{U_n} \setminus \partial U_n$ since if $y \in U_n$, then there exists $x \in U$ with $y = j_n\mu_n(x)$ and this together with $U = pseudo - int\,U$ yields $j_n\mu_n(x) \in \overline{U_n} \setminus \partial U_n$ i.e., $y \in \overline{U_n} \setminus \partial U_n$. In addition, notice

$$\overline{U_n} \setminus \partial U_n = (int\, U_n \cup \partial U_n) \setminus \partial U_n = int\, U_n \setminus \partial U_n = int\, U_n$$

since $int\, U_n \cap \partial U_n = \emptyset$. Consequently,

$$U_n \subseteq \overline{U_n} \setminus \partial U_n = int\, U_n, \quad \text{so} \quad U_n = int\, U_n.$$

As a result U_n is open. Finally, U_n is bounded since U is bounded (note if $y \in U_n$, then there exists $x \in U$ with $y = j_n \mu_n(x)$).

We begin with a result for Volterra type operators.

Theorem 2.1. *Let E and E_n be as described above, $F : \Omega \to 2^E$ where Ω is a pseudo-open bounded subset of E. Also assume for each $n \in N$ that $F : \overline{\Omega_n} \to CK(E_n)$. Suppose the following conditions are satisfied:*

$$\text{(2.5)} \qquad \begin{cases} \textit{for each } \ n \in N, \ F : \overline{\Omega_n} \to CK(E_n) \ \textit{ is an} \\ \textit{upper semicontinuous countably condensing map} \end{cases}$$

$$\text{(2.6)} \qquad \textit{for each } \ n \in N, \ 0 \notin (I - F)\,(\partial \Omega_n)$$

$$\text{(2.7)} \qquad \textit{for each } \ n \in N, \ \deg(I - F, \Omega_n, 0) \neq 0$$

and

$$\text{(2.8)} \qquad \begin{cases} \textit{for each } \ n \in \{2, 3, \dots\} \ \textit{ if } \ y \in \overline{\Omega_n} \ \textit{ solves } \ y \in F\, y \ \textit{ in } \ E_n \\ \textit{then } \ y \in \overline{\Omega_k} \ \textit{ for } \ k \in \{1, \dots, n-1\}. \end{cases}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. Now there exists $y_n \in \overline{\Omega_n}$ with $y_n \in F\, y_n$. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in \overline{\Omega_1}$ and $y_k \in \overline{\Omega_1}$ for $k \in N \backslash \{1\}$ from (2.8). As a result $y_n \in \overline{\Omega_1}$ for $n \in N$, $y_n \in F\, y_n$ in E_n together with (2.5) implies there is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{\Omega_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in \overline{\Omega_2}$ for $n \in N_1$ together with (2.5) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{\Omega_2}$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note from (2.4) that $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in \overline{\Omega_k}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \ldots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F\, y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with the fact that $F : \overline{\Omega_k} \to 2^{E_k}$ is upper semicontinuous (note $y_n \in \overline{\Omega_k}$ for $n \in N_k$) implies $y \in F\, y$ in E_k. We can do this for each $k \in N$ so $y \in F\, y$ in E. ■

Our next result was motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (2.14).

Theorem 2.2. *Let E and E_n be as described in the beginning of Section 2, Ω a pseudo-open bounded subset of E and $F : \Omega \to 2^E$. Also assume for each $n \in N$ that $F_n : \overline{\Omega_n} \to CK(E_n)$. Suppose the following conditions are satisfied:*

(2.9) $$\overline{\Omega_1} \supseteq \overline{\Omega_2} \supseteq \ldots$$

(2.10) $$\begin{cases} \text{for each } n \in N,\ F_n : \overline{\Omega_n} \to CK(E_n) \text{ is an} \\ \text{upper semicontinuous map} \end{cases}$$

(2.11) $$\text{for each } n \in N,\ 0 \notin (I - F_n)\,(\partial \Omega_n)$$

(2.12) $$\text{for each } n \in N,\ \deg(I - F_n, \Omega_n, 0) \neq 0$$

(2.13) $$\begin{cases} \text{for each } n \in N, \text{ the map } \mathcal{K}_n : \overline{\Omega_n} \to 2^{E_n}, \text{ given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \ \text{(see Remark 2.2), is} \\ \text{countably condensing} \end{cases}$$

and

(2.14) $$\begin{cases} \text{if there exists a } w \in E \text{ and a sequence } \{y_n\}_{n \in N} \\ \text{with } y_n \in \overline{\Omega_n} \text{ and } y_n \in F_n\, y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \ldots\} \text{ of } N \text{ with } y_n \to w \text{ in } E_k \\ \text{as } n \to \infty \text{ in } S, \text{ then } w \in F\, w \text{ in } E. \end{cases}$$

Then F has a fixed point in E.

Remark 2.2. The definition of $\mathcal{K}_n$ in (2.13) is as follows. If $y \in \overline{\Omega_n}$ and $y \notin \overline{\Omega_{n+1}}$, then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in \overline{\Omega_{n+1}}$ and $y \notin \overline{\Omega_{n+2}}$, then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in N$. Now there exists $y_n \in \overline{\Omega_n}$ with $y_n \in F_n\, y_n$ in E_n. Let us look at $\{y_n\}_{n\in N}$. Now Theorem 1.1 (with $Y = E_1$, $G = \mathcal{K}_1$, $D = \overline{\Omega_1}$ and note $d_1(y_n, \mathcal{K}_1(y_n)) = 0$ for each $n \in N$ since $|x|_1 \le |x|_n$ for all $x \in E_n$ and $y_n \in F_n\, y_n$ in E_n; here $d_1(x, Z) = \inf_{y\in Z} |x - y|_1$ for $Z \subseteq Y$) guarantees that there exists a subsequence $N_1^\star$ of N and a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n\in N_1}$. Now Theorem 1.1 (with $Y = E_2$, $G = \mathcal{K}_2$ and $D = \overline{\Omega_2}$) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in E_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1^\star$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \dots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n\, y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (2.14) implies $y \in F\, y$ in E. ∎

The results in Theorem 2.1 and Theorem 2.2 clearly extend for countably condensing J maps. For completeness we just state the analogue of Theorem 2.1 for compact J maps.

Theorem 2.3. *Let E and E_n be as described in the beginning of Section 2, $F : \Omega \to 2^E$ where Ω is a pseudo-open bounded subset of E. Also assume for each $n \in N$ that $F : \overline{\Omega_n} \to 2^{E_n}$. Suppose the following condition is satisfied:*

(2.15) *for each* $n \in N$, $F \in J(\overline{\Omega_n}, E_n)$ *is a compact map.*

Also assume (2.6), (2.7) *and* (2.8) *hold. Then F has a fixed point in E.*

We now obtain another result for Volterra type operators. However, before we prove this result we show that if C is a convex subset of the Fréchet space E described in the beginning of Section 2, then for each $n \in N$ we have that $\overline{C_n}$ is convex. To see this let $\hat{x}$, $\hat{y} \in \mu_n(C)$ and $\lambda \in [0, 1]$.

Then for every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$ we have $\lambda x + (1-\lambda)y \in C$ since C is convex and so $\lambda\hat{x} + (1-\lambda)\hat{y} = \lambda\mu_n(x) + (1-\lambda)\mu_n(y)$. It is easy to check that $\lambda\mu_n(x) + (1-\lambda)\mu_n(y) = \mu_n(\lambda x + (1-\lambda)y)$ so as a result

$$\lambda\hat{x} + (1-\lambda)\hat{y} = \mu_n(\lambda x + (1-\lambda)y) \in \mu_n(C),$$

and so $\mu_n(C)$ is convex. Now since j_n is linear we have $C_n = j_n(\mu_n(C))$ is convex and as a result $\overline{C_n}$ is convex.

Theorem 2.4. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E, U a pseudo-open bounded subset of E and $F : U \cap C \to 2^E$. Also assume for each $n \in N$ that $F : \overline{W_n} \to CK(\overline{C_n})$ where $W_n = U_n \cap \overline{C_n}$; here in this situation $\overline{W_n}$ denotes the closure of W_n in $\overline{C_n}$. Suppose the following conditions are satisfied:*

$$(2.16) \qquad \begin{cases} \text{for each } \ n \in N, \ F : \overline{W_n} \to CK(\overline{C_n}) \ \text{ is an} \\ \text{upper semicontinuous countably condensing map} \end{cases}$$

$$(2.17) \qquad \begin{cases} \text{for each } \ n \in N, \ x \notin F\,x \ \text{ for } \ x \in \partial W_n \\ (\text{here } \ \partial W_n \ \text{ denotes the boundary of } \ W_n \ \text{ in } \ \overline{C_n}\,) \end{cases}$$

$$(2.18) \qquad \text{for each } \ n \in N, \ ind(F, \overline{C_n}, W_n) \neq 0$$

and

$$(2.19) \qquad \begin{cases} \text{for each } \ n \in \{2,3,\ldots\} \ \text{ if } \ y \in \overline{W_n} \ \text{ solves } \ y \in F\,y \ \text{ in } \ E_n \\ \text{then } \ y \in \overline{W_k} \ \text{ for } \ k \in \{1,\ldots,n-1\}. \end{cases}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. Now there exists $y_n \in W_n \cap \overline{C_n}$ with $y_n \in F\,y_n$. Essentially the same argument as in Theorem 2.1 establishes the result. ∎

We next obtain another result for Urysohn type operators.

Theorem 2.5. *Let E and E_n be as described in the beginning of Section 2, C a closed convex subset of E, U a pseudo-open bounded subset of E and $F : U \cap C \to 2^E$. Also assume for each $n \in N$ that $F_n : \overline{W_n} \to CK(\overline{C_n})$*

where $W_n = U_n \cap \overline{C_n}$ *and in this situation* $\overline{W_n}$ *denotes the closure of* W_n *in* $\overline{C_n}$*. Suppose the following conditions are satisfied:*

$$\overline{W_1} \supseteq \overline{W_2} \supseteq \dots \tag{2.20}$$

$$\begin{cases} \text{for each } \ n \in N, \ F_n : \overline{W_n} \to CK(\overline{C_n}) \ \text{ is an} \\ \text{upper semicontinuous map} \end{cases} \tag{2.21}$$

$$\begin{cases} \text{for each } \ n \in N, \ x \notin F_n\, x \ \text{ for } \ x \in \partial W_n \\ (\text{here } \ \partial W_n \ \text{ denotes the boundary of } \ W_n \ \text{ in } \ \overline{C_n}\,) \end{cases} \tag{2.22}$$

$$\text{for each } \ n \in N, \ ind(F_n, \overline{C_n}, W_n) \neq 0 \tag{2.23}$$

$$\begin{cases} \text{for each } \ n \in N, \ \text{ the map } \ \mathcal{K}_n : \overline{W_n} \to 2^{E_n}, \ \text{ given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \ \ (\text{see Remark } 2.3), \text{ is} \\ \text{countably condensing} \end{cases} \tag{2.24}$$

and

$$\begin{cases} \text{if there exists a } \ w \in E \ \text{ and a sequence } \ \{y_n\}_{n\in N} \\ \text{with } \ y_n \in \overline{W_n} \ \text{ and } \ y_n \in F_n\, y_n \ \text{ in } \ E_n \ \text{ such that} \\ \text{for every } \ k \in N \ \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \dots\} \ \text{ of } \ N \ \text{ with } \ y_n \to w \ \text{ in } \ E_k \\ \text{as } \ n \to \infty \ \text{ in } \ S, \ \text{ then } \ w \in F\, w \ \text{ in } \ E. \end{cases} \tag{2.25}$$

Then F *has a fixed point in* E*.*

Remark 2.3. The definition of $\mathcal{K}_n$ in (2.24) is as follows. If $y \in \overline{W_n}$ and $y \notin \overline{W_{n+1}}$, then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in \overline{W_{n+1}}$ and $y \notin \overline{W_{n+2}}$, then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in N$. Now there exists $y_n \in W_n \cap \overline{C_n}$ with $y_n \in F_n\, y_n$. Essentially the same argument as in Theorem 2.2 establishes the result. ∎

Remark 2.4. It is easy to obtain the analoge of Theorem 2.4 and Theorem 2.5 (the details are left to the reader) if the Kakutani maps are replaced by J maps.

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