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CONTROLLABILITY OF IMPULSIVE SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH FINITE DELAY IN FRÉCHET SPACES

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Abstract

In this paper, we use the extrapolation method combined with a recent nonlinear alternative of Leray-Schauder type for multivalued admissible contractions in Fréchet spaces to study the existence of a mild solution for a class of first order semilinear impulsive functional differential inclusions with finite delay, and with operator of nondense domain in original space.

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1. Introduction

In this paper, we consider the existence of mild solutions of first order semilinear impulsive functional differential inclusions in Fréchet spaces. More precisely in Section 3, we will consider the first order semilinear impulsive functional differential inclusions of the form:

$$y'(t) - Ay(t) \in F(t, y_t) + Bu(t), \quad \text{a.e.} \quad t \in J \backslash \{t_1, t_2, \ldots\} \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, \tag{2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{3}$$

where $J := [0, \infty)$, $F : J \times \mathcal{D} \to \mathcal{P}(E)$ is a multivalued map with compact values, $(E, |\cdot|)$ is a separable Banach space, $\mathcal{D} = \{\psi : [-r, 0] \to E, \psi$ is continuous everywhere except for a finite number of points at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, B is a bounded linear operator from E into E, and the control parameter $u(\cdot)$ belongs to $L^2(J, U)$ a space of admissible controls with U a Banach space, and $\phi \in \mathcal{D}$. For every $t \in [0, \infty)$, the history function $y_t : [-r, 0] \to E$ is defined by

$$y_t(\theta) = y(t + \theta), \text{ for } \theta \in [-r, 0].$$

We assume that the histories $y_t : [-r, 0] \to E$, $y_t(\theta) = y(t + \theta)$, belong to the phase space $\mathcal{D}$.

In recent years, much effort has been devoted to study impulsive differential and partial differential equations and inclusions, in particular, in the areas of impulsive differential equations and inclusions with fixed moments; see the monographs of Benchohra *et al.* [8], Lakshmikantham *et al.* [18], and Samoilenko and Perestyuk [25], and the references therein. The problem (1)–(3) when A is nondensely defined and generates an integral semigroup on compact intervals with finite delay was considered by Benchohra *et al.* [7] and Gatsori *et al.* [14]. Our aim here is to give global existence results and controllability for the above problem. These results extend some existing ones in the previous literature in the case of densely defined linear operators. The fundamental tools applied here are the recent nonlinear alternative of Leray Schauder type on Frechet spaces due to Frigon [13] and the semigroup theory [1, 23] as used by Benchohra and Ouahab [9] and Henderson and Ouahab [16].

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts from the semigroup theory, extrapolation spaces and multivalued analysis, which are used throughout this paper.

$C([-r,0],E)$ is the Banach space of all continuous functions from $[-r,0]$ into E endowed with the norm $\|.\|$ defined by

$$\|y\|_\infty := \sup\{|y(\theta)| : \theta \in [-r,0]\}$$

Also, $B(E)$ is the Banach space of all linear bounded operators from E into E with the norm

$$\|N\|_{B(E)} := \sup\{|N(y)| : |y| = 1\}.$$

$L^1([0,b],E)$ denotes the space of a measurable functions $y : [0,b] \longrightarrow E$ which are Bochner integrable (i.e., $|y|$ is Lebesgue integrable) normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

Definition 2.1. We say that a linear operator A satisfies the "Hille-Yosida condition" if there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \rho(A)$ and

$$\sup\{(\lambda-\omega)^n|(\lambda I - A)^{-n}| : n \in \mathbb{N}, \lambda > \omega\} \leq M.$$

Here and hereafter, we assume:

(HY) A satisfies the Hille-Yosida condition.

Let A_0 be a part of A in $X_0 = \overline{D(A)}$ which is defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \text{ and } A_0x = Ax, \text{ for } x \in D(A_0).$$

Lemma 2.1 [12]. *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on X_0 and $|T_0(t)| \leq N_0e^{\omega t}$, for $t \geq 0$. Moreover, $\rho(A) \subset \rho(A_0)$ and $R(\lambda,A_0) = R(\lambda,A)/X_0$, for $\lambda \in \rho(A)$.*

For a fixed $\lambda_0 \in \rho(A)$, we introduce on X_0 a new norm defined by

$$\|x\|_1 = |R(\lambda_0,A_0)x| \text{ for } x \in \overline{D(A_0)}.$$

The completion X_1 of $(X_0, \|\cdot\|_1)$ is called the *extrapolation space of X associated with A*. Note that $\|\cdot\|_1$ and the norm on X_0 given by $|R(\lambda, A_0)x|$, for $\lambda \in \rho(A)$, are equivalent. Let $T_1(t)$ be the extension to the Banach space X_1 of $(T_0(t))_{t\geq 0}$. $(T_1(t))_{t\geq 0}$ is a strongly continuous semigroup on X_1 and is called *the extrapolated semigroup of* $(T_0(t))_{t\geq 0}$, and we denote its generator by $(A_1, D(A_1))$.

Lemma 2.2 [15]**.** *The following properties hold:*

(i) $|T(t)|_{B(X_1)} = |T_0(t)|_{L(X_0)}$.

(ii) $D(A_1) = X_0$.

(iii) $A_1 : X_0 \to X_1$ *is the unique continuous extension of* $A_0 : D(A_0) \subset (X_0, |\cdot|) \to (X_0, \|\cdot\|_1)$, *and* $(\lambda - A_1)^{-1}$ *is an isometry from* $(X_0, |\cdot|) \to (X_0, \|\cdot\|_1)$.

(iv) *If* $\lambda \in \rho(A_0)$, *then* $(\lambda - A_1)$ *is invertible and* $(\lambda - A_1)^{-1} \in B(X_1)$. *In particular,* $\lambda \in \rho(A_1)$ *and* $R(\lambda, A_1)/X_0 = R(\lambda, A_0)$.

(v) *The space* $X_0 = \overline{D(A)}$ *is dense in* $(X_1, \|\cdot\|_1)$. *Hence the extrapolation space* X_1 *is also the completion of* $(X, \|.\|_1)$ *and* $X \hookrightarrow X_1$.

(vi) *The operator* A_1 *is an extension of* A. *In particular, if* $\lambda \in \rho(A)$, *then* $R(\lambda, A_1)/X = R(\lambda, A)$ *and* $(\lambda - A_1)X = D(A)$.

More details on abstract extrapolation spaces can be found in the books by Da Prato and Grisvard [10], Engel and Nagel [12] and Lunardi [19] and used for various purposes [2, 3, 4, 20, 21, 22].

We finish this section with notation, definitions, and some results from the multivalued analysis. Hereafter, we will use the following notation. Given a space X and metrics $d_\alpha, \alpha \in \bigwedge$, denote $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$. We denote by D_α, $\alpha \in \bigwedge$, the Hausdorff pseudo-metric induced by d_α; that is, for $V, W \in \mathcal{P}(X)$,

$$D_\alpha(V, W) = \inf \Big\{\varepsilon > 0 : \forall x \in V,\ \forall y \in W,\ \exists \bar{x} \in V,\ \bar{y} \in W \text{ such that } d_\alpha(x, \bar{y}) < \varepsilon, d_\alpha(\bar{x}, y) < \varepsilon\Big\}$$

with $\inf \emptyset = \infty$. In the particular case where X is a complete locally convex space, we say that a subset $V \subset X$ is bounded if $D_\alpha(\{0\}, V) < \infty$ for every $\alpha \in \bigwedge$. More details can be found in [13].

Definition 2.2. A multivalued map $F : X \to \mathcal{P}(E)$ is called an admissible contraction with constant $\{k_\alpha\}_{\alpha\in\bigwedge}$ if for each $\alpha \in \bigwedge$ there exists $k_\alpha \in (0,1)$ such that

(i) $D_\alpha(F(x), F(y)) \leq k_\alpha d_\alpha(x,y)$ for all $x, y \in X$.

(ii) for every $x \in X$ and every $\varepsilon \in (0,\infty)^{\bigwedge}$, there exists $y \in F(x)$ such that

$$d_\alpha(x,y) \leq d_\alpha(x, F(x)) + \varepsilon_\alpha \ for\ every\, \alpha \in \bigwedge.$$

Lemma 2.3 (Nonlinear Alternative, [13])**.** *Let E be a Fréchet space and U an open neighborhood of the origin in E, and let $N : \overline{U} \to \mathcal{P}(E)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements holds:*

(C1) *N has at least one fixed point;*

(C2) *there exists $\lambda \in [0,1)$ and $x \in \partial U$ such that $x \in \lambda N(x)$.*

For applications of Lemma 2.3 we consider $H_d : \mathcal{P}(X)\times\mathcal{P}(X) \longrightarrow \mathbb{R}_+\cup\{\infty\}$, given by

$$H_d(V,W) = \max\left\{\sup_{a\in V} d(a,W),\ \sup_{b\in W} d(V,b)\right\},$$

where $d(V,b) = \inf_{a\in V} d(a,b),\ d(a,W) = \inf_{b\in W} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space ([17]).

Definition 2.3. A multivalued map $F : [0,\infty) \times \mathcal{D} \to \mathcal{P}(E)$ is said to be L^1- Carathéodory if:

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathcal{D}$;

(ii) $x \longmapsto F(t,x)$ is continuous for almost all $t \in [0,\infty)$; here $\mathcal{P}(E)$ is endowed with the Hausdorff metric H_d;

(iii) For each $q > 0$, there exists $h_q \in L^1([0,\infty), \mathbb{R}_+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} := \{\sup |v| : v \in F(t,x)\} \leq h_q(t)$$

for each $x \in \mathcal{D}$ with $\|x\|_{\mathcal{D}} \leq q$, and for almost each $t \in [0,\infty)$.

3. Main result

We shall consider the space

$$PC = \Big\{ y : [-r, \infty) \to E : y(t) \text{ is continuous everywhere except for some } t_k \text{ at which } y(t_k^-), y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-), k = 1, \dots \Big\}.$$

Set

$$\Omega = \{ y : [-r, \infty) \to E : y \in PC \cap \mathcal{D} \}.$$

Let us start by defining what we mean by a solution of the problem (1)–(3).

Definition 3.1. We say that the function $y \in \Omega$ is a mild solution of the system (1)–(3) if $y(t) = \phi(t)$ for all $t \in [-r, 0]$, the restriction of $y(\cdot)$ to the interval $[0, \infty)$ is continuous and there exists $v(\cdot) \in L^1_{loc}([0, \infty), E)$, such that $v(t) \in F(t, y_t)$ a.e $[0, \infty)$, and such that y satisfies the integral equation,

$$\begin{aligned} y(t) &= T_0(t)\phi(0) + \int_0^t T_1(t-s)v(s)ds + \int_0^t T_1(t-s)Bu_y(s)ds \\ &\quad + \sum_{0<t_k<t} T_1(t-t_k)I_k(y(t_k^-)), \quad 0 \le t < \infty. \end{aligned} \tag{4}$$

Definition 3.2. The system (1)–(3) is said to be infinite controllable on the interval $[-r, \infty)\backslash\{t_k\}, k = 1, \dots$ if for every initial function $\phi \in \mathcal{D}$ and every $y_1 \in E$, and for each $n \in \mathbb{N}$, there exists a control $u \in L^2([0, n], U)$, such that the mild solution y of (1)–(3) satisfies $y(n) = y_1$.

Let us introduce the following hypotheses:

(H1) $F : J \times \Omega \to \mathcal{P}_{cp}(E)$ is an L^1-Carathéodory multivalued map.

(H2) There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$\|F(t, x)\|_{\mathcal{P}} \le p(t)\psi(\|x\|_{\mathcal{D}}) \text{ for a.e. } t \in J \text{ and each } x \in \mathcal{D},$$

with

$$\int^{\infty} \frac{ds}{s + \psi(s)} = \infty.$$

(H3) There exists $M > 0$ such that

$$\|T_1(t)\|_{B(E)} \leq M \text{ for each } \ t > 0.$$

(H4) For all $R > 0$ there exists $l_R \in L^1_{loc}([0,\infty), \mathbb{R}_+)$ such that

$$H_d(F(t,x), F(t,\overline{x})) \leq l_R(t)\|x - \overline{x}\|_{\mathcal{D}} \ \text{ for all } x, \ \overline{x} \in \mathcal{D}$$
$$\text{with } \|x\|_{\mathcal{D}}, \|\overline{x}\|_{\mathcal{D}} \leq R,$$

and

$$d(0, F(t,0)) \leq l_R(t) \ \text{ for a.e. } \ t \in J.$$

(H5) There exist constants $c_k \geq 0$, $k = 1, \ldots,$ such that

$$|I_k(y) - I_k(x)| \leq c_k|x - \overline{x}| \ \text{ for each } \quad x, \overline{x} \in E.$$

(H6) For every $n > 0$, the linear operator $W : L^2(J_n, U) \to E$ ($J_n = [0,n]$), defined by

$$Wu = \int_0^n T(n-s)Bu(s)ds,$$

has an invertible operator W^{-1} which takes values in $L^2(J_n, U)\backslash KerW$, and there exist positive constants $\overline{M}$, $\overline{M}_1$ such that $\|B\| \leq \overline{M}$ and $\|W^{-1}\| \leq \overline{M}_1$.

Remark 3.1. For the construction of W see [24].

Theorem 3.1. *Assume that the hypotheses* (H1)–(H6) *hold. If* $M \sum_{k=1}^{\infty} c_k < 1$, *then the IVP* (1)–(3) *is infinite controllable on* $[-r, \infty)$.

Proof. Using the hypothesis (H6) for each $y(\cdot)$ define the control

$$u_y(t) =$$
$$= W^{-1}\Big[y_1 - T(n)\phi(0) - \int_0^n T(n-s)v(s)ds - \sum_{0<t_k<s} T(s-t_k)I_k(y(t_k^-))\Big](t),$$

where

$$v \in S_{F,y} = \Big\{v \in L^1(J,E) : v(t) \in F(t, y_t) \ a.e \ t \in J\Big\}.$$

We shall now show that when using this control, the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ T_0(t)\phi(0) + \displaystyle\int_0^t T_1(t-s)v(s)ds & \\ + \displaystyle\int_0^t T_1(t-s)(Bu_y)(s)ds & \\ + \displaystyle\sum_{0<t_k<t} T_1(t-t_k)I_k(y(t_k^-)), & \text{if } t \in J, \end{cases} \right\}$$

has a fixed point, which is clearly the mild solution of the IVP (1)–(3).

We define on Ω a family of semi-norms, thus rendering Ω into a Fréchet space. Let τ be sufficiently large, then $\forall n \in \mathbb{N}$ we define in Ω the semi-norm:

$$\|y\|_n = \sup\{e^{-\tau L_n(t)}|y(t)| : -r \le t \le n\},$$

where

$$L_n = \int_{-r}^t \hat{l}_n(s)ds$$

and

$$\hat{l}_n(t) = \begin{cases} 0, & \text{if } t \in [-r, 0], \\ l_n(t)[M^2\overline{M}\ \overline{M_1}n + M] + M^2\overline{M}\ \overline{M}_1 \displaystyle\sum_{k=0}^t c_k, & \text{if } t \in [0, n]. \end{cases}$$

Thus $\Omega = \bigcup_{n\ge1} \Omega_n$, where

$$\Omega_n = \Big\{ y : [-r, n] \to E : y \in \mathcal{D} \bigcap PC_n(J, E) \Big\}$$

and

$$PC_n = \Big\{ y : [0, n] \to E : \ y(t) \text{ is continuous everywhere except for some } t_k \text{ at which } y(t_k^-), y(t_k^+) \text{ exist with } \ y(t_k) = y(t_k^-), k = 1, \dots n-1 \Big\}.$$

Then Ω is a Fréchet space with the family of the semi-norms $\{\|.\|_n\}_{n\in\mathbb{N}}$.

Now, using Frigon's alternative, we are able to prove that the operator N has a fixed point.

Let $y \in \lambda N(y)$ for some $\lambda \in [0,1]$, and for some $v \in S_{F,y}$. For each $n \in \mathbb{N}$ and $t \in [0,n]$ we have:

$$y(t) = \lambda\Big[T_0(t)\phi(0) + \int_0^t T_1(t-s)v(s)ds + \int_0^t T_1(t-s)Bu_y(s)ds + \sum_{0<t_k<t} T_1(t-t_k)I_k(y(t_k^-))\Big].$$

Then, we have

$$\begin{aligned}
&|y(t)| \leq \\
&\leq \|T_0(t)\|\|\phi\|_{\mathcal{D}} + \int_0^t |T_1(t-s)|\|v(s)\|ds + \int_0^t |T_1(t-s)||Bu_y(s)|ds \\
&\quad + \sum_{0<t_k<t} |T_1(t-t_k)||I_k(y(t_k^-))| \\
&\leq M\|\phi\|_{\mathcal{D}} + M\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds \\
&\quad + M\int_0^t \|B\|\|u_y(s)\|ds + M\sum_{k=1}^n |I_k(y(t_k^-))| \\
&\leq M\|\phi\|_{\mathcal{D}} + M\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds + M\overline{M}\int_0^t \Big|W^{-1}\Big[y_1 - T_0(n)\phi(0) \\
&\quad - \int_0^n T_1(n-\tau)v(\tau)d\tau - \sum_{0<t_k<\tau} T_1(\tau-t_k)I_k(y(t_k^-))\Big](s)\Big|ds \\
&\quad + M\sum_{k=1}^n |I_k(y(t_k^-))| \\
&\leq M\|\phi\|_{\mathcal{D}} + M\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds + M\overline{M}\ \overline{M_1}\int_0^t \Big[|y_1| + \|T_0(n)\|\|\phi(0)\| \\
&\quad + M\int_0^n p(\tau)\psi(\|y_\tau\|_{\mathcal{D}})d\tau + M\sum_{0<t_k<s} |I_k(y(t_k^-))|\Big]ds \\
&\quad + M\sum_{k=1}^n |I_k(y(t_k^-))| \\
&\leq M\|\phi\|_{\mathcal{D}} + M\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds + M\overline{M}\ \overline{M_1}n|y_1| + M\overline{M}\ \overline{M_1}nM\|\phi\|_{\mathcal{D}}
\end{aligned}$$

$$
\begin{aligned}
&+ M^2\overline{M}\ \overline{M_1}n\int_0^t p(\tau)\psi(\|y_\tau\|_{\mathcal{D}})d\tau + M^2\overline{M}\ \overline{M}_1\int_0^t \sum_{0<t_k<s}|I_k(y(t_k^-))|ds \\
&+ M\sum_{k=1}^n |I_k(y(t_k^-))| \\
\leq\ & M\overline{M}\ \overline{M_1}n|y_1| + \Big[M + M\overline{M}\ \overline{M}_1 n\Big]\|\phi\|_{\mathcal{D}} \\
&+\Big[M + M^2\overline{M}\ \overline{M}_1 n\Big]\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds \\
&+ M^2\overline{M}\ \overline{M}_1\int_0^t \sum_{0<t_k<s}|I_k(y(t_k^-))|ds + M\sum_{k=1}^n |I_k(y(t_k^-))| \\
\leq\ & M\overline{M}\ \overline{M}_1 n|y_1| + \Big[M + M\overline{M}\ \overline{M}_1 n\Big]\|\phi\|_{\mathcal{D}} \\
&+\Big[M + M^2\overline{M}\ \overline{M}_1 n\Big]\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds \\
&+ M^2\overline{M}\ \overline{M}_1\int_0^t \sum_{0<t_k<s}(|I_k(y(t_k^-)) - I_k(0)| + |I_k(0)|)ds \\
&+ M\sum_{k=1}^n(|I_k(y(t_k^-)) - I_k(0)| + |I_k(0)|) \\
\leq\ & M\overline{M}\ \overline{M}_1 n|y_1| + \Big[M + M\overline{M}\ \overline{M}_1 n\Big]\|\phi\|_{\mathcal{D}} \\
&+\Big[M + M^2\overline{M}\ \overline{M}_1 n\Big]\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds \\
&+ M^2\overline{M}\ \overline{M}_1\int_0^t \sum_{0<t_k<s}|I_k(y(t_k^-)) - I_k(0)|ds \\
&+ M\sum_{k=1}^n |I_k(y(t_k^-)) - I_k(0)| + M^2\overline{M}\overline{M}_1\int_0^t \sum_{0<t_k<s}|I_k(0)|ds + M\sum_{k=1}^n |I_k(0)| \\
\leq\ & M\overline{M}\ \overline{M}_1 n|y_1| + \Big[M + M\overline{M}n\overline{M}_1\Big]\|\phi\|_{\mathcal{D}} + \Big[M + M^2\overline{M}\ \overline{M}_1 n\Big]\sum_{k=1}^n \|I_k(0)\| \\
&+\Big[M + M^2\overline{M}n\overline{M_1}\Big]\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds + M^2\overline{M}\ \overline{M_1}\int_0^t \sum_{k=1}^n c_k|y(t_k^-)|ds \\
&+ M\sum_{k=1}^n c_k|y(t_k^-)|.
\end{aligned}
$$

Set

$$c = M\overline{M}\ \overline{M_1}n|y_1| + \Big[M + M\overline{M}n\overline{M_1}\Big]\|\phi\|_{\mathcal{D}} + \Big[M + M^2\overline{M}n\overline{M_1}\Big]\sum_{k=1}^{n}|I_k(0)|.$$

Now, we consider the function μ defined by

$$\mu(t) = \sup\Big\{|y(s)| : -r \le s \le t\Big\},\ t \le n.$$

Let $t^* \in [-r, n]$ be such that $\mu(t) = |y(t^*)|$. It is clear that if $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$. If $t^* \in [0, n]$, we have for each $t \in [0, n]$

$$\begin{aligned}\mu(t) \le\ & c + \Big[M + M^2\overline{M}n\overline{M_1}\Big]\int_0^t p(s)\psi(\mu(s))ds \\ & + M^2\overline{M}\ \overline{M_1}\int_0^t \sum_{0<t_k<s} c_k\mu(s)ds + M\sum_{k=1}^{n} c_k\mu(t).\end{aligned}$$

Then

$$\begin{aligned}\Big[1 - M\sum_{k=1}^{n} c_k\Big]\mu(t) \le\ & c + \Big[M + M^2\overline{M}n\overline{M_1}\Big]\int_0^t p(s)\psi(\mu(s))ds \\ & + M^2\overline{M}\ \overline{M_1}\int_0^t \sum_{0<t_k<s} c_k\mu(s)ds.\end{aligned}$$

Thus we have

$$\mu(t) \le c_* + \int_0^t \hat{M}(s)[\mu(s) + \psi(\mu(s))]ds,$$

where

$$c_* = \frac{c}{1 - M\sum_{k=1}^{n} c_k},$$

and

$$\widehat{M}(s) = \frac{1}{1 - M\sum_{k=1}^{n} c_k}\left[\Big[M + M^2\overline{M}\ \overline{M}_1 n\Big]p(s) + M^2\overline{M}\ \overline{M}_1\sum_{0<t_k<s} c_k\right].$$

Let us take the right hand side of the above inequality as $v(t)$, then we have:

$$v(0) = c_*, \quad \mu(t) \leq v(t) \qquad \forall t \in [0, n]$$

and

$$v'(t) = \widehat{M}(t)(\mu(t) + \psi(\mu(t))).$$

Using the nondecreasing character of ψ, we get:

$$v'(t) \leq \widehat{M}(t)(v(t) + \psi(v(t)) \text{ a.e } t \in [0, n].$$

This implies that for each $t \in [0, n]$

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \psi(s)} \leq \int_0^n \widehat{M}(s)ds \leq \int_{v(0)}^{\infty} \frac{ds}{s + \psi(s)}.$$

Thus from (H2) there exists a constant M_n such that

$$v(t) \leq M_n, \quad \forall t \in [0, n].$$

From the definition of μ, we conclude that

$$\sup \Big\{ |y(t)|, \quad t \in [0, n] \Big\} \leq M_n.$$

Set

$$U_0 = \Big\{ y \in \Omega_n, \quad \|y\|_n \leq M_n + 1 \Big\}.$$

Clearly, U_0 is a closed subset of Ω_n. We shall show that $N : U_0 \to \mathcal{P}(U_0)$ is a contraction and an admissible operator.

First, we prove that N is a contraction; that is, there exists $\gamma < 1$, such that

$$H_d(N(y), N(y^*)) \leq \gamma \|y - y^*\|_n, \quad \text{for each} \quad y, y^* \in U_0.$$

Let $y, y^* \in U_0$ and $h \in N(y)$. Then there exists $v(t) \in F(t, y_t)$ such that for each $t \in [0, n]$

$$h(t) = T_0(t)\phi(0) + \int_0^t T_1(t-s)v(s)ds + \int_0^t T_1(t-s)Bu_y(s)ds$$
$$+ \sum_{0<t_k<t} T_1(t - t_k)I_k(y(t_k^-)).$$

From (H4) it follows that

$$H_d(F(t,y_t),F(t,y_t^*)) \le l_n(t)\|y_t - y_t^*\|_{\mathcal{D}}.$$

Hence there exists $v^* \in F(t,y_t^*)$ such that

$$|v(t) - v^*(t)| \le l_n(t)\|y_t - y_t^*\|_{\mathcal{D}}, \quad \forall t \in [0,n].$$

Let us define for each $t \in [0,n]$

$$\begin{aligned} h^*(t) = T_0(t)\phi(0) &+ \int_0^t T_1(t-s)v^*(s)ds + \int_0^t T_1(t-s)Bu_{y^*}(s)ds \\ &+ \sum_{0<t_k<t} T_1(t-t_k)I_k(y^*(t_k^-)). \end{aligned}$$

Then we have

$$\begin{aligned} &|h(t) - h^*(t)| = \\ &= \Big| \int_0^t T_1(t-s)[v(s) - v^*(s)]ds + \int_0^t T_1(t-s)[B(u_y - u_{y^*})(s)]ds \\ &\quad + \sum_{0<t_k<t} T_1(t-t_k)[I_k(y(t_k^-) - I_k(y^*(t_k^-))]\Big| \\ &\le M\int_0^t l_n(s)\|y_s - y_s^*\|_{\mathcal{D}}ds + M\overline{M}\int_0^t |u_y(s) - u_{y^*}(s)|ds \\ &\quad + M\sum_{0<t_k<t} c_k|y(t_k^-) - y^*(t_k^-)| \\ &\le M\int_0^t l_n(s)\|y_s - y_s^*\|_{\mathcal{D}}ds \\ &\quad + M\overline{M}\int_0^t \Big| W^{-1}\Big[\int_0^n T_1(t-\tau)[v(\tau) - v^*(\tau)]d\tau \\ &\quad + \sum_{0<t_k<\tau} T_1(t-\tau)I_k(y(t_k^-) - y^*(t_k^-))\Big]\Big| ds \\ &\quad + M\sum_{0<t_k<t} c_k|y(t_k^-) - y^*(t_k^-)| \end{aligned}$$

$$
\begin{aligned}
&\leq M \int_0^t l_n(s)\|y_s - y_s^*\|_{\mathcal{D}} ds \\
&\quad + M^2 \overline{M}\ \overline{M}_1 n \int_0^t l_n(s)\|y_s - y_s^*\|_{\mathcal{D}} ds \\
&\quad + M^2 \overline{M}\ \overline{M}_1 \int_0^t \sum_{0<t_k<s} c_k |y(t_k^-) - y^*(t_k^-)| ds \\
&\quad + M \sum_{0<t_k<t} c_k |y(t_k^-) - y^*(t_k^-)| \\
&\leq \Big[M + M^2 \overline{M}\ \overline{M}_1 n\Big] \int_0^t l_n(s)\|y_s - y_s^*\|_{\mathcal{D}} ds \\
&\quad + M^2 \overline{M}\ \overline{M}_1 \int_0^t \sum_{0<t_k<s} c_k |y(t_k^-) - y^*(t_k^-)| ds \\
&\quad + M \sum_{0<t_k<t} c_k |y(t_k^-) - y^*(t_k^-)| \\
&\leq \Big[M + M^2 \overline{M}\ \overline{M}_1 n\Big] \int_0^t l_n(s) e^{\tau L_n(s)} \|y - y^*\|_n ds \\
&\quad + M^2 \overline{M}\ \overline{M}_1 \int_0^t e^{\tau L_n(s)} \sum_{0<t_k<s} c_k \|y - y^*\|_n ds \\
&\quad + M e^{\tau L_n(t)} \sum_{0<t_k<t} c_k \|y - y^*\|_n ds \\
&\leq \left[3 \int_0^t \hat{l}_n(s) e^{\tau L_n(s)} ds + M e^{\tau L_n(t)} \sum_{0<t_k<t} c_k\right] \|y - y^*\|_n \\
&\leq \left[\frac{3}{\tau} e^{\tau L_n(s)}|_0^t + M e^{\tau L_n(t)} \sum_{k=1}^n c_k\right] \|y - y^*\|_n \\
&\leq \left[\frac{3}{\tau} e^{\tau L_n(t)} - \frac{3}{\tau} + M e^{\tau L_n(t)} \sum_{k=1}^n c_k\right] \|y - y^*\|_n.
\end{aligned}
$$

As τ is sufficiently large, thus

$$
|h(t) - h^*(t)| \leq \left[\frac{3}{\tau} + M \sum_{k=1}^n c_k\right] e^{\tau L_n(t)} \|y - y^*\|_n.
$$

Then, it follows

$$|h(t) - h^*(t)|e^{-\tau L_n(t)} \leq \left[\frac{3}{\tau} + M\sum_{k=1}^{n} c_k\right]\|y - y^*\|_n.$$

Therefore,

$$\|h - h^*\|_n \leq \left[\frac{3}{\tau} + M\sum_{k=1}^{n} c_k\right]\|y - y^*\|_n.$$

By an analogous relation, obtained by interchanging the roles of y and y^*, it follows that

$$H_d(N(y), N(y^*)) \leq \left(\frac{3}{\tau} + M\sum_{k=1}^{n} c_k\right)\|y - y^*\|_n.$$

So, N is a contraction. Now, $N : \Omega_n \to \mathcal{P}_{cp}(\Omega_n)$ is given by

$$N(y) = \left\{ h \in \Omega_n : h(t) = \begin{cases} 0, & \text{if } t \in [-r, 0], \\ \displaystyle\int_0^t T_1(t-s)v(s)ds \\ \displaystyle + \int_0^t T_1(t-s)(Bu_y^n)(s)ds \\ \displaystyle + \sum_{0<t_k<t} T_1(t-t_k)I_k(y(t_k^-)), & \text{if } t \in [0, n], \end{cases} \right\}$$

where $v \in S_{F,y}^n = \{u \in L^1([0,n], E) : u(t) \in F(t, y_t) \ a.e. \ t \in [0,n]\}$. From (H4)–(H6) and since F is compact valued, we can prove that for every $y \in \Omega_n$, $N(y) \in \mathcal{P}_{cp}(\Omega_n)$, and there exists $y^* \in \Omega_n$ such that $y^* \in N(y^*)$. (For the proof see Benchohra *et al.* [6]). Let $h \in \Omega_n$, $y^* \in U_0$ and $\varepsilon > 0$. Now, if $\bar{y} \in N(y^*)$, then we have

$$\|y^* - \bar{y}\|_n \leq \|y^* - h\|_n + \|\bar{y} - h\|_n.$$

Since h is arbitrary we may suppose that

$$h \in B(\bar{y}, \varepsilon) = \{k \in \Omega_n : \|k - \bar{y}\|_n \leq \varepsilon\}.$$

Therefore,

$$\|y^* - \bar{y}\|_n \leq \|y^* - N(y^*)\|_n + \varepsilon.$$

On the other hand, if $\bar{y} \notin N(y^*)$, then $\|\bar{y} - N(y^*)\| \neq 0$. Since $N(y^*)$ is compact, there exists $x \in N(y^*)$ such that $\|\bar{y} - N(y^*)\|_n = \|\bar{y} - x\|_n$. Then we have

$$\|y^* - x\|_n \leq \|y^* - h\|_n + \|x - h\|_n.$$

Therefore,

$$\|y^* - x\|_n \leq \|y^* - N(y^*)\|_n + \varepsilon.$$

So, N is an admissible operator contraction. Finally, by Lemma 2.3, N has at least one fixed point, y, which is a mild solution to (1)–(3).

4. Example

As an application of our result, we consider the following impulsive partial functional differential inclusion,

$$\frac{\partial z(t,x)}{\partial t} - d\triangle z(t,x) \in F(t-r,x) + Bu(t), \text{ a.e. } t \in J\backslash\{t_1, t_2, \ldots\}, x \in \Omega \tag{5}$$

$$b_k z(t_k^-, x) = z(t_k^+, x) - z(t_k^-, x), \quad k = 1, \ldots, \ x \in \partial\Omega \tag{6}$$

$$z(t,x) = 0, \quad t \in [0,\infty)\backslash\{t_1, t_2, \ldots\}, \ x \in \overline{\Omega} \tag{7}$$

$$z(t,x) = \phi(t,x), \quad t \in [-r, 0], \ x \in \overline{\Omega}, \tag{8}$$

where d, r, b_k are positive constants, Ω is a bounded open in $\mathbb{R}^n$ with regular boundary $\partial\Omega$, $\triangle = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $\phi \in \mathcal{D} = \{\psi : [-r,0] \times \overline{\Omega} \to \mathbb{R}^n; \psi$ is continuous everywhere except for a countable number of points at which $\psi(s^-), \psi(s^+)$ exist with $\psi(s^-) = \psi(s)$ and $|\psi(\theta, x)| < \infty\}, 0 = t_0 < t_1 < t_2 < \ldots < t_m < \ldots, t_m \to \infty$ as $m \to \infty, z(t_k^+) = \lim_{(h,x)\to(0^+,x)} z(t_k + h, x), z(t_k^-) = \lim_{(h,x)\to(0^-,x)} z(t_k + h, x), F : [0,\infty) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact values.

Consider $E = C(\overline{\Omega}, \mathbb{R}^n)$ the Banach space of continuous functions on $\overline{\Omega}$ with values in $\mathbb{R}^n$, $y(t) = z(t, .)$. Let A be the operator defined in E by $Ay = d\triangle y$, $I_k : E \to \overline{D(A)}$ such that $I_k(y(t_k^-)) = b_k y(t_k^-)$, then the problem (5)–(8) can be written as

$$y'(t) - Ay(t) \in F(t, y_t) + Bu(t), \quad \text{a.e. } t \in J\backslash\{t_1, t_2, \ldots\}, \tag{9}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k \in \{1, 2, \ldots\} \tag{10}$$

$$y(t) = \phi(t), \quad t \in [-r, 0]. \tag{11}$$

We have

$$D(A) = \{y : y \in E, \triangle y \in E, \text{ and } y|_{\partial\Omega} = 0\},$$

and

$$X_0 = \overline{D(A)} = \{y :\ y \in E,\ y|_{\partial\Omega} = 0\} \neq E.$$

So, we can apply the extrapolation method. It is well known from [11] that $\triangle$ satisfies the following properties

(i) $(0, \infty) \subset \rho(\triangle)$

(ii) $\|R(\lambda, \triangle)\| \leq \frac{1}{\lambda}$ for some $\lambda > 0$.

It follows that $\triangle$ satisfies the Hille Yosida Condition (HY). Also, from [5, 12, 19], the family

$$T_0(t)f(s) = (4\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{\frac{-|s-\tau|^2}{4t}} f(\tau)d\tau$$

for $t > 0, s \in \mathbb{R}^n$, and $f \in X_0$ with $T(0) = I$, defines a strongly continuous semigroup on E, its generator A_0 coincides with the closure of the Laplacian operator with domain X_0, and there exist constants $N_0 > 0, \omega > 0$ such that $\|T_0(t)\| \leq N_0 e^{\omega t}$ for $t > 0$.

Thus under appropriate conditions on the function F and the operator B as those mentioned in the hypotheses (H1)–(H6) the problem (5)–(8) has at least one mild solution.

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