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RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES AND HENSTOCK-KURZWEIL-PETTIS INTEGRALS

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Abstract

We prove an existence theorem for the equation $x' = f(t, x_t)$, $x(\Theta) = \varphi(\Theta)$, where $x_t(\Theta) = x(t + \Theta)$, for $-r \le \Theta < 0$, $t \in I_a$, $I_a = [0, a]$, $a \in R_+$ in a Banach space, using the Henstock-Kurzweil-Pettis integral and its properties. The requirements on the function f are not too restrictive: scalar measurability and weak sequential continuity with respect to the second variable. Moreover, we suppose that the function f satisfies some conditions expressed in terms of the measure of weak noncompactness.

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1. Introduction

The Henstock-Kurzweil integral encompasses the Newton, Riemann and Lebesgue integrals ([18, 21, 27]). A particular feature of this integral is that integrals of highly oscillating functions such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on $(0, 1]$ and $F(0) = 0$, can be defined. This integral was introduced by Henstock and Kurzweil independently in 1957–58 and has since proved useful in the study of ordinary differential equations ([1, 7, 8, 24]).

A further step of generalization was done in [9] which applies the Henstock-Kurzweil integrals to the study of Retarded Functional Differential Equations with finite delays, i.e., equations of the form

$$x'(t) = f(t, x_t), \tag{1.1}$$

where $x_t(\Theta) = x(t+\Theta)$ and Θ takes values from $[-r, 0]$ for some finite positive number r, subject to some initial function φ, where φ is some Henstock-Kurzweil integrable function over $[-r, 0]$.

The theory of Retarded Functional Differential Equations of (1.1) has been well-understood when φ and f are continuous functions, hence Riemann integrable.

Hale in [20] notes that the results still hold true when continuity of f is weakened to satisfy Carathéodory conditions. M.C. Deflour and S.K. Mitter in [14] further generalized the theory to the case where the initial function φ and f are Lebesgue integrable.

The further step of generalization which was made in [9] is such that f and φ are only assumed to be Henstock-Kurzweil integrable. In [10] T.S. Chew and T.L. Toh showed that the result of [9] can be generalized to Retarded Functional Differential Equations with unbounded delays under Henstock-Kurzweil integral settings. A. Sikorska-Nowak in [29] generalized previous results to Retarded Functional Differential Equations in Banach spaces, using the HL integral, which was defined by S.S. Cao in [6].

In this paper, we are going to prove the existence theorem for the problem (1.1), where the requirements on the function f are not too restrictive: scalar measurability and weak sequential continuity with respect to the second variable. We generalize both concepts of integrals: Pettis and Henstock-Kurzweil, introducing the Henstock-Kurzweil-Pettis integral.

Let $(E, \|\cdot\|)$ be a Banach space and let E^* be its dual space. Moreover, let $(C(I_a, E), \omega)$ denote the space of all continuous functions from I_a to E endowed with the topology $\sigma(C(I_a, E), C(I_a, E)^*)$.

Let r, a be nonnegative real numbers, $I_a = [0, a]$, $a \in R_+$. Let x be some function defined on $[-r, a]$. For any $t \in I_a$, the function x_t is defined as $x_t(\Theta) = x(t+\Theta)$, where $-r \leq \Theta < 0$. Here Θ may be a function involving t.

Let $f : I_a \times C([-r, 0], E) \to E$ and

$$\begin{cases} x'(t) = f(t, x_t) \\ x(\Theta) = \varphi(\Theta), \end{cases} \tag{1.2}$$

where φ is some specified function.

We will consider the problem

$$(1.3)\qquad \begin{cases} x(t) = \varphi(0) + \int_0^t f(t, x_s)ds & t \in I_a, \\ x_0 = \varphi \end{cases}$$

where the integral is taken in the sense of Henstock-Kurzweil-Pettis.

Our fundamental tool is the measure of weak noncompactness developed by DeBlasi ([5], see also [4]).

The measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{ there exists } C \in K^\omega \text{ such that } A \subset C + tB_0\},$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball in E. We use the following properties of the measure of weak noncompactness $\beta(A)$:

(i) if $A \subset B$ then $\beta(A) \leq \beta(B)$;

(ii) $\beta(A) = \beta(\bar{A})$, where $\bar{A}$ denotes the closure of A;

(iii) $\beta(A) = 0$ if and only if A is relatively weakly compact;

(iv) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;

(v) $\beta(\lambda A) = |\lambda|\beta(A)$, $(\lambda \in R)$;

(vi) $\beta(A + B) \leq \beta(A) + \beta(B)$;

(vii) $\beta(\text{conv}A) = \beta(A)$.

It is necessary to remark that if β has these properties, then the following Lemma is true.

Lemma 1.1 ([26]). *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_a, H(t) = \{h(t) \in E, h \in H\}$. Then $\beta_C(H) = \sup_{t \in I_a} \beta(H(t)) = \beta(H(I_a))$, where $\beta_C(H)$ denotes the measure of noncompactness in $C(I_a, E)$ and the function $t \mapsto \beta(H(t))$ is continuous.*

Fix $x^* \in E^*$ and consider the problem

$$(1.2')\qquad (x^*x)'(t) = x^*f(t, x_t), \quad x(\Theta) = \varphi(\Theta), \quad t \in I_a\,.$$

Let us introduce the following definitions.

Definition 1.2 ([28])**.** Let $F : [a, b] \to E$ and let $A \subset [a, b]$. The function $f : A \to E$ is a *pseudoderivative* of F on A if for each x^* in E^* the real-valued function x^*F is differentiable almost everywhere on A and $(x^*F)' = x^*f$ almost everywhere on A.

From the above definition it is clear that the left-hand side of (1.2') can be rewritten to the form $x^*(x'(t))$ where x' denotes the pseudoderivative.

Definition 1.3 ([18, 27])**.** A family $\mathcal{F}$ of functions F is said to be *uniformly absolutely continuous in the restricted sense on* X or, in short, uniformly $AC_*(X)$ if for every $\varepsilon > 0$ there is $\eta > 0$ such that for every F in $\mathcal{F}$ and for every finite or infinite sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in X$ and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F, [a_i, b_i]) < \varepsilon$ where ω denotes the oscillation of F over $[a_i, b_i]$ (i.e., $\omega(F, [a_i, b_i]) = \sup\{|F(r) - F(s)| : r, s \in [a_i, b_i]\}$).

A family $\mathcal{F}$ of functions F is said to be *uniformly generalized absolutely continuous in the restricted sense on* $[a, b]$ or uniformly ACG_* on $[a, b]$ if $[a, b]$ is the union of a sequence of closed sets X_i such that on each X_i the family $\mathcal{F}$ is uniformly $AC_*(X_i)$.

We will use the following results.

Theorem 1.4 ([23])**.** *Let E be a metrizable locally convex topological vector space. Let D be a closed convex subset of E, and let F be a weakly sequentially continuous map of D into itself. If for some $x \in D$ the implication*

$$\bar{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \quad \Rightarrow \quad V \text{ is relatively weakly compact,} \tag{1.4}$$

holds for every subset V of D, then F has a fixed point.

Let us recall that a function $f : I_a \to E$ is said to be *a weakly continuous* if it is continuous from I_a to E endowed with its weak topology.

A function $g : E \to E_1$, where E and E_1 are Banach spaces, is said to be *a weakly-weakly sequentially continuous* if for each weakly convergent sequence x_n in E, a sequence $(g(x_n))$ is weakly convergent in E_1. The fact that the sequence x_n tends weakly to x_0 in E will be denoted by $x_n \xrightarrow{\omega} x_0$.

A very interesting discussion (including examples) about different types of continuity can be found in [2] and [3]. The notion of weak sequential continuity seems to be most convenient. It is not always possible to show that a given operator between Banach spaces is weakly continuous, quite often its weak sequential continuity offers no problem. This follows from the fact that the Lebesgue dominated convergence theorem is valid for sequences but not for nets.

2. Henstock-Kurzweil-Pettis integrals in Banach spaces

In this part, we present the definition of the Henstock-Kurzweil-Pettis integral and we give properties of this integral. For basic definitions we refer the reader to [18, 21] or [25].

Definition 2.1 ([6])**.** The function $[a,b] \to E$ is *Henstock-Kurzweil integrable* on $[a,b]$ if there exists $A \in E$ with the following property: for every $\varepsilon > 0$ there exists a positive function $\delta(\cdot)$ on $[a,b]$ such that for every division D of $[a,b]$ given by $a = x_0 < x_1 < ... < x_n = b$ and $\{\xi_1, \xi_2, ..., \xi_n\}$ satisfying $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, ..., n$, we have

$$\left\| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right\| < \varepsilon.$$

We write $(HK) \int_a^b f(t)dt = A$. We say that D is δ-fine and we can write $D = \{[u,v]; \xi\}$ with $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$. We will write $f \in HK([a,b], E)$ if f is Henstock-Kurzweil integrable on $[a,b]$.

This definition includes the generalized Riemann integral defined by Gordon ([16]).

Definition 2.2 ([6])**.** A function $f : [a,b] \to E$ is *HL integrable* on $[a,b]$ ($f \in HL([a,b], E)$ if there exists a function $F : [a,b] \to E$, defined on the subintervals of $[a,b]$, satisfying the following property: given $\varepsilon > 0$ there exists a positive function $\delta(\cdot)$ on $[a,b]$ such that if $D = \{[u,v]; \xi\}$ is a δ-fine division of $[a,b]$, we have

$$\sum_{D} \|f(\xi)(v-u) - (F(v) - F(u))\| < \varepsilon.$$

Remark 2.3. We note that $f \in HL([a,b], E)$ implies $f \in HK([a,b], E)$ by the triangle inequality. In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

Definition 2.4 ([28])**.** The function $f : I_a \to E$ is Pettis integrable (P integrable for short) if

(i) $\underset{x^* \in E^*}{\forall}\ x^*f$ is Lebesgue integrable on I_a,

(ii) $\underset{A \subset I_a, A \text{ measurable}}{\forall} \exists g \in E \underset{x^* \in E^*}{\forall} x^*g = (L)\int_A x^*f(s)ds.$

Now we present a definition of the integral which is a generalization of both Pettis and Henstock-Kurzweil integrals.

Definition 2.5 ([13])**.** A function $f : I_a \to E$ is *Henstock-Kurzweil-Pettis integrable* (HKP integrable for short) if there exists a function $g : I_a \to E$ with the following properties:

(i) $\underset{x^* \in E^*}{\forall}\ x^*f$ is Henstock-Kurzweil integrable on I_a and

(ii) $\underset{t \in I_a}{\forall}\ \underset{x^* \in E^*}{\forall}\quad x^*g(t) = (HK)\int_0^t x^*f(s)ds.$

This function g will be called *a primitive of* f and by $g(a) = \int_0^a f(t)dt$ we will denote the Henstock-Kurzweil-Pettis integral of f on the interval I_a.

Remark 2.6. Our notion of integral is essentially more general than the previous ones (in Banach spaces):

(a) Pettis integral. By the definition of the Pettis integral and since each Lebesgue integrable function is HK integrable we can put the Lebesgue integral in condition (i) of Definition 2.4 and as a consequence we obtain, that P integrable function is HKP integrable.

(b) Bochner, Riemann, and Riemann-Pettis integrals ([16]).

(c) McShane integral ([19]).

(d) Henstock-Kurzweil integral, HL integral: we present an example below.

Example. We present an example of a function which is HKP integrable and neither HL integrable nor P integrable.

Let $f : [0,1] \to (L^\infty[0,1], \|\cdot\|_\infty)$ and let $f(t) = \chi_{[0,t]} + A(t) \cdot F'(t)$, where

$$\begin{matrix} F(t) = t^2 \sin t^{-2} \\ F(0) = 0 \end{matrix} \quad , \quad \chi_{[0,t]}(\tau) = \begin{cases} 1, & \tau \in [0,t] \\ 0, & \tau \notin [0,t], \end{cases} \qquad t, \tau \in [0,1],$$

$A(t)(\tau) = 1$ for $t, \tau \in [0,1]$.

Put $f_1(t) = \chi_{[0,t]}$, $f_2(t) = A(t)F'(t)$.

We will show that a function $f(t) = f_1(t) + f_2(t)$ is integrable in the sense of Henstock-Kurzweil-Pettis.

Let us observe that

$$x^*f(t) = x^*(f_1(t) + f_2(t)) = x^*(f_1(t)) + x^*(f_2(t)).$$

The function $x^*(f_1(t))$ is Lebesgue integrable (in fact f_1 is Pettis integrable [15]), so it is Henstock-Kurzweil integrable, and the function $x^*(f_2(t))$ is Henstock-Kurzweil integrable by Definition 2.5.

For each $x^* \in E^*$ the function x^*f is not Lebesgue integrable because x^*f_2 is not Lebesgue integrable. So f is not Pettis integrable. Moreover, the function f_1 is not strongly measurable ([15]) and the function f_2 is strongly measurable. So their sum f is not strongly measurable. Then by Theorem 9 from [6] f is not HL integrable.

In the sequel, we will investigate some properties of the HKP integral which are important in the next part of our paper.

Theorem 2.7 ([13])**.** *Let $f : [a,b] \to E$ be HKP integrable on $[a,b]$ and let $F(x) = \int_a^x f(s)ds$.*

(i) *For each x^* in E^* the function x^*f is HK integrable on $[a,b]$ and $(HK)\int_a^x x^*f(s)ds = x^*F(x)$.*

(ii) *The function F is weakly continuous on $[a,b]$ and f is a pseudoderivative of F on $[a,b]$.*

Theorem 2.8 ([11])**.** *Let $f_n, f : I_a \to E$ and assume that $f_n : I_a \to E$ are HKP integrable on I_a. Let F_n be a primitive of f_n. If we assume that:*

(i) $\underset{x^*\in E^*}{\forall}$ *$x^*f_n(t) \to x^*f(t)$ a.e. on I_a,*

(ii) *for each $x^* \in E^*$ the family $G = \{x^*F_n : n = 1, 2, \ldots)$ is uniformly ACG_* on I_a (i.e., weakly uniformly ACG_* on I_a),*

(iii) *for each $x^* \in E^*$ the set G is equicontinuous on I_a,*

then f is HKP integrable on I_a and $\int_0^t f_n(s)ds$ tends weakly in E to $\int_0^t f(s)ds$ for each $t \in I_a$.

Theorem 2.9 ([13]). *If the function $f : I_a \to E$ is HKP integrable, then*

$$\int_I f(t)dt \in |I| \cdot \overline{\text{conv}}\, f(I),$$

where $\overline{\text{conv}}\, f(I)$ is the closure of the convex of $f(I), I$ is an arbitrary subinterval of I_a and $|I|$ is the lenght of I.

3. Main result

Now we prove an existence theorem for the problem (1.2) under the weakest assumptions of f, as it is known.

Two functions φ_1, φ_2 which are HKP integrable on some interval $[u, v]$ are said to belong to the same equivalence class if $\varphi_1(t) = \varphi_2(t)$ almost everywhere in $[u, v]$.

Let $H[u, v]$ denote the space of equivalence classes of functions which are HKP integrable on $[u, v]$. The norm $\|\cdot\|_H$ on $H[u, v]$ is defined as follows: for $P \in H[u, v]$, $\|P\|_H = \sup_{t\in[u,v]} \|\Phi(t)\|$, where $\Phi(t) = \int_u^t \psi(s)ds$ for any $\psi \in P$.

Let φ be some function fixed in $H[-r, 0]$, where $r > 0$. The sets Ω_b and $R_{a,b}$ are defined as $\Omega_b = \{x \in H[-r, 0], \|x - \varphi\|_H \leq b\}$, $R_{a,b} = I_a \times \Omega_b$, where a, b are positive numbers.

Continuity here is understood in the sense that if $\{x_n\}$, $n = 1, 2, ...$ is a sequence in Ω_b and $x_n(s)$converges uniformly on $[-r, 0]$ to some $x_0 \in \Omega_b$ as $n \to \infty$, then for almost all $t \in I_a$, $f(t, x_n)$ converges to $f(t, x_0)$ as $n \to \infty$.

It is convenient here to introduce an auxiliary function $\widehat{x}$: if x is defined on $I_\alpha(0 < \alpha < a)$ with $x(0) = \varphi(0)$, the function $\widehat{x}$ is defined as:

$$\widehat{x}_t = \begin{cases} x(t), & t \in [0, \alpha], \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

The set $A(\varphi, a) \subset C(I_a, E)$ is defined as

$$A(\varphi, a) = \{x \in C(I_a, E) : x(0) = \varphi(0), \|x\| \leq b + \|\varphi(0)\|, \hat{x}_t \in \Omega_b\}.$$

It is easy to see that the set $A(\varphi, a)$ is bounded, closed and convex.

Let $F : C(I_a, E) \to C(I_a, E)$ be defined by $F(x)(t) = x_0 + \int_0^t f(s, \hat{x}_s)ds$, for $t \in I_a$ and $x \in A(\varphi, a)$, where the integral is taken in the sense of HKP. Moreover, let $K = \{F(x) \in C(I_a, E) : x \in A(\varphi, a)\}$.

Now we are able to introduce the definition of pseudo-solution which we will use in the sequel.

Definition 3.1 ([22]). A function $x : I_a \to E$ is said to *be a pseudo-solution* of the problem (1.2) if it satisfies the following conditions:

(i) $x(\cdot)$ is ACG_*,

(ii) $x(\Theta) = \varphi(\Theta)$,

(iii) for each $x^* \in E^*$ there exists a set $A(x^*)$, with a Lebesgue measure zero, such that for each $t \notin A(x^*)$

$$x^*(x'(t)) = x^*(f(t, x_t)).$$

Here "$'$" denotes a pseudoderivative (see [26]).

Theorem 3.2. *Let φ be some fixed function in $H[-r, 0]$. Assume that for each ACG_* function $x : I_a \to E, f(t, x_t)$ is HKP integrable, $f(t, \cdot)$ is a weakly-weakly sequentially continuous function defined on $R_{a,b}$ for some positive numbers and*

$$\beta(f(I, X)) \le d \cdot \beta(X), 0 \le da < 1, \tag{3.1}$$

for each bounded subset $X \subset E$, and $I \subset I_a$, where β is DeBlasi measure of weak noncompactness. Suppose that K is equicontinuous and uniformly ACG_* on I_a. Then there exists a pseudo- solution of the problem (1.2) *on I_α, for some $0 < \alpha \le a$ with initial function φ.*

Proof. We will prove, in fact, the existence of a solution for the problem (1.3). By Theorem 2.7(i) each solution of the problem (1.3) is a solution of the problem (1.2). Fix an arbitrary $b \ge 0$. By equicontinuity of K, there exists a number α, $0 < \alpha \le a$ such that

$$\left\| \int_0^t f(s, \hat{x}_s) ds \right\| \le b, \quad \text{for} \quad t \in I_a, \qquad \left\| \int_{-r}^{\tau} [\varphi(0) - \varphi(s)] ds \right\| < k,$$

$$\left\| \int_{-r}^{\tau} \int_0^{t+s} f(p, \hat{x}_p) dp ds \right\| < l\,, \qquad k + l = b \quad \text{and} \quad x \in A(\varphi, \alpha).$$

By our assumptions the operator F is well defined and maps $A(\varphi, \alpha)$ into $A(\varphi, \alpha)$ because:

(i) $$\left\| \varphi(0) + \int_0^t f(s, \hat{x}_s) ds \right\| \le \|\varphi(0)\| + \left\| \int_0^t f(s, \hat{x}_s) ds \right\| \le \|\varphi(0)\| + b$$

$$\text{(ii)} \quad \left\|\hat{F}(x_t) - \varphi\right\|_H = \sup_{\tau\in[-r,0]} \left\|\int_{-r}^{\tau} [\hat{F}(x_t)(s) - \varphi(s)]ds\right\|$$

$$= \sup_{\tau\in[-r,0]} \left\|\int_{-r}^{\tau} [\hat{F}(x)(t+s) - \varphi(s)]ds\right\|$$

$$= \sup_{\tau\in[-r,0]} \left\|\int_{-r}^{\tau} \left[\varphi(0) + \int_{0}^{t+s} f(p, \hat{x}_p)dp - \varphi(s)\right] ds\right\|$$

$$\le \sup_{\tau\in[-r,0]} \left\|\int_{-r}^{\tau} [\varphi(0) - \varphi(s)]ds + \int_{-r}^{\tau}\int_{0}^{t+s} f(p, \hat{x}_p)dpds\right\|$$

$$= \sup_{\tau\in[-r,0]} \left\|\int_{-r}^{\tau} [\varphi(0) - \varphi(s)]ds\right\| + \sup_{\tau\in[-r,0]} \left\|\int_{-r}^{\tau}\int_{0}^{t+s} f(p, \hat{x}_p)dpds\right\|$$

$$\le k + l = b.$$

We will show that the operator F is sequentially continuous.

By Lemma 9 of [25] a sequence $x_n(\cdot)$ is weakly convergent in $C(I_\alpha, E)$ to $x(\cdot)$ iff $x_n(t)$ tends weakly to $x(t)$ for each $t \in I_\alpha$, so if $x_t^n \xrightarrow{\omega} x$ in $C(I_\alpha, E)$ then $f(t, x_t^n) \xrightarrow{\omega} f(t, x_t)$ in E for $t \in I_\alpha$ and by Theorem 2.8 we have

$$\lim_{n\to\infty} \int_0^t f(s, x_s^n)ds = \int_0^t f(s, x_s)ds$$

weakly in E, for each $t \in I_\alpha$.

We see that $F(x_n)(t) \to F(x)(t)$ weakly in E for each $t \in I_\alpha$ so $F(x_n) \to F(x)$ in $C((I_\alpha, E), \omega)$.

Suppose that $V \subset A(\varphi, \alpha)$ satisfies the condition $\bar{V} = \overline{\text{conv}}\,(F(V) \cup \{x\})$ for some $x \in A(\varphi, \alpha)$. We will prove that V is relatively weakly compact in $A(\varphi, \alpha)$, thus (1.4) is satisfied. Theorem 1.4 will ensure that F has a fixed point.

Let, for $t \in I_\alpha$, $V(t) = \{v(t) \in E,\ v \in V\}$. Put

$$\left\{\int_0^t f(s, \hat{x}_s)ds,\ x \in V\right\} = \int_0^t f(s, V_s),$$

where

$$V_s = \{\hat{x}_s : x \in V\},\ F(V(t)) = \varphi(0) + \int_0^t f(s, V_s)ds\,.$$

By the properties of the measure of weak noncompactness, the assumption (3.1) and Theorem 2.9 we have

$$\beta(F(V(t))) = \beta\left(\varphi(0) + \int_0^t f(s, V_s)ds\right) \leq \beta\left(\int_0^t f(s, V_s)ds\right)$$
$$\leq \beta(t \cdot \overline{\text{conv}}\, f([0,t], V_t([0,t])))$$
$$\leq t \cdot \beta(f([0,t], V_t([0,t])) \leq a \cdot \beta(f(I_\alpha, V_t(I_\alpha))) \leq a \cdot d \cdot \beta(V_t(I_\alpha)).$$

By Lemma 1.1 we get

$$\beta(V_t(I_\alpha)) = \sup_{t \in I_\alpha} \beta(V_t),$$

so

$$\beta(F(V(t))) \leq a \cdot d \cdot \beta(V_t(I_\alpha)), \quad \text{for each} \quad t \in I_\alpha.$$

Because $\overline{V} = \overline{\text{conv}}\ (F(V) \cup \{x\})$ then by the property of measure of weak noncompactness we have

$$\beta(V(t)) = \beta(\overline{\text{conv}}\, F(V(t)) \cup \{x\}) \leq \beta(F(V(t))) \leq a \cdot d\beta(V(I_\alpha)).$$

Because $0 \leq a \cdot d < 1$ so $\beta(V(t)) = 0$, for each $t \in I_\alpha$.

By Arzelà-Ascoli theorem, V is relatively weakly compact in $C(I_a, E)$. So, by Theorem 1.4, F has a fixed point which is a pseudo-solution of the problem (1.3). Because each solution of the problem (1.3) is a solution of the problem (1.2), so there exists a pseudo-solution of the problem (1.2).

Remark 3.3. The condition (1.4) in our Theorem 3.2 can be generalized to the Sadovskii condition: $\beta(F(I, X)) < \beta(X)$, whenever $\beta(X) > 0$, where β can be replaced by some axiomatic measure of weak noncompactness.

Remark 3.4. As we generalize both Pettis and Henstock-Kurzweil integrals our existence theorem is an extension of previous results; for example: T.S. Chew, W. van Brunt, G.C. Wake ([9]), T.S. Chew, T.L. Toh ([10]), M.C. Deflour, S.K. Mitter ([14]), A. Sikorska-Nowak ([29]) and others.

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