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A RELAXATION THEOREM FOR PARTIALLY OBSERVED STOCHASTIC CONTROL ON HILBERT SPACE

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Abstract

In this paper, we present a result on relaxability of partially observed control problems for infinite dimensional stochastic systems in a Hilbert space. This is motivated by the fact that measure valued controls, also known as relaxed controls, are difficult to construct practically and so one must inquire if it is possible to approximate the solutions corresponding to measure valued controls by those corresponding to ordinary controls. Our main result is the relaxation theorem which states that the set of solutions corresponding to ordinary controls is weakly dense in the set of solutions corresponding to relaxed controls. This is presented in Theorem 5.3 after giving some existence results on optimal controls for the infinite dimensional Zakai equation used for its proof.

Keywords: partially observed control, infinite dimensional Hilbert space, relaxed controls, Zakai equation.

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1. Introduction

We consider the following controlled system governed by a pair of stochastic differential equations as described below:

$$\begin{aligned} dx &= Axdt + F(x)dt + B(x,u_t)dt + \sqrt{Q}dW, x(0) = x_0 \\ dy &= h(x,y)dt + \sigma_0(y)dw^o, y(0) = 0. \end{aligned} \tag{1.1}$$

The first equation is defined on an infinite dimensional Hilbert space H describing the dynamics of state and the other is defined on a finite dimensional Eucledian space R^d which describes the dynamics of observation and measurement. The process x, which is not directly accessible, is controlled through a controller u which exercises its control actions on the basis of available information about x through the process y which is physically accessible to the controller. The fundamental objective is to find from a suitable class of maps, which are $\mathcal{F}_t^y$-adapted, a control law which minimizes the expected cost functional

$$J(u) \equiv E \int_I \ell(t, y(t), x(t), u_t) dt. \tag{1.2}$$

Generally, A is an unbounded operator with the domain and range in a separable Hilbert space H, F is a nonlinear continuous bounded operator in H and B is also a continuous bounded map from $H \times \mathcal{Z}$ to H where $\mathcal{Z}$ is a suitable set from which controls assume their values. The map Q is a symmetric positive operator in H and W is a cylindrical Brownian motion with values in H. The operator h is a continuous bounded map from $H \times R^d$ to R^d and σ_0 is a continuous bounded map from R^d to the space of symmetric $d \times d$ matrices and w^o is a d-dimensional standard Brownian motion. A precise hypothesis will be introduced shortly.

Partially observed finite dimensional control problems have been studied extensively by many authors over the last two decades [13–17]. Fully observed infinite dimensional control problems have also been extensively studied in several papers through Hamilton -Jacobi- Bellman equations [2, 3, 4, 5, 6]. Zhu and Ahmed [5] considered HJB equations on Banach spaces related to fully observed stochastic control problems. Gatarek and Goldys [12] have studied fully observed control problem using the method of substitution of drifts via Girsanov transformation. In recent years, infinite dimensional partially observed control problems have been considered in [7, 9, 14, 19, 23]. In [9] the author considers a partially observed control problem for stochastic differential inclusions using stochastic approach and the theory of monotone operators. In [1], Da-Prato and Zabczyk developed a generation theorem for Kolmogorov operators on the Hilbert space H constructing a C_0-semigroup of bounded linear operators on $L_2(H, \mu^r)$ where μ^r is a reference measure on H. This opened up the prospects of treating infinite dimensional nonlinear filtering problems and partially observed stochastic control problems using an analytic approach [6, 19, 23, 24] in contrast to the stochastic approach

[7, 9]. Based on the analytic approach, recently some results on partially observed control problems proving existence of optimal relaxed (measure valued) controls and necessary conditions of optimality were developed in [19, 23].

In this paper, we use the results of [19, 23, 24] as the starting point to study the question of relaxability of nonconvex partially observed stochastic control problems in infinite dimension. In other words, given the existence of optimal relaxed controls for a nonconvex problem (convexified), the question is whether there exists a regular control-state pair approximating the relaxed control-state pair to any prescribed degree of accuracy. This is the main concern of this paper and a result in this direction is given in Section 5. Some assumptions in [23] are relaxed at the expense of some others using weak solutions instead of mild solutions.

The rest of the paper is organized as follows: In Section 2, some notations are introduced, in Section 3, basic assumptions and some background results due to Da Prato and Zabczyk [1]; Ahmed and Zabczyk [6, 19]; and Ahmed, Fuhrman, Zabczyk [24] are quoted for convenience of the readers. Here admissible controls are introduced and the partially observed control problem is formulated as an equivalent fully observed one. In Section 4, a priori bounds and a recent result [19] on the existence of a weak solution to the controlled Zakai equation is given. In Section 5, after a brief discussion of the existence of optimal controls, the main result of this paper on relaxation is presented.

2. Basic notations

Let H denote a separable Hilbert space. For any Banach space E, $C(I,E)$ denotes the space of continuous functions on I with values in E. $C_b(H)(B_b(H))$ is the space of bounded continuous (measurable) functions on H. $\mathcal{M}_b(H)$ is the space of countably additive bounded signed measures on the measurable space $(H,\mathcal{B}(H))$ and $\mathcal{M}_1(H) \subset \mathcal{M}_b$ is the space of probability measures. For $\mu \in \mathcal{M}_b(H)$ and $\phi \in B_b(H)$ we use $\mu(\phi)$ to denote the functional $\int_H \phi(x)\mu(dx)$. For any pair of Banach spaces X, Y, $\mathcal{L}(X,Y)$ will denote the space of linear operators from X to Y. For any locally convex topological vector space $\mathcal{Z}$, $2^{\mathcal{Z}} \setminus \emptyset$ will denote the space of nonempty subsets of $\mathcal{Z}$, and $c(\mathcal{Z})$, (cb(Z) cc(Z) $cbc(\mathcal{Z})$, $kc(\mathcal{Z})$) denotes the class of nonempty closed (closed bounded, closed convex, closed bounded convex, compact convex) subsets of $\mathcal{Z}$.

Let $(\Sigma, \mathcal{P})$ be an arbitrary measurable space and X a Polish space. A multifunction $F : \Sigma \longrightarrow 2^X \setminus \emptyset$ is said to be measurable (weakly) if for every closed (open) set $C \subset X$ the set

$$F^-(C) \equiv \{\sigma \in \Sigma : F(\sigma) \cap C \neq \emptyset\} \in \mathcal{P}.$$

Let X, Y be any two topological spaces and $F : X \mapsto c(Y)$ a multifunction. F is said to be [upper] (lower) semicontinuous with respect to inclusion if for every $x_0 \in X$ and every open set $V \subset Y$ satisfying $[F(x_0) \subset V]$ $(V \cap F(x_0) \neq \emptyset)$, there exists an open set $U \subset X$ containing x_0 such that $[F(x) \subset V]$ $(F(x) \cap V \neq \emptyset)$ for all $x \in U$. If Y is a metric space with metric d, then one can introduce a metric d_H, called the Hausdorff metric, on $cb(Y)$ as follows:

$$d_H(C, D) \equiv \max\{\sup_{y \in D} d(C, y),\ \sup_{z \in C} d(z, D)\}$$

for $C, D \in cb(Y)$. If (Y, d) is complete, then so is $(cb(Y), d_H)$. $F : X \mapsto cb(Y)$ is said to be continuous in the Hausdorff metric if, whenever $x_n \longrightarrow x$ in the topology of X,

$$\lim_{n \to \infty} d_H(F(x_n), F(x)) = 0.$$

It is said to be quasi upper semicontinuous if for each $x \in X$

$$d^*(F(x_n), F(x)) \equiv \sup\{d(y, F(x)),\ y \in F(x_n)\} \longrightarrow 0$$

whenever $x_n \longrightarrow x$. In Hou [22] quasi upper semicontinuity is called mild upper semicontinuity. For different types of continuity see [20, 21, 22].

Let $(\Omega, \mathcal{F}, \mathcal{F}_t \uparrow\subset \mathcal{F}, t \geq 0, P)$ denote a complete probability space furnished with an increasing family of right continuous complete sub σ-algebras $\mathcal{F}_t \subset \mathcal{F}$. All random processes considered in the paper will be assumed to be $\mathcal{F}_t$-predictable processes unless stated otherwise.

3. Basic assumptions with background results

To study the control problem we shall make use of Da-Prato-Zabczyk Semigroup [1] which is an extension of the Markov transition operator corresponding to the stochastic evolution equation:

$$\begin{aligned} dx &= Axdt + F(x)dt + \sqrt{Q}dW, \\ x(0) &= x_0. \end{aligned} \tag{3.1}$$

For this we need the following assumptions.

(H1)

(a) A is the infinitesimal generator of a C_0-semigroup, $T(t), t \geq 0$ in H satisfying

$$\| T(t) \|_{\mathcal{L}(H)} \leq Me^{-\omega t},\ t \geq 0,\ \omega > 0,\ M \geq 1$$

(b) Q is a positive, symmetric, bounded operator in H so that the operator Q_t given by

$$Q_t x \equiv \int_0^t T(s)QT^*(s)xds,\ x \in H, t \geq 0$$

is nuclear for all $t \geq 0$ and $\sup_{t\geq 0} TrQ_t < \infty$.

(c) W is a cylindrical Wiener process with values in H with $CovW(1) = I$.

(H2) F is a bounded Lipschitz mapping from H to H.

(H3) For all $t \geq 0$, $RangeT(t) \subset Range(Q_t^{1/2})$ and the operator valued function $\Gamma(t) \equiv (Q_t^{-1/2}T(t)), t \geq 0$, is Laplace transformable.

Let $D\phi$ and $D^2\phi$ denote the first and the second Frechet derivatives of the function $\phi : H \longrightarrow R$, whenever they exist as elements of H and $\mathcal{L}(H)$ respectively. Let $L_2(H, \mu^r)$ denote the equivalence class of real valued functions on H which are square integrable with respect to a reference measure μ^r on $B(H)$, the Borel algebra of subsets of H. Furnished with the natural scalar product and the associated norm it is a Hilbert space. Let $W^{1,2}(H, \mu^r) \subset L_2(H, \mu^r)$ denote the Sobolev space furnished with the norm topology

$$\| \phi \|_{W^{1,2}(H,\mu^r)} \equiv \left(\| \phi \|^2_{L_2(H,\mu^r)} + \| D\phi \|^2_{L_2(H,\mu^r)} \right)^{1/2}.$$

Define the operators $\mathcal{A}_0, \mathcal{B}$ and $\mathcal{A}$ by

$$\begin{aligned} D(\mathcal{A}_0) \equiv \Big\{ & \phi \in C_b^2(H) : D^2\phi \in \mathcal{L}_1(H),\ \sup_{x\in H} \| D^2\phi \|_{\mathcal{L}_1(H)} < \infty \\ & \text{and there exists } \ \psi \in C_b^2(H) : \phi(x) = \psi(A^{-1}x),\ x \in H \Big\}, \\ \mathcal{A}_0\phi \equiv & (1/2)Tr(QD^2\phi) + (x, A^*D\phi), \ \text{ for } \phi \in D(\mathcal{A}_0), \end{aligned}$$

$$(\mathcal{B}\phi)(\cdot) \equiv \langle F(\cdot),\ D\phi(\cdot)\rangle,\ \phi \in W^{1,2}(H, \mu^r)$$

$$\text{and } \ \mathcal{A} \equiv (\bar{\mathcal{A}}_0 + \mathcal{B}),\ D(\mathcal{A}) = D(\bar{\mathcal{A}}_0),$$

where $\mathcal{L}_1(H)$ is the space of nuclear operators in H and $\bar{\mathcal{A}}_0$ denotes the (strongly) closed extension of $\mathcal{A}_0$. We consider the semigroup $S(t), t \geq 0$, corresponding to the Kolmogorov operator $\mathcal{A}$ associated with the nonlinear stochastic evolution equation (3.1). Recently it was shown by Da Prato and Zabczyk [1, Theorem 2.10] that under the assumptions (H1)–(H3), the linear version of (3.1) ($F \equiv 0$) has a unique invariant measure μ^0 on $\mathcal{B}(H)$ and the operator $\mathcal{A}$ generates a C_0-semigroup of bounded linear operators, $S(t), t \geq 0$, in $L_2(H, \mu^0)$ which is the extension of the associated Markov semigroup from $B_b(H)$ to $L_2(H, \mu^0)$. Further $D(\mathcal{A}) \subset W^{1,2}(H, \mu^0)$ and for $t > 0$, $S(t)$ is a family of compact operators in $L_2(H, \mu^0)$. This result was then used in [6, Ahmed-Zabczyk; 24, Ahmed-Fuhrman-Zabczyk] to develop nonlinear filtering equations for the system (1.1) with $B \equiv 0$ giving Kushner and Zakai equations on the Hilbert space H. Based on these results partially observed control problem, as stated in the introduction, was formulated as an equivalent fully observed control problem with Zakai equation considerd as the state equation see [19, 23]. For convenience of reference we quote this result as Theorem 3.1.

Throughout the rest of the paper we take the invariant measure μ^0 as our reference measure μ^r without further notice. In fact, one can choose any measure which is equivalent to μ^0. First, we introduce the admissible controls.

Admissible Controls

Let $I \equiv [0, T], T < \infty$, and $Y \equiv C(I, R^d)$ and $\mathcal{B}(Y)$ denote the Borel σ-algebra on Y and $\mathcal{B}_t(Y), t \in I$, denote the family of increasing subsigma algebras of the sigma algebra $\mathcal{B}(Y)$. Consider the set $\Sigma \equiv I \times Y$ and the class of nonanticipating subsets of Σ given by the rectangles of the form

$$\mathcal{R} \equiv \{\{0\} \times K, K \in \mathcal{B}_0(Y) \text{ and } (s, t] \times K \subset \Sigma : K \in \mathcal{B}_s(Y), 0 \leq s < t < \infty\}.$$

Let $\mathcal{P} \equiv \sigma(\mathcal{R})$ denote the σ-algebra of predictable subsets of the set Σ generated by the Borel completion of $\mathcal{R}$. For any probability measure η on $\mathcal{B}(Y)$, let $\hat{\eta}$ denote the restriction of the product measure $dt \times \eta(d\omega)$ on the predictable σ-field $\mathcal{P}$. We assume that $\mathcal{P}$ has been completed with respect to the measure $\hat{\eta}$. Let $\mathcal{Z}$ be a compact Polish space and $\mathcal{U}(\Sigma, \mathcal{Z})$ denote the class of functions from Σ to $\mathcal{Z}$ which are Borel measurable with respect to the predictable σ-field $\mathcal{P}$. In other words, $v \in \mathcal{U}(\Sigma, \mathcal{Z})$ if and only if $v^{-1}(\Delta) \in \mathcal{P}$ for every $\Delta \in \mathcal{B}(\mathcal{Z})$. Let $U : \Sigma \longrightarrow kc(\mathcal{Z})$ be a $\mathcal{P}$-measurable multifunction in the sense defined in section 2. For a $\mathcal{P}$-measurable multifunction U, it is

clear that, for each $(t,\omega) \in \Sigma$, $U(t,\omega)$ is $\mathcal{B}_t(Y)$ adapted for each $t \geq 0$. We take for the admissible controls the set

$$\mathcal{U}_{ad} \equiv \{u \in \mathcal{U}(\Sigma, \mathcal{Z}) : u(t,\omega) \in U(t,\omega), \quad \hat{\eta} \ \ a.e \ \ (t,\omega) \in \quad \Sigma\}.$$

In other words, the admissible controls are given by the $\mathcal{P}$-measurable selections of the multifunction U. Intuitively, a control in this class has the representation of the form: $u_t = C(t,\omega) = C(t,\pi_t y)$, where $C \in \mathcal{U}_{ad}$ is determined by u alone and $\pi_t y$ denotes the history or a part thereof of the process $\{y(s), 0 \leq s \leq t\}$ up to the current time t. By definition, it is a causal and non anticipative map.

For the formulation of the control problem we need some additional assumptions. Define $H_d \equiv H \times R^d$ with the standard scalar product and norm.

(H4) B and $Q^{-1/2}B : H \times \mathcal{Z} \longrightarrow H$ are bounded Borel measurable maps, Lipschitz in the first variable and continuous in the second.

(H5) $h : H_d \longrightarrow R^d$ is a bounded Lipschitz map and $\sigma_0 : R^d \longrightarrow \mathcal{L}(R^d)$ is a bounded continuous map having bounded inverse.

The partially observed control problem as stated in the introduction can be formulated as an equivalent fully observed control problem as presented below.

Theorem 3.1. *Suppose the hypotheses* (H1)–(H5) *hold and that the initial measure* π_0, *corresponding to the initial state* x_0, *is absolutely continuous with respect to the reference measure* μ^0 *with the Radon-Nikodyme derivative* $\frac{d\pi_0}{d\mu^0} \equiv \rho^0 \in L_2(H,\mu^0)$. *Then the partially observed control problem* (1.1)–(1.2) *is equivalent to the following fully observed control problem on the Hilbert space* $\mathcal{H} \equiv L_2(H,\mu^0)$: *Find a control* $u \in \mathcal{U}_{ad}$ *such that*

$$J(u) \equiv \int_\Sigma \int_H \{\ell(t,y,x,u_t)\rho_t(x)\}\mu^0(dx)\hat{\eta}(dt \times d\omega) \Longrightarrow \inf \tag{3.2}$$

where ρ *is the solution of the evolution equation*

$$d\rho(t) = \mathcal{A}^*\rho(t)dt + L_u^*\rho(t)dt + G(\rho(t))dy(t),\ t \geq 0,\ \rho(0) = \rho_0, \tag{3.3}$$

on the Hilbert space $\mathcal{H}$ *corresponding to the control law* u, *and the operators* L_u *and* G *are given by* $L_u\phi \equiv (B(\cdot,u), D\phi(\cdot))$ *and* $G(\rho) \equiv \sigma_0^{-1}h\rho$.

Proof. (see [19, 23]).

For easy reference, in the following section we present a recent result on the existence of solutions of equation (3.3) (see [19]). Throughout the rest of the paper we assume that $\hat{\eta}$ is constructed from the measure η induced by the process y given by the equation $dy = \sigma_0(y)dw^o, y(0) = 0$, or any other measure absolutely continuous with respect to η. This is justified by Girsanov argument on substitution of measures corresponding to change of drifts (see [3, 14, 19, 23]).

4. Existence of weak solutions

In this section, we discuss the question of existence of weak solution of equation (3.3) under some general conditions. For mild solutions see ([23, 24]). For convenience of notation we let $\mathcal{H} \equiv L_2(H, \mu^0), V \equiv W^{1.2}(H, \mu^0)$. Let V^* denote the topological dual of V. Since the embedding $V \hookrightarrow \mathcal{H}$ is continuous and dense, identifying $\mathcal{H}$ with its own dual we obtain the Gelfand triple

$$V \hookrightarrow \mathcal{H} \hookrightarrow V^*$$

where the embeddings are actually compact. Let $< \cdot, \cdot >_{V^*,V}$ denote the duality pairing of elements of V^* with those of V, and $(\cdot, \cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$. Note that for $\xi \in \mathcal{H}$, $< \xi, v >_{V^*,V} = (\xi, v)_{\mathcal{H}}$. Let $L_2^e(I, V) \equiv L_2^e(V), L_2^e(I, \mathcal{H}) \equiv L_2^e(\mathcal{H})$ and $L_2^e(I, V^*) \equiv L_2^e(V^*)$ denote the Banach spaces of $\mathcal{F}_t^y$-predictable or equivalently $\mathcal{P}$ measurable $V, \mathcal{H}$ and V^* valued processes with respective norm topologies given by

$$\| \lambda \|_{L_2^e(X)} \equiv \left(E^\eta \int_I \| \lambda(t) \|_X^2 \, dt \right)^{1/2} \equiv \left(\int_\Sigma \| \lambda(t, \omega) \|_X^2 \, \hat{\eta}(dt \times d\omega) \right)^{1/2} \tag{4.1}$$

for $\lambda \in L_2^e(X), X = V, \mathcal{H}, V^*$. In the following lemma we shall first present an a priori bound for the solutions of the evolution equation (3.3). For this we shall use the hypothesis (H5) of [1] which is reproduced here as our assumption (H6):

(H6)

(i) The operator A of equation (1.1) is negative definite with eigen values and eigenvectors equal to $\{-\alpha_n, e_n, n = 1, 2, \}$.

(ii) There exists a sequence $\{q_n\}$ of positive numbers such that $\{Qe_n = q_n e_n, n = 1, 2, \ldots\}$.

Definition 4.1. For any given $\rho_0 \in \mathcal{H}$, an $\mathcal{H}$-valued process $\rho \equiv \{\rho(t), t \in I\}$ is said to be a weak solution of (3.3) if

(i) $\rho \in L^e_\infty(\mathcal{H}) \cap L^e_2(V)$ and

(ii) for every $t \in I$ and $\phi \in V$, the following identity

$$(\rho(t), \phi)_{\mathcal{H}} = (\rho_0, \phi)_{\mathcal{H}} + \int_0^t \langle \rho(s), (\mathcal{A} + L_u)\phi \rangle_{V,V^*} ds + \int_0^t \langle \phi, G(\rho(s)) \rangle_{V,V^*} dy, \tag{4.2}$$

holds η-a.s, for all $t \in I$.

Lemma 4.2. *Consider the system* (3.3) *with the initial state* $\rho_0 \in \mathcal{H}$ *and suppose the assumptions* (H1b)–(H6) *hold. Then there exists a finite number* $\beta > 0$*, independent of the choice of a control law* $u \in \mathcal{U}_{ad}$*, such that every weak solution of* (3.3)*, if one exists, belongs to* $L^e_\infty(\mathcal{H}) \cap L^e_2(V)$ *and satisfies the following a priori estimates:*

$$\sup \left\{ E^\eta \parallel \rho(t) \parallel^2_{\mathcal{H}}, t \in I \right\} \leq \beta$$

$$E^\eta \int_I \parallel \rho(t) \parallel^2_V \, dt \leq \beta.$$

Proof. See [Ahmed-Zabczyk 19, Lemma 5.2].

Remark. It follows from Fubini's theorem and the above Lemma that $\rho \in L_\infty(I, \mathcal{H}) \cap L_2(I, V)$ η-a.s.

In the next theorem we present an existence and uniqueness result for the system (3.3) with a brief outline of its proof.

Theorem 4.3. *Under the assumptions of Lemma* 4.2*, for each initial state* $\rho_0 \in \mathcal{H}$ *and admissible control* $u \in \mathcal{U}_{ad}$*, the system* (3.3) *has a unique weak solution* $\rho \in L^e_\infty(\mathcal{H}) \cap L^e_2(V)$*. Further, the solution family* Ξ *corresponding to the set of admissible controls* $\mathcal{U}_{ad}$ *is a bounded subset of* $L^e_\infty(\mathcal{H}) \cap L^e_2(V)$*.*

Proof. For a detailed proof see Ahmed-Zabczyk [19]. Here we give an outline. Since the members of the Gelfand triple $\{V, \mathcal{H}, V^*\}$ are separable Hilbert spaces, and the embedding $V \hookrightarrow \mathcal{H}$ is compact, there exists a complete system of basis functions which are orthogonal in V and V^* and

orthonormal in $\mathcal{H}$. Let us denote this set by $\{\psi_i, i \in N\}$. For example, by the results of Da Prato and Zabczyk [1], Hermite Polynomials in infinitely many variables as constructed in their paper form a complete orthonormal system in $\mathcal{H}$. Using this set one can use Fadeo-Galerkin technique to project the infinite dimensional problem (3.3) to a sequence of finite dimensional stochastic differential equations giving a sequence of approximations which clearly satisfy the a priori bounds as given in Lemma 4.2. This sequence is then shown to contain a subsequence that converge weakly in $L_2^e(V)$ and w^* in $L_\infty^e(\mathcal{H})$ to a unique element in $L_2^e(V) \cap L_\infty^e(\mathcal{H})$ satisfying the identity (4.2). Uniqueness follows from linearity of the system. The boundedness of the set Ξ follows from the a priori bound as stated in Lemma 4.2. This completes an outline of the proof.

5. Optimal controls and relaxation theorem

Now we are prepared to consider the question of existence of optimal controls and a relaxation theorem which is the main concern of this paper.

Define for $t \in I, y \in Y, \lambda \in \mathcal{H}, v \in \mathcal{Z}$,

$$\hat{\ell}(t, \omega, \lambda, v) \equiv \int_H \ell(t, y(t, \omega), x, v)\lambda(x)\mu^0(dx). \tag{5.1}$$

The control problem (3.2)–(3.3) as stated in Theorem 3.1 is equivalent to: Find a control $u \in \mathcal{U}_{ad}$ such that

$$J(u) \equiv \int_\Sigma \{\hat{\ell}(t, \omega, \rho_t, u_t)\}\hat{\eta}(dt \times d\omega) \Longrightarrow \inf \tag{5.2}$$

where ρ is the weak solution of the evolution equation

$$d\rho(t) = \mathcal{A}^*\rho(t)dt + L_u^*\rho(t)dt + G(\rho(t))dy(t), t \geq 0, \rho(0) = \rho_0, \tag{5.3}$$

corresponding to the control law u.

Let $\bar{R}$ denote the extended real number system. Define the set valued map

$$\mathcal{Q} : \Sigma \times \mathcal{H} \longrightarrow 2^{\bar{R} \times V^*} \setminus \emptyset$$

as follows:

$$\mathcal{Q}(t,\omega,\lambda) \equiv \Big\{(r,\beta) \in \bar{R} \times V^* : r \geq \hat{\ell}(t,\omega,\lambda,v) \quad \text{and} \quad \beta = L_v^* \lambda \ \text{ for some } \ v \in U(t,\omega)\Big\}, \tag{5.4}$$

$(t,\omega) \in \Sigma$ and $\lambda \in \mathcal{H}$.

For any $\lambda^0 \in \mathcal{H}$, let $N_\epsilon(\lambda^0)$ denote the ϵ-neighborhood of λ^0 in $\mathcal{H}$. The multifunction $\mathcal{Q}$ is said to satisfy the weak Cesari property on $\Sigma \times \mathcal{H}$ if, for each $\lambda^0 \in \mathcal{H}$,

$$\bigcap_{\epsilon>0} C\ell Co\ \mathcal{Q}(t,\omega,N_\epsilon(\lambda^0)) \subset \mathcal{Q}(t,\omega,\lambda^0), \quad \text{for} \quad (t,\omega) \in \Sigma.$$

If a multifunction satisfies the Cesari property, then it must be necessarily closed convex valued. On the other hand, a closed convex valued Hausdorff continuous multifunction satisfies the Cesari property. In fact, the Cesari property holds for upper semicontinuous (even less, quasi-upper semicontinuous) closed convex valued multifunctions [20, 21].

Regular Controls

For regular controls a result on the existence of optimal controls is given in the following theorem. Its proof is based on similar technique as in [8, 9, 19, 23]. For a detailed proof see [19, 23].

Theorem 5.1. (Convex Case) *Consider the optimal control problem* (5.2)–(5.3) *and suppose the following assumptions hold in addition to the basic hypotheses* (H1)–(H6):

(a1) $U : (\Sigma, \mathcal{P}) \longmapsto kc(\mathcal{Z})$ *is weakly measurable.*

(a2) *the integrand* $\hat{\ell}(t,\omega,\lambda,v)$ *is* $\mathcal{P}$*-measurable on* Σ *in the first two arguments, and lower semicontinuous in the last two variables, convex in the last argument, and further, there exists a real number* $\delta \geq 0$ *and a* $g \in L_1(\Sigma, \hat{\eta}; R)$ *such that*

$$\hat{\ell}(t,\omega,\lambda,v) + \delta \parallel \lambda \parallel_{\mathcal{H}}^2 \geq g(t,\omega), \ \ \hat{\eta}-a.e, \ \ \textit{for all } \lambda \in \mathcal{H} \ \textit{ and } v \in U(t,\omega).$$

(a3) *In addition to* $(H4)$, $B(x, U(t,\omega)) \in cc(H)$, *for all* $(x,t,\omega) \in H \times \Sigma$. *Then there exists an optimal control for the problem.*

Proof. (outline, for details see Ahmed-Zabczyk, 19) Using assumptions (a2) and (a3), one can verify that $\mathcal{Q}(t,\omega,\lambda) \in cc(\bar{R} \times V^*)$ for $(t,\omega,\lambda) \in \Sigma \times \mathcal{H}$ and that $\lambda \longrightarrow \mathcal{Q}(t,\omega,\lambda)$ is quasi upper semicontinuous with respect to d^* associated to the natural metric $d((r_1,v_1^*),(r_2,v_2^*)) \equiv |r_1 - r_2| + \| v_1 - v_2 \|_{V^*}$ on $R \times V^*$. Hence $\mathcal{Q}$ satisfies the weak Cesari property [22]. Based on this fact the proof is identical to that of Theorem 5.3 [23] where relaxed controls are used.

In case the convexity assumption of Theorem 5.1 is not satisfied, it is well known that there may not exist any optimal control from the admissible class of regular controls as defined. Often this problem is overcome by convexifying the multifunction Q [21]. This leads to a relaxed system which can be realized by use of measure valued controls replacing the regular controls (measurable functions of their arguments). Under some mild assumptions this modification allows one to prove the existence of optimal controls from the class of relaxed (measure valued) controls. Then the natural question is whether the generalized optimal control and the associated optimal state trajectory can be approximated to any specified degree of accuracy by a regular control-state pair. If it is impossible to do so the relaxation does not serve any practical purpose. On the other hand, if it is possible then a suboptimal control satisfying a pre-specified degree of accuracy can be constructed and in this case we may say that the system is relaxable. Thus in control theory often one attempts to prove the relaxation theorem asserting relaxability. We present below one such result.

Suppose that our original control problem is nonconvex, in the sense that neither the function $v \in \mathcal{Z} \mapsto \ell(\omega,x,v)$ nor the set valued map $B(x,U(t,\omega))$ are convex. We convexify the problem by use of measure valued controls as follows.

Relaxed controls

Let $\mathcal{Z}$ be a Polish space and Γ any compact subset of $\mathcal{Z}$ and $\mathcal{M}(\Gamma)$ the space of Radon measures on the Borel field of Γ, and $\mathcal{M}_1(\Gamma) \subset \mathcal{M}(\Gamma)$ the space of probability measures. For convenience of notation, we let $\hat{\Sigma}$ denote the measure space $(\Sigma, \mathcal{P}, \hat{\eta})$. For admissible controls, we consider a proper subset of the vector space $L_\infty^w(\hat{\Sigma}, \mathcal{M}(\Gamma))$ which is the dual of $L_1(\hat{\Sigma}, C(\Gamma))$. Note that

$$L_\infty^w(\hat{\Sigma}, \mathcal{M}(\Gamma)) \supset L_\infty(\hat{\Sigma}, \mathcal{M}(\Gamma))$$

where the later space consists of elements which are strongly measurable in contrast to those of the first space which are only weak star measurable. It is known that the dual of $L_1(\hat{\Sigma}, X)$ is isometrically isomorphic to $L_\infty(\hat{\Sigma}, X^*)$ if and only if X^* has the Radon-Nikodyme property. The space $\mathcal{M}(\Gamma)$ does not have this property and the duality $L^w_\infty(\hat{\Sigma}, \mathcal{M}(\Gamma)) = (L_1(\hat{\Sigma}, C(\Gamma)))^*$ is obtained by use of the theory of lifting [20, p115], see also the references therein. We take for the admissible controls the set

$$\mathcal{U}_r \equiv \mathcal{U}(\hat{\Sigma}, \mathcal{M}_1(\Gamma)). \tag{5.5}$$

In other words, the admissible controls are given by w^* $\mathcal{P}$-measurable functions taking values from the Polish space $\mathcal{M}_1(\Gamma)$. In view of this and Alaoglu's theorem $\mathcal{U}_r \subset L^w_\infty(\hat{\Sigma}, \mathcal{M}(\Gamma))$ is compact in the w^*-topology.

Returning to our control problem, we modify and rewrite the functions $\hat{\ell}$ and B as follows:

$$\begin{aligned} \tilde{\ell}(t, \omega, \lambda, \nu) &\equiv \int_\Gamma \hat{\ell}(t, \omega, \lambda, \xi)\ \nu(d\xi), \\ \tilde{B}(x, \nu) &\equiv \int_\Gamma B(x, \xi)\ \nu(d\xi),\ \ \nu \in \mathcal{M}_1(\Gamma), \\ (\tilde{L}_\nu \phi)(x) &\equiv (\tilde{B}(x, \nu), D\phi(x)), \end{aligned} \tag{5.6}$$

where $\hat{\ell}$ is as defined by (5.1). Consider now the relaxed problem : Find a control $\gamma \in \mathcal{U}_r$ such that

$$J(\gamma) = \int_\Sigma \tilde{\ell}(t, \omega, \rho_t, \gamma_t)\hat{\eta}(dt \times d\omega) \Longrightarrow \inf \tag{5.7}$$

where ρ is the (relaxed) weak solution of the evolution equation

$$d\rho(t) = \mathcal{A}^*\rho(t)dt + \tilde{L}^*_\gamma \rho(t)dt + G(\rho(t))dy(t), t \geq 0, \rho(0) = \rho_0, \tag{5.8}$$

corresponding to the relaxed control γ. Clearly, this covexifies the nonconvex problem.

Theorem 5.2. (Relaxed Problem) *Consider the optimal control problem* (5.7)–(5.8) *and suppose the following assumptions hold in addition to the basic hypotheses* (H1)–(H6):

(a1) $\mathcal{U}_r = \mathcal{U}(\hat{\Sigma}, \mathcal{M}_1(\Gamma))$.

(a2) *the integrand* $\tilde{\ell}(t,\omega,\lambda,\nu)$ *is* $\mathcal{P}$*-measurable in the first two variables and lower semicontinuous in the last two arguments on* $\mathcal{H}\times\mathcal{M}_1(\Gamma)$ *and further, there exists a real number* $\delta\geq 0$ *and* $g\in L_1(\hat{\Sigma},R)$ *such that*

$$\tilde{\ell}(t,\omega,\lambda,\nu)+\delta\parallel\lambda\parallel^2_{\mathcal{H}}\geq g(t,\omega),\ \hat{\eta}-a.e,\quad \textit{for all}\ \ \lambda\in\mathcal{H}\ \ \textit{and}\ \nu\in\mathcal{M}_1(\Gamma)).$$

Then there exists an optimal control (relaxed) for the problem.

Proof. The assumptions imply quasi upper semicontinuity of the multifuction $\lambda\in\mathcal{H}\mapsto\mathcal{Q}(t,\omega,\lambda)\in cc(\bar{R}\times V^*)$ which in turn implies the weak Cesari property. Hence the conclusion follows from [23, Theorem 5.3].

Theorem 5.1 proves the existence of optimal controls for a convex problem while Theorem 5.2 proves their existence for a relaxed problem which was a nonconvex problem at its beginning. The basic problem then is to show that under some suitable assumptions the original nonconvex problem is relaxable. Let Ξ_0 and Ξ_r denote the solution trajectories corresponding to the regular controls $\mathcal{U}_o\equiv\mathcal{U}(\hat{\Sigma},\Gamma)$ and the relaxed controls $\mathcal{U}_r\equiv\mathcal{U}(\hat{\Sigma},\mathcal{M}_1(\Gamma))$ respectively. Note that here the regular controls are not subject to the constraints as described in the statement of Theorem 5.1. Here Γ is a fixed compact, possibly nonconvex, subset of the Polish space $\mathcal{Z}$. The main result of this paper is given in the following theorem.

Theorem 5.3. *Suppose the basic assumptions of Lemma* 4.2 *hold and further the map* $\xi\in\Gamma\mapsto B(x,\xi)\in H$ *is continuous on* Γ *uniformly with respect to* $x\in H$ *and let* $\mathcal{U}_o$ *and* $\mathcal{U}_r$ *denote the class of regular and relaxed controls respectively. Consider the two control problems:*

(P_0) (5.2)–(5.3) *subject to regular controls where neither* ℓ *nor* B *is convex with* Ξ_0 *denoting the corresponding set of solution trajectories,*

(P_r) (5.7)–(5.8) *subject to relaxed controls with* Ξ_r *denoting the set of associated relaxed trajectories.*

Then, considered as subsets of $L^e_2(V)$, Ξ_0 *is weakly dense in* Ξ_r.

Proof. Let $\rho^0\in\Xi_r$ and $\gamma^0\in\mathcal{U}_r$ denote the corresponding relaxed control. Then by Krien-Milman theorem [Dunford-Schwartz, 25, Theorem V.8.4] or a result due to Fattorini [11, Theorem 3.2], see also Balder [10], there exists a sequence of regular controls $u_n\in\mathcal{U}_o$ such that the corresponding sequence of Dirac measures denoted by $\{\delta_{u_n}\equiv\gamma^n\}$ converges in the w^*-topology to the

relaxed control γ^0. Let $\{\rho^n\} \in \Xi_0$ denote the sequence of regular trajectories corresponding to the controls $\{u_n\}$. We show that

$$\rho^n \xrightarrow{w} \rho^0 \quad \text{in} \quad L_2^e(V).$$

First, note that by virtue of the a priori bounds as given by Lemma 4.2, the set $\{\rho^n\}$ is a bounded subset of $L_2^e(V) \cap L_\infty^e(\mathcal{H})$. Hence there exists a subsequence relabeled as such and an element $\tilde{\rho} \in L_2^e(V) \cap L_\infty^e(\mathcal{H})$ such that

$$\begin{aligned} &\rho^n \xrightarrow{w} \tilde{\rho} \text{ in } L_2^e(V) \\ &\rho^n \xrightarrow{w^*} \tilde{\rho} \text{ in } L_\infty^e(\mathcal{H}). \end{aligned} \tag{5.9}$$

By virtue of uniqueness of the weak solution it suffices to show that $\tilde{\rho}$ is also a weak solution of (5.8) in the sense of Definition 4.1. Clearly, by definition of a weak solution, for each $\phi \in V$, ρ^n satisfies the following equation for all $t \in I$, η-a.s

$$\begin{aligned} (\rho^n(t), \phi)_{\mathcal{H}} = (\rho_0, \phi)_{\mathcal{H}} &+ \int_0^t \langle\, \rho^n(s),\, (\mathcal{A} + \tilde{L}_{\gamma^n})\phi \,\rangle_{V,V^*} ds \\ &+ \int_0^t \langle\, \phi, G(\rho^n(s)) \,\rangle_{V,V^*} dy. \end{aligned} \tag{5.10}$$

For an arbitrary $\vartheta \in C_0^1(I)$, (C^1 functions with compact support in the interior of the set I) this is equivalent to

$$\begin{aligned} -\int_I (\rho^n(s), \phi)\dot{\vartheta}(s) ds = &\int_I \langle\, \rho^n(s), (\mathcal{A} + \tilde{L}_{\gamma^n})\phi \,\rangle\, \vartheta(s) ds \\ &+ \int_I \vartheta(s)(G(\rho^n(s)), \phi)_{V^*,V} dy(s). \end{aligned} \tag{5.11}$$

Now for each $z \in L_\infty(\mathcal{F}_T^y, R)$, it follows from (5.9) that

(5.12)

$$\begin{aligned} &\lim_{n\to\infty} E\Big\{ z \int_I \dot{\vartheta}(s)(\rho^n(s), \phi)_{\mathcal{H}} ds \Big\} = E\Big\{ z \int_I \dot{\vartheta}(s)(\tilde{\rho}(s), \phi)_{\mathcal{H}} ds \Big\}, \\ &\lim_{n\to\infty} E\Big\{ z \int_I \vartheta(s) \langle\, \rho^n(s), \mathcal{A}\phi \,\rangle_{V,V^*} ds + z \int_I \vartheta(s) \langle\, G(\rho^n(s)),\, \phi \,\rangle_{V^*,V} dy \Big\} \\ &= E\Big\{ z \int_I \vartheta(s) \langle\, \tilde{\rho}(s), \mathcal{A}\phi \,\rangle_{V,V^*} ds + z \int_I \vartheta(s) \langle\, G(\tilde{\rho}(s)),\, \phi \,\rangle_{V^*,V} dy \Big\}. \end{aligned}$$

Thus it suffices to verify that

$$
\begin{aligned}
&\lim_{n\to\infty} E\Big\{ z \int_I \vartheta \, \langle \, \rho^n, \tilde{L}_{\gamma^n}\phi \, \rangle_{V,V^*} ds \Big\} \\
&= E\Big\{ z \int_I \vartheta \, \langle \, \tilde{\rho}, \tilde{L}_{\gamma^0}\phi \, \rangle_{V,V^*} ds \Big\},
\end{aligned} \tag{5.13}
$$

since this, along with (5.11)–(5.12), will imply that $\tilde{\rho}$ is also a weak solution of equation (5.8) corresponding to the relaxed control γ^0. Defining

$$
z(t,\omega) \equiv E\Big\{ z | \mathcal{F}_t^y \Big\}
$$

and setting $z(t,\omega)\vartheta(t) \equiv \alpha(t,\omega)$, it is clear that α is $\mathcal{P}$-measurable and it follows from the properties of conditional expectations that we can rewrite expression (5.13) in terms of the measure $\hat{\eta}$, (see Section 3), as follows

$$
\begin{aligned}
&\lim_{n\to\infty} \int_\Sigma \alpha(t,\omega) \, \langle \, \rho^n, \tilde{L}_{\gamma^n}\phi \, \rangle_{V,V^*} \hat{\eta}(dt \times d\omega) \\
&= \int_\Sigma \alpha(t,\omega) \, \langle \, \tilde{\rho}, \tilde{L}_{\gamma^0}\phi \, \rangle_{V,V^*} \hat{\eta}(dt \times d\omega).
\end{aligned} \tag{5.14}
$$

Hence we must justify the validity of (5.14). Writing

$$
\begin{aligned}
&\int_\Sigma \alpha(t,\omega) \, \langle \, \rho^n, \tilde{L}_{\gamma^n}\phi \, \rangle_{V,V^*} \hat{\eta}(dt \times d\omega) \\
&= \int_\Sigma \alpha(t,\omega) \, \langle \, \rho^n, \, (\tilde{L}_{\gamma^n} - \tilde{L}_{\gamma^0})\phi \, \rangle_{V,V^*} \hat{\eta}(dt \times d\omega) \\
&+ \int_\Sigma \alpha(t,\omega) \, \langle \, \rho^n, \, \tilde{L}_{\gamma^0}\phi \, \rangle_{V,V^*} \hat{\eta}(dt \times d\omega),
\end{aligned} \tag{5.15}
$$

it is evident from the weak convergence of ρ^n to $\tilde{\rho}$ that for (5.14) to hold the first term on the right hand side of (5.15) must converge to zero. Using Fubini's theorem the first term of equation (5.15) can be written as

$$
\begin{aligned}
R_n &\equiv \int_\Sigma \alpha(t,\omega) \, \langle \, \rho^n, (\tilde{L}_{\gamma^n} - \tilde{L}_{\gamma^0})\phi \, \rangle_{V,V^*} \hat{\eta}(dt \times d\omega) \\
&= \int_\Sigma \langle \, g_n(t,\omega,\cdot), \, (\gamma^n - \gamma^0)(t,\omega) \, \rangle_{C(\Gamma),\mathcal{M}(\Gamma)} \hat{\eta}(dt \times d\omega)
\end{aligned} \tag{5.16}
$$

where

$$g_n(t,\omega,\xi) \equiv \int_H \alpha(t,\omega)\rho^n(t,\omega)(x)(D\phi(x), B(x,\xi))_H \ \mu^0(dx),$$

and

$$\begin{aligned} &\langle\, g_n(t,\omega,\cdot), (\gamma^n - \gamma^0)(t,\omega)\,\rangle_{C(\Gamma),\mathcal{M}(\Gamma)} \\ &\equiv \int_\Gamma g_n(t,\omega,\xi)(\gamma^n(t,\omega) - \gamma^0(t,\omega))(d\xi). \end{aligned} \tag{5.17}$$

Since α is an essentially bounded $\mathcal{P}$-measurable function on Σ and by our assumption (H4) the operator B is a bounded Borel measurable map from $H \times \Gamma$ to H, there exists a constant $K_{\alpha,\phi} > 0$ such that

$$|g_n(t,\omega,\cdot)|_{C(\Gamma)} \leq K_{\alpha,\phi} \parallel \rho^n(t,\omega) \parallel_{\mathcal{H}} . \tag{5.18}$$

Let $r > 0$ and define

$$\Sigma_r \equiv \{\omega \in \Sigma : \parallel \rho^n(t,\omega) \parallel_{\mathcal{H}} \leq r\},$$

and let $\Sigma_r^{'}$ denote its compliment. Since by the a priori bounds given by Lemma 4.2

$$\sup_n \left\{ \parallel \rho^n \parallel^2_{L^e_2(V)}, \parallel \rho^n \parallel^2_{L^e_2(\mathcal{H})} \right\} \ \leq \ \beta$$

and because the embedding $V \hookrightarrow \mathcal{H}$ is continuous it follows from Chevyshev inequality that for any $\epsilon > 0$, there exists $0 < r_\epsilon < \infty$ such that $\hat{\eta}(\Sigma^{'}_{r_\epsilon}) < \epsilon$ for all $r \geq r_\epsilon$. We rewrite (5.16) as

$$\begin{aligned} R_n &\equiv \int_\Sigma \langle g_n(t,\omega,\cdot), (\gamma^n - \gamma^0)(t,\omega)\,\rangle_{C(\Gamma),\mathcal{M}(\Gamma)} \hat{\eta}(dt \times d\omega) \\ &= \int_{\Sigma_r} \langle\, g_n(t,\omega,\cdot), (\gamma^n - \gamma^0)(t,\omega)\,\rangle_{C(\Gamma),\mathcal{M}(\Gamma)} \hat{\eta}(dt \times d\omega) \\ &+ \int_{\Sigma_r^{'}} \langle\, g_n(t,\omega,\cdot), (\gamma^n - \gamma^0)(t,\omega)\,\rangle_{C(\Gamma),\mathcal{M}(\Gamma)} \hat{\eta}(dt \times d\omega) \\ &= R_{n,1} + R_{n,2}. \end{aligned} \tag{5.19}$$

We show that by choosing n sufficiently large, R_n can be made as small as desired. On Σ_r, we have

$$|g_n(t,\omega,\cdot)|_{C(\Gamma)} \leq r\ K_{\alpha,\phi} < \infty,$$

$$|g_n(t,\omega,\xi) - g_n(t,\omega,\zeta)| \leq r\ K_{\alpha,\phi}\ \ \sup\{\| B(x,\xi) - B(x,\zeta) \|_H, x \in H\}.$$

Since $\xi \mapsto B(x,\xi)$ is continuous on Γ uniformly with respect to $x \in H$, it follows from this that the family of functions $\{g_n(t,\omega,\cdot), (t,\omega) \in \Sigma_r, n \in N\}$ is bounded and equicontinuous on Γ. Hence by weak convergence of γ^n to γ^0 it follows from [Theorem 6.8, 26] that

$$\int_\Gamma g_n(t,\omega,\xi)\gamma^n(t,\omega)(d\xi) \longrightarrow \int_\Gamma \tilde{g}(t,\omega,\xi)\gamma^0(t,\omega)(d\xi), \quad \hat{\eta}\ a.e\ (t,\omega) \in \Sigma_r,$$

where

$$\tilde{g}(t,\omega,\xi) \equiv \int_H \alpha(t,\omega)\tilde{\rho}(t,\omega)(x)(D\phi(x), B(x,\xi))_H \mu^0(dx).$$

Using the Lebesgue dominated convergence theorem we conclude from this that

$$\begin{aligned} &\int_{\Sigma_r\times\Gamma} g_n(t,\omega,\xi)\gamma^n(t,\omega)(d\xi)\ \hat{\eta}(dt\times d\omega) \\ &\longrightarrow \int_{\Sigma_r\times\Gamma} \tilde{g}(t,\omega,\xi)\gamma^0(t,\omega)(d\xi)\ \hat{\eta}(dt\times d\omega). \end{aligned} \tag{5.20}$$

In other words, $\lim_{n\to\infty} R_{n,1} = 0$. Further, for $r \geq r_\epsilon$, using (5.18) and the fact that $\{\gamma^n\}$ and γ^0 are $\mathcal{M}_1(\Gamma)$-valued, we have

$$\begin{aligned} |R_{n,2}| &\leq \int_{\Sigma_r'} |\langle g_n(t,\omega,\cdot), (\gamma^n - \gamma^0)(t,\omega)\rangle_{C(\Gamma),\mathcal{M}(\Gamma)}|\hat{\eta}(dt\times d\omega) \\ &\leq 2\int_{\Sigma_r'} |g_n(t,\omega,\cdot)|_{C(\Gamma)}\hat{\eta}(dt\times d\omega) \\ &\leq \left(2cK_{\alpha,\phi}\sqrt{\beta}\right)\sqrt{\epsilon} \end{aligned} \tag{5.21}$$

where c is the embedding constant $V \hookrightarrow \mathcal{H}$. Since by virtue of (5.20) $\lim_{n\to\infty} R_{n,1} = 0$, there exists an integer $n_\epsilon \in N$ such that $|R_{n,1}| \leq \sqrt{\epsilon}$, for $n \geq n_\epsilon$. Hence for any choice of $r \geq r_\epsilon$ and $n \geq n_\epsilon$ we have

$$|R_n| \leq \left(1 + 2cK_{\alpha,\phi}\sqrt{\beta}\right)\sqrt{\epsilon}. \tag{5.22}$$

Since $\epsilon > 0$ is arbitrary, this shows that $\lim_{n\to\infty} R_n = 0$. Thus (5.14) holds and hence $\tilde{\rho}$ is also a weak solution of (5.8). By uniqueness $\tilde{\rho} = \rho^0$. Hence, considered as subsets of $L_2^e(V)$, Σ_0 is weakly dense in Σ_r. This completes the proof.

Remark. It is clear from the relaxation result of Theorem 5.3 that admissible controls and their constraints are dependent on available information about the observable process y. This is precisely what is desirable in applications.

References

[1] G. Da Prato and J. Zabczyk, *Regular Densities of Invariant Measures in Hilbert Spaces*, Journal of Functional Analysis **130** (1995), 427–449 .

[2] V. Barbu and G. Da Prato, Hamilton Jacobi Equations in Hilbert Spaces, Pitman Res. Notes in Math. 86, Pitman, 1983.

[3] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimension, Encyclopedia of Mathematics and it's Applications series, 44, Cambridge University Press, 1992.

[4] G. Da Prato, Parabolic Equations in Infinitely Many Variables, Preprint n. 140, Scuola Normale Superiore, Pisa, Sept. 1992.

[5] Q. Zhu and N.U. Ahmed, *Some results on parabolic equations in Banach space*, Nonlinear Analysis, Theory and Methods **24** (9) (1995), 1305–1319.

[6] N.U. Ahmed, *Generalized solutions of HJB equations applied to stochastic control on Hilbert space*, Nonlinear Analysis **54** (2003), 495–523.

[7] N.U. Ahmed, *Relaxed controls for stochastic boundary value problems in infinite dimension*, Lect. Notes in Contr. and Inf. Sciences **149** (1990), 1–10.

[8] N.U. Ahmed, *Existence of optimal relaxed controls for a class of systems governed by differential inclusions on a Banach space*, JOTA **50** (2) (1986), 213–237.

[9] N.U. Ahmed, *Optimal relaxed controls for nonlinear infinite dimensional stochastic differential inclusions*, International Symposium on Optimal Control of infinite Dimensional Systems, Lect. Notes in Pure and Applied Math, Marcel Dekker, New York and Basel **160** (1994), 1–19.

[10] E.J. Balder, *A general denseness result for relaxed control theory*, Bull. Austral. Math. Soc. **30** (1984), 463–475.

[11] H.O. Fattorini, *Relaxation theorems, differential inclusions, and Filippov's theorem for relaxed controls in semilinear infinite dimensional systems*, Journal of Differential Equations **112** (1994), 131–153.

[12] D. Gątarek and B. Gołdys, *On solving stochastic evolution equations by the change of drift with applications to optimal control*, Polish Academy of Sciences, (personal communication).

[13] W.H. Fleming and E. Pardoux, *Optimal control for partially observed diffusions*, SIAM J. Contr. Optim. **20** (1982), 261–285.

[14] N.J. Cutland and T. Lindström, *Random relaxed controls and partially observed stochastic systems*, Acta Applicandae Mathematicae **32** (1993), 157–182.

[15] O. Hijab, *Partially observed control of Markov processes* IV, J. Func. Anal. **109** (1992), 215–256.

[16] A. Bensoussan, Stochastic Control of Partially Observable Systems, Cambridge University press, 1992.

[17] V.S. Borkar, *Existence of optimal controls for partially observed diffusions*, Stochastics **11** (1983), 103–141.

[18] N.U. Ahmed and X. Xiang, *Admissible relaxation in optimal control problems for infinite dimensional uncertain systems*, J. Appl. Math. and Stoch. Anal. **5** (3) (1993), 227–236.

[19] N.U. Ahmed and J. Zabczyk, *Partially observed optimal controls for nonlinear infinite dimensional stochastic systems*, Dynamic Systems and Applications **5** (1996), 521–538.

[20] J. Diestel and J.J. Uhl, Jr., Vector Measures, Mathematical Surveys, 15, AMS, American Mathematical Society, Providence, Rhode Island, 1977.

[21] N.U. Ahmed and K.L. Teo, Optimal Control of Distributed Parameter Systems, North Holland, New York Oxford, 1981.

[22] S.H. Hou, *On property (Q) and other semicontinuity properties of multifunctions*, Pacific J. Math. **103** (1) (1992), 39–56.

[23] N.U. Ahmed, *Optimal relaxed controls for infinite dimensional stochastic systems of Zakai type*, SIAM J. Control and Optimization **34** (5) (1996), 1592–1615.

[24] N.U. Ahmed, M. Fuhrman and J. Zabczyk, *On filtering equations in infinite dimensions*, J. Func. Anal. **143** (1) (1997), 180–204.

[25] N. Dunford and J.T. Schwartz, Linear Operators, Part 1, Interscience Publishers, Inc., New York, 1964.

[26] K.R. Parthasarathy, Probability Measures on Metric Spaces, Academic Press, New York and London, 1967.

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