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CONTINUOUS SELECTIONS AND APPROXIMATIONS IN α-CONVEX METRIC SPACES

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Abstract

In the paper, the notion of a generalized convexity was defined and studied from the view-point of the selection and approximation theory of set-valued maps. We study the simultaneous existence of continuous relative selections and graph-approximations of lower semicontinuous and upper semicontinuous set-valued maps with α-convex values having nonempty intersection.

Keywords: generalized convexity, selections, relative selections, graph-approximations.

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1. Introduction

In the paper we study criteria for the existence of the so-called relative graph-approximations, selections and approximate selections of set-valued maps admitting closed and α-convex values in an α-convex metric space. The notion of generalized convexity under consideration constitutes a slight generalization of α-convexity defined by de Blasi and Pianigiani (see [3, 4]). Given an upper semicontinuous map $\varphi : X \multimap Y$, where X is a finite-dimensional metric space and Y is a complete α-convex metric space, a lower semicontinuous map $\psi : X \multimap Y$ (both with closed and α-convex values) such that, $\varphi(x) \cap \psi(x) \neq \emptyset$, a closed set $A \subset X$ and $\varepsilon > 0$, we establish the existence of $\delta > 0$ such that any continuous map $f : A \to Y$ being simultaneously a selection of ψ and a δ-approximation of φ admits a continuous extension $f^* : X \to Y$ such that f^* is a selection of ψ and an ε-approximation of φ. This relative generalized version of (a convex valued) result due to Ben-El-Mechaiekh and Kryszewski (see [1]) constitutes also an α-convex counterpart of a result from [16] and those from [14]. Moreover, we prove a result concerning the existence of the so-called *strong* approximations (see e.g. [7]) of α-convex upper semicontinuous maps. As in the case of any approximation results, the theory presented here has important applications in the fixed point theory.

2. Notation and preliminaries

Let X be a metric space with a distance d. Given $a \in X$ and $r > 0$, by $B_X(a,r)$ (resp. $D_X(a,r)$) we denote an open (resp. closed) ball in X centered at a with the radius r; if $A \subset X$, then $B_X(A,r) := \{x \in X \mid d(x,A) := \inf_{a\in A} d(x,a) < r\}$. By $\overline{A}$ we denote the closure of A, and by $\operatorname{diam}(A) := \sup_{a,\, b\in A} d(a,b)$ its *diameter*. If $A, B \subset X$, then $\mathfrak{D}(A,B) := \max\{\mathfrak{D}^+(A,B), \mathfrak{D}^-(A,B)\}$, where $\mathfrak{D}^+(A,B) := \sup_{b\in B} d(b,A)$ and $\mathfrak{D}^-(A,B) := \sup_{a\in A} d(a,B)$, is the Hausdorff distance between A and B. Recall (see [10]) that the *covering* dimension $\dim X \leq p$, where $p \geq -1$, if any covering $\mathcal{V}$ admits a refinement $\mathcal{V}'$ of order $\leq p$, i.e., such that each collection of $p+2$ elements from $\mathcal{V}'$ has an empty intersection. If we write a covering, we mean an open covering. By the *star* of a set $A \subset X$ with respect to a covering $\mathcal{A} = \{A_j\}_{j\in J}$ (open or closed) of X we understand the set $\operatorname{st}(A,\mathcal{A}) := \bigcup\{A_j \mid A_j \cap A \neq \emptyset\}$.

Let X and Y be metric spaces. By a set-valued map $F : X \multimap Y$ we understand a function which assigns to any $x \in X$ a nonempty set $F(x) \subset Y$. A map $F : X \multimap Y$ is *lower semicontinuous* (abbr. l.s.c.) (resp. *upper semicontinuous* (abbr. u.s.c.)) if, for every open (resp. closed) $A \subset Y$, the *preimage* $F^{-1}(A) := \{x \in X | F(x) \cap A \neq \emptyset\}$ is open (resp. closed) in X. We say that $F : X \multimap Y$ is $\mathfrak{D}$-*lower semicontinuous* at a point $x_0 \in X$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{D}^+(F(x), F(x_0)) < \varepsilon$ for all $x \in B_X(x_0,\delta)$. By the *graph* of F we understand the set

$$\operatorname{Gr}(F) := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

Let $\varepsilon > 0$. We say that a function $f : X \to Y$ is a *selection* (resp. an ε-*selection*) of a map $F : X \multimap Y$ if $f(x) \in F(x)$ (resp. $f(x) \in B_Y(F(x),\varepsilon)$) for all $x \in X$. A function f is an ε-*graph-approximation* (resp. a $(\mu(\cdot),\varepsilon)$-*approximation*, where $\mu : X \to (0,\infty)$ is a continuous function) of F if the graph Gr (f) of f is contained in the ε-neighborhood of the graph F, i.e., $f(x) \in B(F(B(x,\varepsilon)),\varepsilon)$ (resp. $f(x) \in B(F(B(x,\mu(x))),\varepsilon)$) for each $x \in X$.

Let $\mathcal{U}$ be a neighborhood of the graph of the map F in $X \times Y$ and let A be a subset of Y. A map $f : A \to Y$ is called an *$\mathcal{U}$-approximation* of F over A provided Gr $(f) \subset \mathcal{U}$.

Given a function $f : X \to [0,1]$, $\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}}$ is the *support* of f and $\operatorname{carr} f := \{x \in X \mid f(x) > 0\}$ is the *carrier* of f.

The following two propositions are well-known, but are included for the sake of completeness of the paper.

Proposition 2.1 (see [9]). *Let X, Y be metric spaces. Let $\{A_\beta\}_{\beta\in B}$ be a covering of X and let $\{g_\beta\}_{\beta\in B}$ be family continuous functions $g_\beta : A_\beta \to Y$ such that, for every $\beta, \beta' \in B$ with $A_\beta \cap A_{\beta'} \neq \emptyset$,*

$$g_\beta(x) = g_{\beta'}(x) \quad for\ every\ \ x \in A_\beta \cap A_{\beta'}.$$

Then there is a unique continuous function $g : X \to Y$ such that, for all $\beta \in B$,

$$g(x) = g_\beta(x) \quad for\ every\ \ x \in A_\beta.$$

Proposition 2.2 (see [4]). *Let X, Y be metric spaces. Let a map $\psi : X \multimap Y$ be lower semicontinuous, a function $f : X \to Y$ continuous and $\varepsilon > d_Y(f(x), \psi(x))$ for every $x \in X$. Then the map $\Phi : X \multimap Y$ given by*

$$\Phi(x) := \psi(x) \cap B(f(x), \varepsilon), \quad x \in X,$$

is lower semicontinuous.

Now we are going to introduce the concept of an α-convex metric space. An earlier version of this notion has been presented by F.S. de Blasi and G. Pianigiani in [3].

Definition 2.3. By an α-convex metric space we mean a pair (Y, α), where (Y, d) is a metric space and $\alpha : Y \times Y \times [0,1] \to Y$ is a continuous function such that, for all $x, y \in Y$ and $t \in [0,1]$,

(i) $\alpha(x, y, 0) = x$;

(ii) $\alpha(x, y, 1) = y$;

(iii) $\alpha(x, x, t) = x$;

and, moreover, satisfying the following Hausdorff continuity property:

(iv) for any $\varepsilon > 0$, there is $\eta > 0$ such that, for any $x_1, x_2, y_1, y_2 \in Y$, if $d(x_1, y_1) < \eta$ and $d(x_2, y_2) < \eta$, then

$$\mathfrak{D}(\Lambda_\alpha(x_1, x_2; [0,1]), \Lambda_\alpha(y_1, y_2; [0,1])) < \varepsilon,$$

where, for any $x, y \in Y$ and $t_1, t_2 \in [0,1]$, $\Lambda_\alpha(x, y; [t_1, t_2]) := \{\alpha(x, y, t) \mid t \in [t_1, t_2]\}$.

In their definition, the authors in [3] require a stronger version of the above mentioned continuity and they demand that:

($*$) for all $\varepsilon > 0$, there is $\eta \in (0, \varepsilon]$ such that, for any $x_1, x_2, y_1, y_2 \in Y$, if $d(x_1, y_1) < \varepsilon$ and $d(x_2, y_2) < \eta$, then

$$\mathfrak{D}(\Lambda_\alpha(x_1, x_2; [0,1]), \Lambda_\alpha(y_1, y_2; [0,1])) < \varepsilon.$$

Our setting seems to be more natural, since – contrary to the definition of de Blasi and Pianigiani – it is just an equivalent of the usual Hausdorff continuity of the map $X \times X \ni (x, y) \multimap \Lambda_\alpha(x, y; [0,1])$. Consider also the following example.

Example 2.4. Consider the two-dimensional sphere $S^2 := \{x = (x_1, x_2, x_2) \in \mathbb{R}^3 \mid \|x\| = 1\}$ and denote by d the metric on S^2 induced by the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^3$. Let $Y = \{x \in S^2 \mid x_1^2 + x_2^2 < \sin^2 \rho\}$, where $\rho \in (0, \frac{\pi}{2})$. It is easy to see that Y is nothing else but the set of points whose northern altitude is greater than ρ. Define $\alpha : Y \times Y \times [0,1] \to X$ for $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in Y$ by the formula

$$\alpha((x_1, y_1, z_1), (x_2, y_2, z_2), t) = \left(x(t), y(t), \sqrt{1 - x^2(t) - y^2(t)}\right),$$

where

$$x(t) = \frac{|B|}{\sqrt{A^2 + B^2}} \cos g_{t_1,t_2}(t) + \sqrt{\frac{B^2 + C^2}{A^2 + B^2 + C^2} - \frac{A^2}{A^2 + B^2}} \sin g_{t_1,t_2}(t),$$

$$y(t) = \frac{|A|}{\sqrt{A^2 + B^2}} \cos g_{t_1,t_2}(t) + \sqrt{\frac{A^2 + C^2}{A^2 + B^2 + C^2} - \frac{B^2}{A^2 + B^2}} \sin g_{t_1,t_2}(t),$$

$$A = y_1 z_2 - y_2 z_1, \quad B = x_2 z_1 - x_1 z_2, \quad C = x_1 y_2 - x_2 y_1,$$

$$g_{t_1,t_2}(t) = \left[\operatorname{arctg}\left[t\left(tg\frac{2\pi t_2 - \pi^2}{2\pi - 4\rho} - tg\frac{2\pi t_1 - \pi^2}{2\pi - 4\rho}\right) + tg\frac{2\pi t_1 - \pi^2}{2\pi - 4\rho}\right] + \frac{\pi^2}{2\pi - 4\rho}\right]\left(1 - \frac{2\rho}{\pi}\right)$$

and

$$t_1 = \arccos\frac{|B|x_1 + |A|y_1}{\sqrt{A^2 + B^2}}, \quad t_2 = \arccos\frac{|B|x_2 + |A|y_2}{\sqrt{A^2 + B^2}}.$$

It is easy to see that, for $x, y \in Y$, the set $\Lambda_\alpha(x, y; [0,1])$ is the arc of the great circle passing through x and y. It is rather obvious (but requiring tedious computation) that α satisfies properties listed in Definition 2.3. At the same time α does not have property $(*)$. De Blasi and Pianigiani study a similar example but they have to assume that $\rho \in (0, \frac{\pi}{4})$ in order to assure that property $(*)$ is satisfied.

It is clear that a space Y together with α satisfying property $(*)$ satisfies (iv). Therefore all examples given in [3] are α-convex in the sense of Definition 2.3.

Let (Y, α) be an α-convex space. We say that $A \subset X$ is *α-convex* if, for any $a_1, a_2 \in A$ and $t \in [0,1]$, $\alpha(a_1, a_2, t) \in A$. The continuity of α implies that if $A \subset X$ is α-convex, then so is $\overline{A}$.

In what follows the concept of a *pseudo-barycenter* plays an important role (compare [5]).

Definition 2.5. Let Y be an α-convex metric space. Let $n \geq 1$ be an integer. For $y_1, \dots, y_n \in Y$ and $(\lambda_1, \dots, \lambda_n) \in \Sigma^n := \{(\lambda_1, \dots, \lambda_n) \mid 0 \leq \lambda_i \leq 1, i = 1, \dots, n, \lambda_1 + \dots + \lambda_n = 1\}$, the corresponding a n-pseudo-barycenter $b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n)$ is defined as follows:

$$b_1(y_1; 1) := y_1$$

$$b_2(y_1, y_2; \lambda_1, \lambda_2) := \alpha(y_1, y_2, \lambda_2)$$

and, for $n \geq 3$,

$$b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) :=$$

$$:= \begin{cases} y_n, & \text{if} \quad \lambda_n = 1; \\ \alpha\left(b_{n-1}\left(y_1, \dots, y_{n-1}; \frac{\lambda_1}{1-\lambda_n}, \dots, \frac{\lambda_{n-1}}{1-\lambda_n}\right), y_n, \lambda_n\right), & \text{if} \quad \lambda_n \neq 1. \end{cases}$$

We shall need the following properties of the pseudo-barycenter (compare [5]).

Property 2.6. *For any $n \geq 1$, the map $b_n : Y^n \times \Sigma^n \to Y$ is continuous.* ∎

Property 2.7. *Let $y_1, \ldots, y_n \in Y$ and $(\lambda_1, \ldots, \lambda_n) \in \Sigma^n, n \geq 2$. Let $\{i_1, \ldots, i_k\}, 1 \leq k \leq n-1$, be a subset of $\{1, \ldots, n\}$ with $i_1 < i_2 < \ldots < i_k$ such that, for all $i \in \{i_1, \ldots, i_k\}$, $\lambda_i > 0$ and if $i \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$, then $\lambda_i = 0$. Then*

$$b_n(y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_n) = b_k(y_{i_1}, \ldots, y_{i_k}; \lambda_{i_1}, \ldots, \lambda_{i_k}). \qquad \blacksquare$$

In what follows we shall frequently make use of the following formalism. Assume that $(J, \prec)$ is a well-ordered set and let $\lambda : J \to [0,1]$ be such that the set $J_0 := \{j \in J \mid \lambda(j) > 0\}$ is finite, i.e., $J_0 = \{j_1, \ldots, j_m\}$, where $j_i \prec j_{i+1}$ for all $i = 1, .., m-1$. For any function $y : J \to Y$, we define

$$b(y, \lambda) := b_m(y_1, \ldots, y_m; \lambda_1, \ldots, \lambda_m), \tag{1}$$

where $y_i := y(j_i)$ and $\lambda_i := \lambda(j_i)$ for $i = 1, \ldots, m$.

Obviously, by the Zermelo theorem [9], every set can be a well-ordered. In this paper, we let each set J of indices be a well-ordered set in the sense of a relation $\prec$.

Proposition 2.8. *Let X be a metric space, Y be an α-convex metric space and assume that functions $y : J \to Y$ and $\lambda : J \times X \to [0,1]$ are given. Suppose that $\{\operatorname{carr} \lambda(j, \cdot)\}_{j \in J}$ is a locally finite covering of X (in particular, for each $x \in X$, the set $J(x) := \{j \in J \mid \lambda(j, x) > 0\}$ is finite) and, for each $j \in J$, the function $\lambda(j, \cdot) : X \to [0,1]$ is continuous. Then, for each $x \in X$, we may define*

$$b(y, x) := b(y, \lambda(\cdot, x)).$$

The map $b(y, \cdot) : X \to Y$ is continuous.

Moreover, if $y : J \times X \to Y$ is such that, for all $j \in J$, $y(j, \cdot) : X \to Y$ is continuous, then the map

$$X \ni x \mapsto b(y(\cdot, x), \lambda(\cdot, x)) \in Y$$

is also continuous.

Proof. For each $j \in J$, let $U_j := \operatorname{carr} \lambda(j, \cdot)$. Obviously U_j is open for any $j \in J$. Moreover, for any $x \in X$, let $J(x) := \{j \in J \mid \lambda(j,x) > 0\}$ and $m_x := \#J(x)$. According to formula (1),

$$(2)\quad b(y,x) = b(y,\lambda(\cdot,x)) = b_{m_x}(y(j_1^x),\dots,y(j_{m_x}^x);\lambda(j_1^x,x),\dots,\lambda(j_{m_x}^x,x)),$$

where $J(x) = \{j_1^x,\dots,j_{m_x}^x\}$ and $j_i^x \prec j_{i+1}^x$ for all $i = 1,\dots,m_x - 1$.

Let $x_0 \in X$. There exists a neighborhood Ω_{x_0} of x_0 such that the set $I(x_0) := \{j \in J \mid \Omega_{x_0} \cap U_j \neq \emptyset\}$ is non-empty, finite and ordered. Let $n_{x_0} := \#I(x_0)$.

Observe that, if $x \in \Omega_{x_0}$, then $J(x) \subset I(x_0)$. By Property 2.7 and in view of (2), we have

$$(3)\qquad b(y,x) = b_{n_{x_0}}(y(j_1),\dots,y(j_{n_{x_0}});\lambda(j_1,x),\dots,\lambda(j_{n_{x_0}},x)),$$

where $I(x_0) = \{j_1,\dots,j_{n_{x_0}}\}$ and this sequence increases (in the sense of $\prec$). In view of Property 2.6 (in fact the continuity of $b_n(y_1,\dots,y_n;\cdot) : \Sigma^n \to Y$ is sufficient) and continuity of the function $\lambda(j,\cdot) : X \to [0,1]$, for all $j \in J$, formula (3) correctly defines a continuous map $b(y,\cdot) : \Omega_{x_0} \to Y$.

Now let $x_0, x_0' \in X$ and suppose that $x \in \Omega_{x_0} \cap \Omega_{x_0'}$. Then $J(x) \subset I(x_0) \cap I(x_0')$. Hence, by (3),

$$\begin{aligned}
&b(y,\cdot)|_{\Omega_{x_0}}(x) = b_{n_{x_0}}(y(j_1),\dots,y(j_{n_{x_0}});\lambda(j_1,x),\dots,\lambda(j_{n_{x_0}},x))\\
&= b_{m_x}(y(j_1^x),\dots,y(j_{m_x}^x);\lambda(j_1^x,x),\dots,\lambda(j_{m_x}^x,x))\\
&= b_{n_{x_0'}}(y(j_1'),\dots,y(j_{n_{x_0'}}');\lambda(j_1',x),\dots,\lambda(j_{n_{x_0}}',x)) = b(y,\cdot)|_{\Omega_{x_0'}}(x).
\end{aligned}$$

In view of Proposition 2.1, $b(y,\cdot)$ is well-defined and continuous.

The proof of continuity of the function $x \mapsto b(y(\cdot,x),\lambda(\cdot,x))$ is analogous if we use the continuity of the map $b_n : Y^n \times \Sigma^n \to Y$. ■

Proposition 2.9. *Let Y be an α-convex metric space and $0 \le p < \infty$ be an integer number. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $1 \le n \le p+1$, any $y_1,\dots,y_n \in Y$ and any non-empty α-convex set $A \subset Y$, if $d(y_i,A) < \delta$ for $i = 1,\dots,n$, then*

$$d(b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n),A) < \varepsilon$$

for all $(\lambda_1,\dots,\lambda_n) \in \Sigma^n$.

Proof. Let $\varepsilon > 0$. Let $\delta_{p+1} := \eta$, where $\eta > 0$ corresponds to ε as in Definition 2.3. Obviously, we may take $\eta < \varepsilon$. Suppose that $\delta_i > 0$, for $2 \le i \le p$, is defined. By definition, there is $\delta_{i-1} > 0$ such that $\delta_{i-1} \le \delta_i$ and, for all $x_1, x_2, y_1, y_2 \in Y$, if $d(x_1, y_1), d(x_2, y_2) < \delta_{i-1}$, then

$$\mathfrak{D}(\Lambda_\alpha(x_1, x_2; [0,1]), \Lambda_\alpha(y_1, y_2; [0,1])) < \delta_i.$$

In this manner the number $\delta := \delta_1$ is well-defined.

Let $1 \le n \le p+1$. Take a set $A \subset Y$ and arbitrary points $y_1, \ldots, y_n \in Y$ such that $d(y_i, A) < \delta$ for $i = 1, 2, \ldots, n$. We shall show that

$$d(b_n(y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_n), A) < \delta_n.$$

If $n = 1$, then $b_1(y_1; 1) = y_1$, and hence $d(b_1(y_1; 1), A) < \delta = \delta_1$. Let $n > 1$ and assume that, for any $(\mu_1, \ldots, \mu_n) \in \Sigma^n$,

$$d(b_n(y_1, \ldots, y_n; \mu_1, \ldots, \mu_n), A) < \delta_n.$$

Hence there is $x_n \in A$ such that

$$d(b_n(y_1, \ldots, y_n; \mu_1, \ldots, \mu_n), x_n) < \delta_n. \tag{4}$$

Since $d(y_{n+1}, A) < \delta \le \delta_n$, there is $x'_n \in A$ such that

$$d(y_{n+1}, x'_n) < \delta_n. \tag{5}$$

Let $(\lambda_1, \ldots, \lambda_{n+1}) \in \Sigma^{n+1}$. If $\lambda_{n+1} < 1$, then

$$b_{n+1}(y_1, \ldots, y_{n+1}; \lambda_1, \ldots, \lambda_{n+1}) = \alpha(b_n(y_1, \ldots, y_n; \mu_1, \ldots, \mu_n), y_{n+1}, \lambda_{n+1}),$$

where $\mu_i := \lambda_i(1 - \lambda_{n+1})^{-1}$. By (4) and (5) and the definition of δ_n, we see that

$$\begin{aligned}&d(b_{n+1}(y_1, \ldots, y_{n+1}; \lambda_1, \ldots, \lambda_{n+1}), A)\\ &\le \mathfrak{D}(\Lambda_\alpha(b_n(y_1, \ldots, y_n; \mu_1, \ldots, \mu_n), y_{n+1}; [0,1]), \Lambda_\alpha(x_n, x'_n; [0,1]) < \delta_{n+1}\end{aligned}$$

since $\lambda_\alpha(x_n, x'_n; [0,1]) \subset A$. If $\lambda_{n+1} = 1$, then $b_{n+1}(y_1, \ldots, y_{n+1}; \lambda_1, \ldots, \lambda_{n+1}) = y_{n+1}$, and then again $d(b_{n+1}(y_1, \ldots, y_{n+1}; \lambda_1, \ldots, \lambda_{n+1}), A) < \delta_n \leq \delta_{n+1}$ by (5). ■

Observe that if $n = 2$, then we can take $\delta := \eta$ from Definition 2.3.

3. Continuous selections and approximations

It is an important fact in this section that, for every continuous function defined on a closed subset of a finite-dimensional metric space with values in an α-convex metric space, there exists a continuous extension onto the whole domain. It means that an α-convex metric space is an "almost" absolute retract (in the paper [2] it is proved that an α-convex metric space defined by De Blasi and Pianigiani is an absolute retract).
Let us sketch the proof:

Let X be a metric space such that $\dim X = p < \infty$ and Y be an α-convex metric space. Let $A \subset X$ be a closed set. Take a continuous function $f : A \to Y$.

Consider the so-called Dugundji system $\{U_j, a_j\}_{j \in J}$ for $X \setminus A$ (see [6]), i.e.,

(i) $U_j \subset X \setminus A$, $a_j \in A$, $j \in J$;

(ii) $\{U_j\}_{j \in J}$ is a locally finite covering of $X \setminus A$;

(iii) if $x \in U_j$, then $d_X(x, a_j) \leq 2d_X(x, A)$, $j \in J$.

Without loss of generality we may assume that $\{U_j\}_{j \in J}$ is such that every collection consisting of $p + 2$ elements $U_{j_1}, \ldots, U_{j_{p+2}}$ of this covering has empty intersection. Let $\{\lambda_j\}_{j \in J}$ be a continuous partition of unity subordinate to $\{U_j\}_{j \in J}$. Define functions $\lambda : J \times [X \setminus A] \to [0,1]$ and $y : J \to Y$ putting $\lambda(j, x) := \lambda_j(x)$ and $y(j) := f(a_j)$ for $j \in J$ and $x \in X \setminus A$. Hence, in view of the Proposition 2.8, we have a continuous function $b(y, \cdot) : X \setminus A \to Y$.

We define the continuous extension $\overline{f} : X \to Y$ of f by the formula

$$\overline{f}(x) = \begin{cases} f(x), & \text{for} \quad x \in A; \\ b(y, \lambda(\cdot, x)), & \text{for} \quad x \in X \setminus A. \end{cases}$$

In particular, by the above assertion, it follows that, for each α-convex closed subset T of an α-convex metric space such that $T \subset M$ and $\dim M < \infty$, there exists a retraction $r : M \to T$ as an extension of an identical function $id_T : T \to T$.

Let us start with the following result saying that any locally selectionable set-valued map with α-convex values admits a continuous selection.

Theorem 3.1. *Let X be a metric space, $\dim X < \infty$, and let Y be an α-convex space. If a set-valued map $F : X \multimap Y$ has α-convex values and, for any point $x \in X$, there is a neighborhood V_x and a continuous function $f_x : V_x \to Y$ such that $f_x(y) \in F(y)$ for $y \in V_x$, then F admits a continuous selection.*

Proof. Let $\mathcal{W} := \{W_j\}_{j \in J}$ be a locally finite refinement of the covering $\mathcal{V} := \{V_x\}_{x \in X}$ and $\{\lambda_j\}_{j \in J}$ be a continuous partition of unity subordinate to $\mathcal{W}$.

For all $j \in J$, let $x_j \in X$ be such that $W_j \subset V_{x_j}$. Without loss of generality we may assume that, for each $j \in J$, $\overline{W}_j \subset V_{x_j}$. Since for $f_{x_j} : \overline{W}_j \to Y$ there exists an extension to the whole space X, we may assume that f_{x_j} is actually defined on X. Set $f_j(x) := f_{x_j}(x)$ for $x \in X$.

Let $\lambda : J \times X \to [0,1]$ be given by $\lambda(j,x) = \lambda_j(x)$, let $y : J \times X \to Y$ be given by $y(j,x) = f_j(x)$. Then all assumptions from Proposition 2.8 are satisfied, hence we may define

$$f(x) = b(y(\cdot,x), \lambda(\cdot,x)), \quad x \in X.$$

Clearly, $f : X \to Y$ is well-defined and continuous. Let $x \in X$. We have that the set $\{j \in J \mid \lambda(j,x) > 0\} = \{j_1, \ldots, j_m\}$ is non-empty, finite and well-ordered. Since, for each $i = 1, \ldots, m$, $f_{j_i}(x) \in F(x)$ and $F(x)$ is α-convex, we see that $f(x) \in F(x)$. ∎

A theorem below shows a connection between a graph-approximation and a selection of set-values maps.

Theorem 3.2. *Let X be a metric space such that $\dim X = p < \infty$ and Y an α-convex metric space. Let $\psi : X \multimap Y$ be a lower semicontinuous map with α-convex values, and $\varphi : X \multimap Y$ be an upper semicontinuous map with α-convex values such that $\varphi(x) \cap \psi(x) \neq \emptyset$ for each $x \in X$. Then, for any $\varepsilon > 0$ and a continuous function $\mu : X \to (0,\infty)$, there is a continuous map $f : X \to Y$ such that f is an ε-selection of the map ψ and a $(\mu(\cdot),\varepsilon)$-approximation of the map φ.*

Proof. Let $\varepsilon > 0$ and $\mu : X \to (0, \infty)$ be a continuous function. By Proposition 2.9, there exists $0 < \delta \leq \varepsilon$ such that, for every $y_1, \dots, y_k \in Y$, $1 \leq k \leq p+1$, and a non-empty α-convex set $A \subset Y$, if $d_Y(y_i, A) < \delta$, $i = 1, \dots, k$, then

$$d_Y(b_k(y_1, \dots, y_k; \lambda_1, \dots, \lambda_k), A) < \varepsilon$$

for any $(\lambda_1, \dots, \lambda_k) \in \Sigma^k$.

For $x \in X$, let

$$U(x) := \mu^{-1}(\mu(x)/2, \infty) \cap B_X(x, \mu(x)/2) \cap \{x' \in X \mid \varphi(x') \subset B_Y(\varphi(x), \delta)\}.$$

Then, in view of the upper semicontinuity of φ and continuity of μ, for each $x \in X$, the set $U(x)$ is open. Thus $\mathcal{U} := \{U(x)\}_{x \in X}$ is a covering of X. Since $\dim X = p$, there exists an open refinement $\mathcal{U}'$ of $\mathcal{U}$ such that every collection consisting of $p+2$ elements $U_1', \dots, U_{p+2}'$ of $\mathcal{U}'$ has an empty intersection. Next let $\mathcal{V} := \{V\}$ be an open star-refinement of $\mathcal{U}'$, i.e., for any $V \in \mathcal{V}$, there is $U' \in \mathcal{U}'$ such that

$$\operatorname{st}(V, \mathcal{V}) = \bigcup \{W \in \mathcal{V} \mid W \cap V \neq \emptyset\} \subset U'.$$

Without loss of generality we may assume that the covering $\mathcal{V}$ is such that each collection of $p+2$ elements from $\mathcal{V}$ has an empty intersection.

For any $x \in X$, choose $z_x \in \varphi(x) \cap \psi(x)$ and consider the covering $\mathcal{T} := \{T_V(x)\}_{V \in \mathcal{V}, x \in X}$ of X, where

$$T_V(x) := \{x' \in V \mid \psi(x') \cap B_Y(z_x, \delta) \neq \emptyset\}.$$

Obviously, by lower continuity of ψ, the set $T_V(x)$ is open for every $V \in \mathcal{V}$ and $x \in X$. Let $\{\lambda_j\}_{j \in J}$ be a partition of unity subordinated to $\mathcal{T}$. Hence, for each $j \in J$, there are $V_j \in \mathcal{V}$ and $x_j \in V_j$ such that $\operatorname{supp} \lambda_j \subset T_{V_j}(x_j)$.

Define functions $\lambda : J \times X \to [0,1]$ and $y : J \to Y$ putting $\lambda(j, x) := \lambda_j(x)$ and $y(j) := z_j := z_{x_j}$ for $j \in J$ and $x \in X$.

Obviously, all assumptions of Proposition 2.8 are satisfied and we may define a map $f : X \to Y$ by the formula

$$f(x) := b(y, \lambda(\cdot, x)).$$

This map is well-defined and continuous.

Let $x \in X$. Since, for each $j \in J(x) := \{j \in J \mid \lambda(j,x) > 0\}$, $x \in \operatorname{supp} \lambda_j \subset T_{V_j}(x_j)$, thus $\psi(x) \cap B_Y(z_j, \delta) \neq \emptyset$. Therefore $d_Y(z_j, \psi(x)) < \delta$ and this implies that

$$d(f(x), \psi(x)) < \varepsilon.$$

On the other hand, $x \in T_{V_j}(x_j) \subset V_j$ and $x_j \in V_j$ for all $j \in J(x)$. Hence, using the fact that $\mathcal{V}$ is a star refinement of $\mathcal{U}'$, there is $U' \in \mathcal{U}'$ such that $x, x_j \in U'$ for all $j \in J(x)$. Since $\mathcal{U}'$ is a refinement of $\mathcal{U}$, there is $\overline{x} \in X$ such that $x, x_j \in U(\overline{x})$ for all $j \in J(x)$. Consequently, for any such j, $z_j \in \varphi(x_j)$ and $\varphi(x_j) \subset B_Y(\varphi(\overline{x}), \delta)$, i.e., $d_Y(z_j, \varphi(\overline{x})) < \delta$. Since the value $\varphi(\overline{x})$ is α-convex, this implies that $d_Y(f(x), \varphi(\overline{x})) < \varepsilon$. Moreover, $x \in \mu^{-1}(\mu(\overline{x})/2, \infty) \cap B_X(\overline{x}, \mu(\overline{x})/2)$, then $d_X(x, \overline{x}) < \frac{\mu(\overline{x})}{2} < \mu(x)$. This completes the proof. ∎

Observe that if $\mu(x) = \varepsilon$, for all $x \in X$, then the above function f is an ε-graph-approximation of φ.

Remark 3.3. Theorem 3.2 is a direct generalization of Lemma 5.1 from [1]. Below we shall show a result even stronger than Theorem 3.2. Namely, we shall show the following theorem.

Theorem 3.4. *Under the assumption of Theorem* 3.2, *suppose that the space Y is complete and, for each $x \in X$, the set $\psi(x)$ is closed. Then, for every $\varepsilon > 0$ and a continuous function $\mu : X \to (0, \infty)$, there exists a continuous selection $f : X \to Y$ of ψ such that f is a $(\mu(\cdot), \varepsilon)$-approximation of φ.*

Proof. Let $\varepsilon > 0$ and let $\mu : X \to (0, \infty)$ be a continuous function.

By Proposition 2.9, there exist strictly decreasing sequences $(\delta_n), (\theta_n)$ such that, for any $n \geq 1$,

$$0 < 2\delta_n < \theta_n < 2\varepsilon_n,$$

where $\varepsilon_n := 2^{-n-1}\varepsilon$, and the following two conditions (C_1), (C_2) are satisfied.

(C_1) For every α-convex $A \subset Y$, $y_1, \dots, y_k \in Y$ and $(\lambda_1, \dots, \lambda_k) \in \Sigma^k$, where $1 \leq k \leq p+1$, if $d_Y(y_i, A) < \theta_n$ for all $i = 1, \dots, k$, then $d_Y(b_k(y_1, \dots, y_k; \lambda_1, \dots, \lambda_k), A) < \varepsilon_n$.

(C_2) For every α-convex $A \subset Y$, $y_1, \dots, y_k \in Y$ and $(\lambda_1, \dots, \lambda_k) \in \Sigma^k$, where $1 \leq k \leq p+1$, if $d_Y(y_i, A) < \delta_n$ for all $i = 1, \dots, k$, then $d_Y(b_k(y_1, \dots, y_k; \lambda_1, \dots, \lambda_k), A) < \theta_n/2$.

By Theorem 3.2, there exists a continuous function $g : X \to Y$ such that, for every $x \in X$,

$$B(g(x), \theta_1/2) \cap \psi(x) \neq \emptyset, \tag{6}$$

$$g(x) \in B(\varphi(B(x, \mu(x))), \theta_1/2). \tag{7}$$

Now we shall construct a sequence $(f_n)_{n=1}^{\infty}$ of continuous functions $f_n : X \to Y$ such that, for every $x \in X$,

$$d_Y(f_n(x), \psi(x)) < \theta_n/2, \quad n \geq 1, \tag{8}$$

$$d_Y(f_n(x), f_{n-1}(x)) < \varepsilon_{n-1}, \quad n \geq 2. \tag{9}$$

Let $f_1 := g$. Suppose that $n \geq 2$ and that continuous functions $f_i : X \to Y$, $i = 1, \dots, n-1$, have been defined. In order to construct $f_n : X \to Y$, let $\Phi_n : X \multimap Y$ be given by

$$\Phi_n(x) := \psi(x) \cap B(f_{n-1}(x), \theta_{n-1}/2) \text{ for } x \in X.$$

It is clear that $\Phi_n(x)$ has nonempty values and, in view of Proposition 2.2, is l.s.c. For each $y \in Y_0 := \psi(X)$, let

$$U_y := \{x \in X \mid \Phi_n(x) \cap B(y, \delta_n) \neq \emptyset\}.$$

Then, in view of the lower semicontinuity of $\Phi_n(x)$, $\mathcal{U} := \{U_y\}_{y \in Y_0}$ is an open covering of X. Since $\dim X \leq p$, there exists an open refinement $\mathcal{V} := \{V_j\}_{j \in J}$ of $\mathcal{U}$ such that every collection consisting of $p+2$ elements $V_1, \dots, V_{p+2}$ of $\mathcal{V}$ has an empty intersection.

Let $\{\lambda_j\}_{j \in J}$ be a partition of unity subordinate to $\mathcal{V}$. As before we assume that the set J is well-ordered. For each $j \in J$, there is $V_j \in \mathcal{V}$ and $y(j) \in Y_0$ such that $\operatorname{supp} \lambda_j \subset V_j \subset U_{y(j)}$.

Define function $\lambda : J \times X \to [0,1]$ putting $\lambda(j, x) := \lambda_j(x)$ for $j \in J$ and $x \in X$. Because the function λ satisfies assumptions of Proposition 2.8, then we may define a map $f_n : X \to Y$ by the formula

$$f_n(x) := b(y, \lambda(\cdot, x)).$$

By Proposition 2.8, f_n is well-defined and continuous.

We shall show that conditions (11), (12) hold. Indeed, let $x \in X$. For each $j \in J$, if $x \in \operatorname{supp} \lambda_j$, then $x \in V_j \subset U_{y(j)}$. Hence $\Phi_n(x) \cap B(y(j), \delta_n) \neq \emptyset$ and therefore

$$d_Y(y(j), \psi(x)) < \delta_n. \tag{10}$$

Observe that $\#\{j \in J \mid x \in \operatorname{supp} \lambda(j, \cdot)\} \leq p+1$. By $(C2)$, we have

$$d_Y(f_n(x), \psi(x)) < \theta_n/2,$$

i.e., f_n satisfies condition (11).

Since $x \in U_{y(j)}$, then $B(f_{n-1}(x), \theta_{n-1}/2) \cap B(y(j), \delta_n) \neq \emptyset$ and we get that

$$d_Y(y(j), f_{n-1}(x)) < \delta_n + \theta_{n-1}/2 < \theta_n/2 + \theta_{n-1}/2 < \theta_{n-1}.$$

Then, by $(C1)$,

$$d_Y(f_n(x), f_{n-1}(x)) < \varepsilon_{n-1}.$$

This shows condition (12) and inductively completes the construction of the sequence $(f_n)_{n=1}^\infty$.

Since, for each $x \in X$, $(f_n(x))_{n=1}^\infty$ is a Cauchy sequence and the space Y is complete, $(f_n(x))_{n=1}^\infty$ converges to a point $f(x) \in Y$. Because the sequence $(f_n)_{n=1}^\infty$ is uniformly convergent, then the function $f : X \to Y$ is continuous. Moreover, it is clear that $f(x) \in \psi(x)$ for any $x \in X$.

For every $x \in X$, we have

$$\begin{aligned} d_Y(f_1(x), f(x)) &= \lim_{n\to\infty} d_Y(f_1(x), f_{n+1}(x)) \\ &\leq \lim_{n\to\infty} \sum_{k=1}^{n} d_Y(f_k(x), f_{k+1}(x)) < \sum_{k=1}^{\infty} \varepsilon_k = \varepsilon \sum_{k=1}^{\infty} 2^{-k-1} = \varepsilon/2. \end{aligned}$$

By (7),

$$f(x) \in B(\varphi(B(x, \mu(x))), \theta_1/2 + \varepsilon/2) \subset B(\varphi(B(x, \mu(x))), \varepsilon).$$

The proof is complete. ■

Remark 3.5. Observe that if in Theorem 3.4 $\psi(x) = Y$, for every $x \in X$, than we have generalization of the Cellina theorem (see [7]), and if $\varphi(x) = Y$, for each $x \in X$, than we have the Michael type continuous selection theorem (see [15]).

Now we present the relative graph-approximations theorem.

Theorem 3.6. *Let X be a metric space,* $\dim X < \infty$*, and Y be an α-convex metric space. Let $\psi : X \multimap Y$ be a $\mathfrak{D}$-lower semicontinuous map with α-convex values and $\varphi : X \multimap Y$ an upper semicontinuous map with α-convex closed values such that $\varphi(x) \cap \psi(x) \neq \emptyset$ for each $x \in X$. Let $A \subset X$ be any closed set.Then, for any $\varepsilon > 0$ and $0 < \delta < \eta$, where $\eta \leq \varepsilon$ corresponds to ε as in Definition* 2.3*, we have that*

(i) *each δ-selection $f : A \to Y$ of φ such that f is a selection of ψ extends to a map $\overline{f} : X \to Y$ such that $\overline{f}$ is an ε-graph-approximation of φ and an ε-selection of ψ,*

(ii) *there exists a continuous function $\delta : X \to (0, \infty)$ such that any $\delta(\cdot)$-graph-approximation $f : A \to Y$ of φ such that f is a selection of ψ extends to a map $\overline{f} : X \to Y$ such that $\overline{f}$ is an ε-graph-approximation of φ and an ε-selection of ψ.*

Proof. Let $\varepsilon > 0$ and $0 < \delta < \eta$, where $\eta \leqslant \varepsilon$ corresponds to ε as in Definition 2.3.

Step 1. We shall construct a continuous function $\mu : X \to (0, \infty)$ such that, for each $x \in X$, there is $\overline{x} \in B_X(x, \eta)$ such that

$$B(\varphi(B(x, \mu(x))), \delta) \subset B(\varphi(\overline{x}), \eta).$$

Because of φ is u.s.c., then we can, for each $x \in X$, choose $r_x \in (0, \eta)$ so that

$$\varphi(B(x, 2r_x)) \subset B(\varphi(x), \eta - \delta).$$

Let $\{\lambda_j\}_{j \in J}$ be a partition of unity subordinated to the covering $\{B(x, r_x)\}_{x \in X}$, that means, for each $j \in J$, there is a point x_j such that $\operatorname{supp} \lambda_j \subset B(x_j, r_j)$, where $r_j := r_{x_j}$, and $\sum_{j \in J} \lambda_j(x) = 1$ for each $x \in X$.

Put

$$\mu(x) := \sum_{j \in J} \lambda_j(x) r_j.$$

Let $x \in X$, then there is $j \in J$ such that $\lambda_j(x) > 0$ and $\mu(x) \leq r_j$. Since $\lambda_j(x) > 0$, then $x \in B(x_j, r_j)$, i.e., $d_X(x, x_j) < r_j$. Thus

$$B(x, \mu(x)) \subset B(x_j, 2r_j),$$

and hence

$$\varphi(B(x, \mu(x))) \subset \varphi(B(x_j, 2r_j)) \subset B(\varphi(x_j), \eta - \delta).$$

We have that

$$B(\varphi(B(x, \mu(x))), \delta) \subset B(\varphi(x_j), \eta)$$

and $d_X(x, x_j) < r_j < \eta$.

We finish the proof if we put $\overline{x} = x_j$.

Step 2. For each $(x, y) \in X \times Y$, put

$$U^\delta(x, y) := [\mu^{-1}((\mu(x)/2, \infty)) \cap B(x, \mu(x)/2)] \times B(y, \delta)$$

and let

$$\mathcal{U}_\varphi^\delta := \bigcup_{(x,y) \in Gr(\varphi)} U^\delta(x, y).$$

We shall show that there is a continuous function $\delta : X \to (0, \infty)$ such that any $\delta(\cdot)$-graph-approximation of φ over A is an $\mathcal{U}_\varphi^\delta$-approximation of φ over A.

The upper semicontinuity of φ implies that, for any $x \in X$, there exists a number $r(x) > 0$ such that

$$B(x, r(x)) \times B(\varphi(B(x, 2r(x))), r(x)) \subset \mathcal{U}_\varphi^\delta.$$

Let $\{\lambda_j\}_{j \in J}$ be a partition of unity inscribed into the covering $\{B(x, r(x))\}_{x \in X}$. Hence, for each $j \in J$, there is $x_j \in X$ such that $\operatorname{supp} \lambda_j \subset B(x_j, r(x_j))$. Let $r_j := r(x_j)$ and define

$$\delta(x) := \sum_{j \in J} \lambda_j(x) r_j, \quad x \in X.$$

Suppose that $f : A \to Y$ is a $\delta(\cdot)$-graph-approximation of φ.

Take $x \in A$. There is $j \in J$ such that $\lambda_j(x) > 0$ and $\delta(x) \leq r_j$. Thus $x \in B(x_j, r_j))$. Since

$$f(x) \in B(\varphi(B(x, \delta(x))), \delta(x)),$$

then there is $x' \in B(x, \delta(x))$ and $y' \in \varphi(x')$ such that

$$f(x) \in B(y', \delta(x)).$$

Therefore $y' \in \varphi(B(x_j, 2r_j))$ and $f(x) \in B(\varphi(B(x_j, 2r_j)), r_j)$. Altogether we have

$$(x, f(x)) \in B(x_j, r_j) \times B(\varphi(B(x_j, 2r_j)), r_j) \subset \mathcal{U}^{\delta}_{\varphi}.$$

Step 3. Take any function $f : A \to Y$ such that f is a δ-selection of φ (adequately a $\delta(\cdot)$-graph-approximation of φ) and a selection of ψ.

We shall construct $\widetilde{f} : X \to Y$ a continuous extension of f such that $\widetilde{f}$ is a δ-selection of ψ.

Let $0 < \xi \leq \delta$ be a number such that, for any $x_1, x_2, y_1, y_2 \in Y$, if $d(x_1, y_1) < \xi$ and $d(x_2, y_2) < \xi$, then

$$\mathfrak{D}(\Lambda_\alpha(x_1, x_2; [0,1]), \Lambda_\alpha(y_1, y_2; [0,1])) < \delta.$$

Since ψ is $\mathfrak{D}$-lower semicontinuous, then, for each $z \in X$, there is a neighbourhood U_z such that $\mathfrak{D}^+(\psi(x), \psi(z)) < \frac{\xi}{2}$ for all $x \in U_z$. For each $(z, y) \in \mathrm{Gr}\,(\psi)$, let

$$V^\xi(z, y) := U_z \times B(y, \xi/2).$$

Obviously, $V^\xi(z, y)$ is a neighbourhood of the point (z, y). Let

$$\mathcal{U}^\xi_\psi := \bigcup_{(x,y)\in \mathrm{Gr}\,(\psi)} V^\xi(x, y).$$

Let $f^* : X \to Y$ be any extension of f. Since $f^*|_A$ is a selection of ψ, then, in particular, it is an $\mathcal{U}^\xi_\psi$-approximation of ψ over A. Because $\mathcal{U}^\xi_\psi$ is an open set in $X \times Y$, then there exists an open neighbourhood M of A such that $f^*|_M : M \to Y$ is an $\mathcal{U}^\xi_\psi$-approximation of ψ over M.

Let $x \in M$. We have that $(x, f^*(x)) \in \mathcal{U}^\xi_\psi$, this means that there exist $z \in X$ and $y \in \psi(z)$ such that $x \in U_z$ and $f^*(x) \in B(y, \xi/2)$. Thus $\mathfrak{D}^+(\psi(x), \psi(z)) < \frac{\xi}{2}$ and $d_Y(f^*(x), y) < \frac{\xi}{2}$. Hence $y \in \psi(z) \subset B(\psi(x), \xi/2)$ and

$$d_Y(f^*(x), \psi(x)) \leq d_Y(f^*(x), y) + d_Y(y, \psi(x)) < \frac{2\xi}{2} = \xi.$$

Therefore $f^*|_M$ is a ξ-selection of ψ over M.

Take any ξ-selection $g : X \to Y$ of the map ψ. Such function exists, by Theorem 3.2.

Let V be an open neighbourhood of the set A such that $A \subset V \subset \overline{V} \subset M$ and let $\{\beta, \kappa\}$ be a continuous partition of unity subordinate to $\{M, X \setminus \overline{V}\}$, that means $\operatorname{supp} \beta \subset M$, $\operatorname{supp} \kappa \subset X \setminus \overline{V}$ and $\beta(x) + \kappa(x) = 1$ for all $x \in X$. Define a continuous function $\widetilde{f} : X \to Y$ as follows

$$\widetilde{f}(x) = \alpha(f^*(x), g(x), \kappa(x)).$$

Let $x \in X$. If $x \notin M$, then $\beta(x) = 0$ and $\kappa(x) = 1$, thus $\widetilde{f}(x) = g(x)$. We have

$$\widetilde{f}(x) = g(x) \in B_Y(\psi(x), \xi) \subset B_Y(\psi(x), \delta).$$

If $x \in M$, then $f^*(x), g(x) \in B_Y(\psi(x), \xi)$ and, by Proposition 2.9, we have

$$\widetilde{f}(x) \in B_Y(\psi(x), \delta).$$

Step 4. We shall show that $\widetilde{f}|_W : W \to Y$ is a $(\mu(\cdot), \delta)$-approximation of φ over an open neighbourhood W of the set A.

Let $x \in A$. Because $\widetilde{f}(x) \in B_Y(\varphi(x), \delta)$, then there is $y \in \varphi(x)$ such that $\widetilde{f}(x) \in B_Y(y, \delta)$. Therefore $(x, \widetilde{f}(x)) \in U^\delta(x, y) \subset \mathcal{U}_\varphi^\delta$ (if $\widetilde{f}|_A$ is a $\delta(\cdot)$-graph-approximation of φ, then it is $\mathcal{U}_\varphi^\delta$-approximation, by step 2).

Since a $\mathcal{U}_\varphi^\delta$ is an open set in $X \times Y$, then there exists an open neighbourhood W of A such that $\widetilde{f}|_W : W \to Y$ is an $\mathcal{U}_\varphi^\delta$-approximation of φ over W.

Let $w \in W$. Since $(w, \widetilde{f}(w)) \in \mathcal{U}_\varphi^\delta$, then there is $(x, y) \in \operatorname{Gr}(\varphi)$ such that $w \in \mu^{-1}((\mu(x)/2, \infty)) \cap B(x, \mu(x)/2)$ and $\widetilde{f}(w) \in B(y, \delta)$. Thus we have that $\widetilde{f}(w) \in B(y, \delta)$ and $y \in \varphi(x)$, where $d_X(x, w) < \frac{\mu(x)}{2} < \mu(w)$.

Therefore $\widetilde{f}(w) \in B_Y(\varphi(B_X(w, \mu(w))), \delta)$ for all $w \in W$.

Step 5. By above steps we have the continuous function $\widetilde{f} : X \to Y$ such that $\widetilde{f}|_A = f$ as well as $\widetilde{f}|_W$ is a $(\mu(\cdot), \delta)$-approximation of φ and a δ-selection of ψ.

Take an open neighbourhood V of A such that $A \subset V \subset \overline{V} \subset W$. Let $\{\beta, \kappa\}$ be a partition of unity inscribed into the covering $\{W, X \setminus \overline{V}\}$, i.e., $\beta, \kappa : X \to [0, 1]$ are continuous and $\operatorname{supp} \beta \subset W$, $\operatorname{supp} \kappa \subset X \setminus \overline{V}$ and $\beta(x) + \kappa(x) = 1$ for all $x \in X$.

By Theorem 3.2 there exists a continuous function $g : X \to Y$ which is a $(\mu(\cdot), \delta)$-approximation of φ and a δ-selection of ψ. Define $\overline{f} : X \to Y$ as follows

$$\overline{f}(x) = \alpha(\widetilde{f}(x), g(x), \kappa(x)).$$

Obviously $\overline{f}$ is continuous. Let $x \in X$. If $x \notin W$, then $\beta(x) = 0$ and $\kappa(x) = 1$, thus $\overline{f}(x) = g(x)$. We have

$$\overline{f}(x) = g(x) \in B_Y(\psi(x), \delta) \subset B_Y(\psi(x), \varepsilon)$$

and, by Step 1,

$$\begin{aligned}\overline{f}(x) = g(x) &\in B_Y(\varphi(B_X(x, \mu(x))), \delta) \subset B_Y(\varphi(B_X(x, \eta)), \eta)\\ &\subset B_Y(\varphi(B_X(x, \varepsilon)), \varepsilon).\end{aligned}$$

If $x \in W$, then

$$\widetilde{f}(x), g(x) \in B_Y(\varphi(B_X(x, \mu(x))), \delta) \cap B_Y(\psi(x), \delta).$$

By Step 1, there is $\overline{x} \in B_X(x, \eta)$ such that

$$B_Y(\varphi(B_X(x, \mu(x))), \delta) \subset B_Y(\varphi(\overline{x}), \eta).$$

$\varphi(\overline{x})$ is α-convex, so, by Proposition 2.9, we have

$$\overline{f} \in B_Y(\varphi(\overline{x}), \varepsilon) \subset B_Y(\varphi(B_X(x, \varepsilon)), \varepsilon)$$

and also, since $B_Y(\psi(x), \delta) \subset B_Y(\psi(x), \eta)$, then

$$\overline{f} \in B_Y(\psi(x), \varepsilon).$$

■

Assumptions of Theorem 3.6 state that the map ψ is $\mathfrak{D}$-lower semicontinuous, because it was necessary in order to get the existence of an extension of a selection of the mapping to a δ-selection. But we know that $\mathfrak{D}$-lower semicontinuity implies lower semicontinuity and if values of the map ψ are compact, then we have a reverse implication.

Remark 3.7. Observe that Theorem 3.6 may be formulated as follows.

Let X be a metric space, $\dim X < \infty$, and Y be an α-convex metric space. Let $\psi : X \multimap Y$ be a $\mathfrak{D}$-lower semicontinuous map with α-convex values and $\varphi : X \multimap Y$ an upper semicontinuous map with α-convex closed values such that $\varphi(x) \cap \psi(x) \neq \emptyset$ for each $x \in X$. Let $A \subset X$ be any closed set. Then, for any $\varepsilon, \iota > 0$ and $0 < \delta < \eta$, where $\eta \leq \varepsilon$ corresponds to ε as in Definition 2.3, we have that

- (i) each δ-selection $f : A \to Y$ of φ such that f is a selection of ψ extends to a map $\overline{f} : X \to Y$ such that $\overline{f}$ is an ε-graph-approximation of φ and an ι-selection of ψ,
- (ii) there exists a continuous function $\delta : X \to (0, \infty)$ such that any $\delta(\cdot)$-graph-approximation $f : A \to Y$ of φ such that f is a selection of ψ extends to a map $\overline{f} : X \to Y$ such that $\overline{f}$ is an ε-graph-approximation of φ and an ι-selection of ψ.

Theorem 3.8. *Let X be a metric space, $\dim X < \infty$, and Y be an α-convex complete metric space. Let $\psi : X \multimap Y$ be a $\mathfrak{D}$-lower semicontinuous map with α-convex closed values and $\varphi : X \multimap Y$ an upper semicontinuous map with α-convex closed values such that $\varphi(x) \cap \psi(x) \neq \emptyset$ for each $x \in X$. Let $A \subset X$ be a non-empty closed set. Then, for any $\varepsilon > 0$ and $0 < \delta < \eta$, where $\eta \leq \frac{\varepsilon}{2}$ corresponds to $\frac{\varepsilon}{2}$ as in Definition 2.3, we have that*

- (i) *each δ-selection $f : A \to Y$ of φ such that f is a selection of ψ extends to a map $\overline{f} : X \to Y$ such that $\overline{f}$ is an ε-graph-approximation of φ and a selection of ψ,*
- (ii) *there exists a continuous function $\delta : X \to (0, \infty)$ such that any $\delta(\cdot)$-graph-approximation $f : A \to Y$ of φ such that f is a selection of ψ extends to a map $\overline{f} : X \to Y$ such that $\overline{f}$ is an ε-graph-approximation of φ and a selection of ψ.*

Proof. This proof is analogous to the proof of Theorem 3.4. Let $\varepsilon > 0$ and $0 < \delta < \eta$, where $\eta \leq \frac{\varepsilon}{2}$ corresponds to $\frac{\varepsilon}{2}$. Let $A \subset X$ be a non-empty closed set.

By Proposition 2.9, there exist strictly decreasing sequences $(\delta_n), (\theta_n)$ such that, for any $n \geq 1$,

$$0 < 2\delta_n < \theta_n < 2\varepsilon_n,$$

where $\varepsilon_n := 2^{-n-1}\varepsilon$, and the following two conditions (C_1), (C_2) are satisfied.

(C_1) For every α-convex $A \subset Y$, $y_1, \dots, y_k \in Y$ and $(\lambda_1, \dots, \lambda_k) \in \Sigma^k$, where $1 \leq k \leq p+1$, if $d_Y(y_i, A) < \theta_n$ for all $i = 1, \dots, k$, then $d_Y(b_k(y_1, \dots, y_k; \lambda_1, \dots, \lambda_k), A) < \varepsilon_n$.

(C_2) For every α-convex $A \subset Y$, $y_1, \dots, y_k \in Y$ and $(\lambda_1, \dots, \lambda_k) \in \Sigma^k$, where $1 \leq k \leq p+1$, if $d_Y(y_i, A) < \delta_n$ for all $i = 1, \dots, k$, then $d_Y(b_k(y_1, \dots, y_k; \lambda_1, \dots, \lambda_k), A) < \theta_n/2$.

By Theorem 3.6 (in fact Remark 3.7), there exists a continuous function $\delta : X \to (0, \infty)$ such that any $\delta(\cdot)$-graph-approximation $g : A \to Y$ of φ such that g is a selection of ψ extends to a map $\overline{g} : X \to Y$ such that $\overline{g}$ is an $\frac{\varepsilon}{2}$-graph-approximation of φ and a $\frac{\theta_1}{2}$-selection of ψ.

Let $f : A \to Y$ be a δ-selection (resp. $\delta(\cdot)$-graph-approximation) of φ and a selection of ψ. By the above theorem, we have an extension $g : X \to Y$ such that g is an $\frac{\varepsilon}{2}$-graph-approximation of φ and a $\frac{\theta_1}{2}$-selection of ψ. Set $f_1 := g$ and construct a sequence $(f_n)_{n=1}^{\infty}$ of continuous functions $f_n : X \to Y$ such that, for every $x \in X$,

$$d_Y(f_n(x), \psi(x)) < \theta_n/2, \quad n \geq 1, \tag{11}$$

$$d_Y(f_n(x), f_{n-1}(x)) < \varepsilon_{n-1}, \quad n \geq 2. \tag{12}$$

For this construction we use the lower semicontinuous function

$$\Phi_n(x) := \begin{cases} f(x), & \text{for} \quad x \in A; \\ \psi(x) \cap B(f_{n-1}(x), \theta_{n-1}/2), & \text{for} \quad x \in X \setminus A. \end{cases}$$

The function $\overline{f} := \lim_{n\to\infty} f_n$ is a required extension of the function f. ■

Under the same assumptions as in the above theorem, we have the following conclusions. Let $\varepsilon > 0$ and A be a closed subset of X.

Corollary 3.9. *Each selection $f : A \to Y$ of φ and ψ over A can extend to an ε-graph-approximation of φ and a selection of ψ.*

Corollary 3.10. *If A is compact, then there exists $\delta > 0$ such that every δ-graph-approximation $f : A \to Y$ of φ such that f is a selection of ψ over A extends to a map $\overline{f} : X \to Y$ such that $\overline{f}$ is an ε-graph-approximation of φ and a selection of ψ.*

Corollary 3.11. *Let $0 < \delta < \eta$, where η corresponds to $\frac{\varepsilon}{2}$ as in Definition 2.3. Let $f : X \to Y$ and $g : X \to Y$ be δ-selections of φ and selections of ψ. Then there exists a homotopy $H : X \times [0,1] \to Y$ between f and g such that $H(\cdot, t)$ is an ε-graph-approximation of φ and a selection of ψ for every $t \in [0,1]$.*

Proof. Let $\delta > 0$, f and g be as above.
Let $\pi : X' \to X$ be a projection, where $X' := X \times [0,1]$, and let $\varphi' := \varphi \circ \pi : X' \multimap Y$, $\psi' := \psi \circ \pi : X' \multimap Y$. It easy to see that φ' and ψ' have properties as φ and ψ adequately. Moreover, $X \times \{0,1\}$ is a closed set in X' and $\dim X' < \infty$.

Now we define a function $H' : X \times \{0,1\} \to Y$ as follows: $H'(x,0) = f(x)$, for all $x \in X$, and $H'(x,1) = g(x)$, for all $x \in X$. Then H' is a δ-selection of φ' and a selection of ψ' over $X \times \{0,1\}$. Thus, by Theorem 3.8, it admits an extension $H : X' \to Y$ being an ε-graph-approximation of φ' and a selection of ψ'. It is clear that H is the required homotopy. ■

Recall that the Cellina theorem [7] concerns an existence of a continuous graph-approximation of an upper semicontinuous map $\varphi : X \multimap Y$. Now we shall present a stronger theorem in the spirit of the second Cellina result [7]. We shall prove that there exists a continuous function such that the Hausdorff distance of its graph to the graph of the map φ with α-convex values is optionally small.

But in order to proceed, we must assume additional properties of the function α:

(i') $\alpha(x, y, \cdot)$ is one-to-one for all $x, y \in X$;

(ii') $\alpha(\alpha(x, y, t_1), \alpha(x, y, t_2)) = \alpha(x, y, (1 - t_3)t_1 + t_2 t_3)$ for all $x, y \in X$ and $t_1, t_2, t_3 \in [0,1]$.

Observe that, by (ii'), the set $\Lambda_\alpha(x_1, x_2; [0,1]) \subset Y$ is α-convex for all $x_1, x_2 \in Y$.

We need to proof the following proposition.

Proposition 3.12. *Let X, Y be α-convex metric spaces such that* $\dim X < \infty$ *and let $K \subset X$ be an α-convex set such that* $\operatorname{diam}(K) < \frac{\varepsilon}{2}$, *where $\varepsilon > 0$, and K contains at least two points. Let $T \subset Y$ be a closed α-convex set for which there exist points $y_1, \ldots, y_n \in Y$, $n \in \mathbb{N}$, such that $T \subset \{b_n(y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_n) \mid (\lambda_1, \ldots, \lambda_n) \in \Sigma^n\}$. Then there is a continuous* *function* $f : K \to T$ *such that*

$$\mathfrak{D}(\mathrm{Gr}\,(f), K \times T) < \varepsilon.$$

Proof. Let $\varepsilon > 0$.
Suppose that T contains only one point, $T = \{y_0\}$. Then $f(x) = y_0$ for all $x \in K$. We have that

$$\mathrm{Gr}\,(f) = K \times \{y_0\} = K \times T,$$

and hence

$$\mathfrak{D}(\mathrm{Gr}\,(f), K \times T) = 0.$$

Suppose that T contains more then one point. Let

$$r : \{b_n(y_1, \ldots, y_n; \Sigma^n)\} \to T$$

be a retraction, where
$\{b_n(y_1, \ldots, y_n; \Sigma^n)\} := \{b_n(y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_n) | (\lambda_1, \ldots, \lambda_n) \in \Sigma^n\}$.

If $x_1, x_2 \in K$, then we have $\Lambda_\alpha(x_1, x_2, [0, 1]) \subset K$. Let

$$\nu : \Lambda_\alpha(x_1, x_2, [0, 1]) \to \{b_n(y_1, \ldots, y_n; \Sigma^n)\}$$

be a continuous surjection. The function ν has a form $\nu = \nu_3 \circ \nu_2 \circ \nu_1$, where

(a) $\nu_1 : \Lambda_\alpha(x_1, x_2; [0, 1]) \to [0, 1]$ and $\nu_1(\alpha(x_1, x_2, t)) = t$, for all $t \in [0, 1]$;

(b) $\nu_2 : [0, 1] \to \Sigma^n$ is a continuous surjection; it exists because Σ^n is a locally connected metric continuum (a continuum is a compact connected set) and hence it is a continuous image of a segment [0,1] (see [11]);

(c) $\nu_3 : \Sigma^n \to \{b_n(y_1, \ldots, y_n; \Sigma^n)\}$ and $\nu_3((\lambda_1, \ldots, \lambda_n)) =$ $b_n(y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_n)$ for all $(\lambda_1, \ldots, \lambda_n) \in \Sigma^n$.

Then $r \circ \nu : \Lambda_\alpha(x_1, x_2; [0, 1]) \to T$ is a continuous surjection too. Let

$$r_1 : K \to \Lambda_\alpha(x_1, x_2; [0, 1])$$

be a retraction.

Define $f : K \to T$ as

$$f(x) := r(\nu(r_1(x))).$$

Because $f(K) = T$, then $\mathrm{Gr}\,(f) \subset K \times T$ and hence $\sup_{z \in \mathrm{Gr}\,(f)} d_{X\times Y}(z, K \times T) = 0$. Take now $(x, y) \in K \times T$. Since $y \in T$, then $r(y) = y$. There is a point $x^* \in \Lambda_\alpha(x_1, x_2, [0,1])$ such that

$$\nu(x^*) = y \quad \text{and} \quad r_1(x^*) = x^*, \quad \text{that means} \quad y = r(\nu(r_1(x^*))).$$

Then $(x^*, y) \in \mathrm{Gr}\,(f)$ and

$$\begin{aligned} d_{X\times Y}((x,y), \mathrm{Gr}\,(f)) &= \inf_{z\in \mathrm{Gr}\,(f)} d_{X\times Y}((x,y), z) \le d_{X\times Y}((x,y),(x^*,y)) \\ &= \max\{d_X(x, x^*), 0\} \le \frac{\varepsilon}{2}, \end{aligned}$$

because $x, x^* \in K$. Therefore

$$\mathfrak{D}(\mathrm{Gr}\,(f), K \times T) < \varepsilon.$$

■

Theorem 3.13. *Let X, Y be α-convex metric spaces such that $\dim X = p < \infty$ and X does not have isolated points. Let $\varphi : X \multimap Y$ be an upper semicontinuous map with closed α-convex values. Suppose that, for each $x \in X$, there exist points $y_1, \dots, y_n \in Y$, $n \in \mathbb{N}$, such that $\varphi(x) \subset \{b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) \mid (\lambda_1, \dots, \lambda_n) \in \Sigma^n\}$. Then, for each $\varepsilon > 0$, there exists a continuous function $f : X \to Y$ such that*

$$\mathfrak{D}(\mathrm{Gr}\,(f), \mathrm{Gr}\,(\varphi)) < \varepsilon.$$

Proof. Let $\varepsilon > 0$. By Proposition 2.9, there exists $0 < \delta \le \frac{\varepsilon}{3}$ such that, for any $n \le p + 1$, any $y_1, \dots, y_n \in Y$ and any non-empty α-convex set $A \subset Y$, if $d_Y(y_i, A) < \delta$ for $i = 1, \dots, n$, then

$$d_Y(b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n), A) < \frac{\varepsilon}{3}$$

for all $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$. Upper semicontinuity of φ implies that, for each $x \in X$, there exists $0 < \theta = \theta(x) < \frac{\varepsilon}{4}$ such that

$$\varphi(B(x, \theta(x))) \subset B(\varphi(x), \delta).$$

Since $\dim X = p$, then there exists $\{U_j\}_{j\in J}$ an open refinement of the covering $\{B(x, \theta(x)/3)\}_{x\in X}$ such that every subfamily consisting of $p + 2$ sets in $\{U_j\}_{j\in J}$ has empty intersection.

For each $j \in J$, choose $x_j \in U_j$ and τ_j' such that $D(x_j, \tau_j') \subset U_j$. Suppose that $D(x_j, \tau_j')$ has a nonempty intersection with sets $U_{j_1}, \ldots, U_{j_n}$ for $n \leq p$ and $j_k \neq j, k = 1, \ldots, n$. Let

$$\tau_j'' := \min_{1 \leq k \leq n} \{d_X(x_j, x_{j_k})\} \quad \text{and} \quad \tau_j := \frac{1}{3} \min\{\tau_j', \tau_j''\}.$$

For every $j \in J$, let $D(x_j)$ be a closed α-convex set such that

$$x_j \in D(x_j) \subset D(x_j, \tau_j)$$

and $D(x_j)$ contains at least two points. For all $j \in J$ such set $D(x_j)$ exists. Indeed, let $0 < \eta \leq \tau_j$ corresponds to τ_j as in Definition 2.3. Take a point $a \in B(x_j, \eta)$, $a \neq x_j$. Since $d_X(a, x_j) < \eta$ and $d_X(x_j, x_j) = 0 < \eta$, then $\mathfrak{D}(x_j, \Lambda_\alpha(x_j, a, [0,1])) < \tau_j$ and we can assume $D(x_j) = \Lambda_\alpha(x_j, a, [0,1])$. Consider the family $\{D(x_j)\}_{j \in J}$. It easy to see $D(x_j) \cap D(x_i) = \emptyset$ for $j, i \in J$, $j \neq i$.

Let $C_j := \bigcup_{i \in J, i \neq j} D(x_i)$. Clearly, C_j is a closed set for $j \in J$. For all $j \in J$, let $V_j := U_j \setminus C_j$. The family $\{V_j\}_{j \in J}$ is a locally finite covering of X such that, for each $j \in J$,

$$D(x_j) \subset V_j \quad \text{and} \quad D(x_j) \cap V_i = \emptyset \quad \text{for} \quad i \neq j.$$

Let $\{\lambda_j\}_{j \in J}$ be a partition of unity inscribed into the covering $\{V_j\}_{j \in J}$. For each $j \in J$, choose $\zeta_j \in X$ such that

$$V_j \subset U_j \subset B(\zeta_j, \theta(\zeta_j)/3).$$

Because $\operatorname{diam} D(x_j) < \frac{\varepsilon}{6}$ and all assumption of Proposition 3.12 (for $K = D(x_j)$ and $T = \varphi(\zeta_j)$) are true, then there exists a continuous function $\gamma_j : D(x_j) \to \varphi(\zeta_j)$ such that

$$\mathfrak{D}(\operatorname{Gr}(\gamma_j), D(x_j) \times \varphi(\zeta_j)) < \frac{\varepsilon}{3}.$$

Let $f_j : X \to \varphi(\zeta_j)$ be an extension of γ_j.

Let $\lambda : J \times X \to [0,1]$ be given by $\lambda(j,x) = \lambda_j(x)$, let $y : J \times X \to Y$ be given by $y(j,x) = f_j(x)$. Then all assumptions from Proposition 2.8 are satisfied, hence we may define

$$f(x) = b(y(\cdot, x), \lambda(\cdot, x)), \quad x \in X.$$

Clearly, $f : X \to Y$ is well-defined and continuous.

Let $x \in X$. Suppose that $x \in V_{j_1}, \dots, V_{j_k}$ and $x \notin V_i$, for $i \neq j_1, \dots, j_k$, where $1 \leq k \leq p+1$. Take number $k_0 \in \{j_1, \dots, j_k\}$ such that $\theta(\zeta_{j_{k_0}}) \geq \theta(\zeta_{j_m})$ for all $1 \leq m \leq k$. Observe that, for $1 \leq m \leq k$,

$$d_X(\zeta_{j_m}, \zeta_{j_{k_0}}) \leq d_X(\zeta_{j_m}, x) + d_X(x, \zeta_{j_{k_0}}) < \frac{\theta(\zeta_{j_m})}{3} + \frac{\theta(\zeta_{j_{k_0}})}{3} < \theta(\zeta_{j_{k_0}}),$$

that means $\zeta_{j_m} \in B(\zeta_{j_{k_0}}, \theta(\zeta_{j_{k_0}}))$. Hence

$$f_{j_m}(x) \in \varphi(\zeta_{j_m}) \subset \varphi(B(\zeta_{j_{k_0}}, \theta(\zeta_{j_{k_0}})) \subset B(\varphi(\zeta_{j_{k_0}}), \delta).$$

Thus $f(x) \in B(\varphi(\zeta_{j_{k_0}}), \frac{\varepsilon}{3})$. Then

$$\begin{aligned} d_{X\times Y}((x, f(x)), \mathrm{Gr}\,(\varphi)) \;&\leq\; d_{X\times Y}((x, f(x)), (\zeta_{j_{k_0}}, f(x))) \\ &\quad + d_{X\times Y}((\zeta_{j_{k_0}}, f(x)), \mathrm{Gr}\,(\varphi)) \leq \frac{\theta(\zeta_{j_{k_0}})}{3} + \frac{\varepsilon}{3} < \frac{\varepsilon}{2}. \end{aligned}$$

Therefore

$$\sup_{z \in \mathrm{Gr}\,(f)} d_{X\times Y}(z, \mathrm{Gr}\,(\varphi)) < \varepsilon.$$

Take $(x, y) \in \mathrm{Gr}\,(\varphi)$. There exist $j \in J$ and $\zeta_j \in X$ such that

$$x \in V_j \subset B(\zeta_j, \theta(\zeta_j)/3) \quad \text{and} \quad y \in \varphi(x) \subset \varphi(B(\zeta_j, \theta(\zeta_j)/3)) \subset B(\varphi(\zeta_j), \delta).$$

Thus there exists $y_j \in \varphi(\zeta_j)$ such that

$$d_Y(y_j, y) < \delta.$$

By construction of γ_j, it follows that there exists $x' \in D(x_j)$ such that $y_j = \gamma_j(x')$. Thus we have that

$$(x', \gamma_j(x')) \in \mathrm{Gr}\,(\gamma_j) \quad \text{and} \quad d_{X\times Y}((x', \gamma_j(x')), (x_j, y_j)) < \frac{\varepsilon}{6}.$$

Obviously, $f|_{D(x_j)} = f_j|_{D(x_j)} = \gamma_j$. Therefore

$$\begin{aligned} d_{X\times Y}((x, y), (x', f(x'))) \;&\leq\; d_{X\times Y}((x, y), (x_j, y_j)) \\ &\quad + d_{X\times Y}((x_j, y_j), (x', f(x'))) < \frac{\varepsilon}{3} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}, \end{aligned}$$

that means

$$\sup_{z\in \mathrm{Gr}\,(\varphi)} d_{X\times Y}(z, \mathrm{Gr}\,(f)) < \varepsilon.$$

Then we have

$$\mathfrak{D}(\mathrm{Gr}\,(f), \mathrm{Gr}\,(\varphi)) < \varepsilon.$$

■

Definition 3.14 (compare [18])**.** Let X, Y be metric spaces. A map $F : X \multimap Y$ is called *sub-lower semicontinuous* if, for each $x \in X$ and each $\varepsilon > 0$, there is $z \in F(x)$ and a neighborhood U_x of x in X such that $z \in B_Y(F(y), \varepsilon)$ for each $y \in U_x$.

It is clear that the sub-lower semicontinuity is weaker than the lower semicontinuity.

Example 3.15. Let $F : X \multimap Y$ be given by the formula

$$F(x) = \begin{cases} 1, & \text{for} \quad x \in \mathbb{R} \setminus \{1\}; \\ [1,3], & \text{for} \quad x = 1. \end{cases}$$

This function is sub-lower semicontinuous, but it is not lower semicontinuous (at the point $x = 1$).

The following theorem concerns an existence of a continuous ε-selection for set-valued maps.

Theorem 3.16. *Let X be a metric space and Y be an α-convex metric space,* $\dim X < \infty$*. Let $F : X \multimap Y$ be a map with α-convex values. Then F is sub-lower semicontinuous if and only if, for each $\varepsilon > 0$, there is a continuous function $f : X \to Y$ such that $d_Y(f(x), F(x)) < \varepsilon$ for each $x \in X$.*

Proof. Let $\varepsilon > 0$. Suppose that F is sub-lower semicontinuous.
For each $x \in X$, let $z_x \in F(x)$ and U_x be an open neighborhood of x in X such that $z_x \in B_Y(F(y), \delta)$, for each $y \in U_x$, where δ is as in Proposition 2.9 and corresponding to ε and $\dim X$.

Let $\{V_j\}_{j\in J}$ be an open refinement of the covering $\{U_x\}_{x\in X}$ such that every subfamily consisting of $p+2$ sets in $\{V_j\}_{j\in J}$ has an empty intersection. Hence, for each $j \in J$, there is $x_j \in X$ such that $V_j \subset U_{x_j}$. Let $\{\lambda_j\}_{j\in J}$ be a partition of unity subordinated to $\{V_j\}_{j\in J}$.

Define functions $\lambda : J \times X \to [0,1]$ and $y : J \to Y$ putting $\lambda(j,x) := \lambda_j(x)$ and $y(j) := z_{x_j}$ for $j \in J$ and $x \in X$. It is clear that the function λ satisfies assumptions of Proposition 2.8. Therefore the function $f : X \to Y$ defined by

$$f(x) := b(y, \lambda(\cdot, x))$$

is well-defined and continuous.

For each $j \in J(x) := \{j \in J \mid \lambda(j,x) > 0\}$, we have $x \in \operatorname{supp} \lambda_j \subset V_j \subset U_{x_j}$. As $z_{x_j} \in B(F(x), \delta)$, for all $j = 1, \ldots, n$, and $F(x)$ is α-convex, then

$$d_Y(f(x), F(x)) < \varepsilon.$$

Conversely, suppose that, for each $\varepsilon > 0$, there is a continuous function $f : X \to Y$ such that $d_Y(f(x), F(x)) < \frac{\varepsilon}{3}$ for each $x \in X$. Then, for each $x \in X$, there is $z_x \in F(x)$ such that

$$d_Y(f(x), z_x) < \frac{\varepsilon}{3}.$$

Let $x \in X$. By continuity of f, we have that there is a neighborhood U_x of x such that

$$d_Y(f(x), f(y)) < \frac{\varepsilon}{3}$$

for each $y \in U_x$. Therefore

$$d_Y(z_x, F(y)) \leq d_Y(z_x, f(x)) + d_Y(f(x), f(y)) + d_Y(f(y), F(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

that is $z_x \in B_Y(F(y), \varepsilon)$ for every $y \in U_x$.

Clearly, it means that F is sub-lower semicontinuous. ■

Theorem 3.17. *Let X be a compact metric space and let Y be an α-convex metric space. Assume that $F, T : X \multimap Y$ are two maps with the following properties:*

(i) *for each $x \in X$, there is $\varepsilon_x > 0$ and a neighborhood U_x of x such that*

$$F(U_x) \cap B(T(x), \varepsilon_x) = \emptyset,$$

(ii) *T is upper semicontinuous,*

(iii) *F is sub-lower semicontinuous and $F(x)$ is a non-empty α-convex set for each $x \in X$. Then, for each $\varepsilon > 0$, there is a continuous function $f : X \to Y$ such that*

$$d_Y(f(x), F(x)) < \varepsilon \quad \textit{and} \quad f(x) \notin T(x)$$

for each $x \in X$.

Proof. Let $\varepsilon > 0$. By (i), for every $x \in X$, there is a neighborhood U_x and $0 < \varepsilon_x \leq \varepsilon$ such that

$$F(y) \cap B(T(x), \varepsilon_x) = \emptyset$$

for each $y \in U_x$.

Again by (ii), there is an open neighborhood $O_x \subset U_x$ of x in X such that

$$T(y) \subset B(T(x), \varepsilon_x/2)$$

for each $y \in O_x$. Hence, for each $y \in O_x$, we have

$$B_Y(F(y), \varepsilon_x/2) \cap T(y) = \emptyset.$$

By the compactness of X, there are finitely many points, say $x_1, \ldots, x_n$, such that $X = \bigcup_{k=1}^{n} O_{x_k}$. Let $s := \min_{1 \leq k \leq n} \frac{\varepsilon_{x_k}}{2}$. Clearly $s < \varepsilon$.

For each $x \in X$, we have

$$B_Y(F(x), s) \cap T(x) = \emptyset.$$

By Theorem 3.16, there is a continuous function $f : X \to Y$ such that

$$d_Y(f(x), F(x)) < s$$

for each $x \in X$. Therefore

$$d_Y(f(x), F(x)) < \varepsilon \quad \text{and} \quad f(x) \notin T(x)$$

for all $x \in X$. ■

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