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GALERKIN PROPER ORTHOGONAL DECOMPOSITION METHODS FOR PARAMETER DEPENDENT ELLIPTIC SYSTEMS

Martin Kahlbacher and Stefan Volkwein

Institute for Mathematics and Scientific Computing
University of Graz
Heinrichstrasse 36, 8010 Graz, Austria

Abstract

Proper orthogonal decomposition (POD) is a powerful technique for model reduction of linear and non-linear systems. It is based on a Galerkin type discretization with basis elements created from the system itself. In this work, error estimates for Galerkin POD methods for linear elliptic, parameter-dependent systems are proved. The resulting error bounds depend on the number of POD basis functions and on the parameter grid that is used to generate the snapshots and to compute the POD basis. The error estimates also hold for semi-linear elliptic problems with monotone nonlinearity. Numerical examples are included.

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1. Introduction

The proper orthogonal decomposition is a method for deriving low order models for systems of differential equations. It is based on projecting the system onto subspaces consisting of basis elements that contain characteristics of the expected solution. This is in contrast to, e.g., finite element techniques, where the elements of the subspaces are uncorrelated to the physical properties of the system that they approximate.

In this work, POD is applied to linear and semi-linear elliptic equations with varying parameters. The POD basis is computed for the snapshots consisting of solutions to the elliptic equation for different parameter values. Error estimates for Galerkin POD type approximations are proven, where we proceed as in [13, 14], where POD error estimates for parabolic systems were derived. Asymptotic results are presented in the sense that the constants appearing in the estimates do not depend on the snapshot set. Moreover, two grids for the parameters are utilized, one for the set of snapshots and the other for the numerical integration. The resulting error bounds depend on the number of POD basis functions and on the snapshot grid.

Let us finally briefly comment on the literature containing applications of POD. It was successfully used in different fields including signal analysis and pattern recognition (see e.g., [6]), fluid dynamics and coherent structures (see e.g., [9, 21]) and more recently in control theory (see e.g., [2, 12, 16]) and inverse problems [3]. The relationship between POD and balancing was considered in [15, 20, 23]. Error analysis for nonlinear dynamical systems in finite dimensions was carried out in [10, 19]. Reduced-basis element methods for parameter dependent elliptic were investigated, for instance, in [4, 17, 18].

The paper is organized in the following manner: In Section 2, we introduce our abstract elliptic problem. The POD approximation is described in Section 3. Section 4 is devoted to present the POD error estimates. A semi-linear elliptic problem is considered in Section 5. In Section 6, numerical examples are shown.

2. Linear parameter dependent elliptic system

In this section, we introduce our abstract linear parameter dependent elliptic system. Let V and H be real separable Hilbert spaces and suppose that V is dense in H with compact embedding. By $\langle \cdot\,,\cdot\rangle_H$ and $\langle \cdot\,,\cdot\rangle_V$ we denote the inner products in H and V, respectively. Since V is continuously injected into H, there exists a constant $c_V > 0$ such that

$$\|\varphi\|_H \le c_V\,\|\varphi\|_V \quad \text{for all } \varphi \in V. \tag{1}$$

For $\mu_a, \mu_b \in \mathbb{R}$ with $\mu_a < \mu_b$ we introduce the interval $\mathcal{I} = [\mu_a, \mu_b]$ containing the admissible values for the parameters. Then we define the parametrized bilinear form $a : V \times V \times \mathcal{I} \to \mathbb{R}$ as

$$a(\varphi, \phi; \mu) = \langle \varphi, \phi \rangle_V + \mu \langle \varphi, \phi \rangle_H \quad \text{for } \varphi, \phi \in V \text{ and } \mu \in \mathcal{I}.$$

For any $\mu \in \mathcal{I}$ we obtain

$$|a(\varphi, \phi; \mu)| \leq \big(1 + c_V^2 \max\{|\mu_a|, |\mu_b|\}\big) \|\varphi\|_V \|\phi\|_V \quad \text{for all } \varphi, \phi \in V,$$

i.e., the bilinear form $a(\cdot\,,\cdot\,;\mu)$ is continuous on $V \times V$ for any $\mu \in \mathcal{I}$. Since

$$a(\varphi, \varphi; \mu) = \|\varphi\|_V^2 + \mu \|\varphi\|_H^2 \quad \text{for all } \varphi \in V \text{ and } \mu \in \mathcal{I},$$

it follows that $a(\cdot\,,\cdot\,;\mu)$ is coercive on $V \times V$ for every $\mu \in \mathcal{I}$ provided

$$\eta_a = 1 + 2c_V^2 \min\{0, \mu_a\} > 0. \tag{2}$$

Let $f \in V'$ be given. For a given parameter $\mu \in \mathcal{I}$ we consider the following variational problem: Find $u = u(\mu) \in V$ such that

$$a(u, \varphi; \mu) = \langle f, \varphi \rangle_{V',V} \quad \text{for all } \varphi \in V, \tag{3}$$

where $\langle \cdot\,,\cdot \rangle_{V',V}$ stands for the duality pairing of V and its dual space V'. For examples we refer the reader to (40) and also to [5].

The following theorem ensures that (3) admits a unique solution.

Theorem 2.1. *Suppose that* (2) *holds. For every $\mu \in \mathcal{I}$ there exists a unique solution $u = u(\mu) \in V$ to* (3) *satisfying*

$$\|u\|_V \leq \frac{1}{\sqrt{\eta_a}} \|f\|_{V'}. \tag{4}$$

Moreover, the mapping $u : \mathcal{I} \to V$, $\mu \mapsto u(\mu)$ is Lipschitz-continuous.

Proof. Since the bilinear form $a(\cdot\,,\cdot\,;\mu)$ is continuous and coercive on $V \times V$ for every parameter $\mu \in \mathcal{I}$, the existence of a unique solution to (3) follows directly from the Lax-Milgram lemma; see [5], for instance. ∎

Together with (3) we will consider a discretized variational problem, where we apply the POD for the discretization of V. In the next section, we will describe the POD Galerkin approximation of the variational problem (3).

3. The POD Galerkin discretization

In this section, we follow the arguments in [14] for time-dependent systems. Henceforth, we denote by $u = u(\mu) \in V$ the associated solution to (3) for a parameter $\mu \in \mathcal{I}$.

3.1. The POD method

We define the bounded linear operator $\mathcal{C} : L^2(\mathcal{I}) \to V$ (i.e., $\mathcal{C} \in L^2(\mathcal{I}; V)$) by

$$\mathcal{C}\varphi = \int_{\mathcal{I}} \varphi(\mu) u(\mu)\,\mathrm{d}\mu \quad \text{for } \varphi \in L^2(\mathcal{I}).$$

Its Hilbert space adjoint $\mathcal{C}^* : V \to L^2(\mathcal{I})$ satisfying

$$\langle \mathcal{C}\varphi, z\rangle_V = \langle \varphi, \mathcal{C}^* z\rangle_{L^2(\mathcal{I})} \quad \text{for all } (\varphi, z) \in L^2(\mathcal{I}) \times V$$

is given by

$$\big(\mathcal{C}^* z\big)(\mu) = \langle z, u(\mu)\rangle_V \quad \text{for } z \in V \text{ and } \mu \in \mathcal{I}.$$

Furthermore, we find that the bounded, linear, symmetric and non-negative operator $\mathcal{R} = \mathcal{C}\mathcal{C}^* : V \to V$ has the form

$$\mathcal{R}z = \int_{\mathcal{I}} \langle z, u(\mu)\rangle_V u(\mu)\,\mathrm{d}\mu \quad \text{for } z \in V. \tag{5}$$

The operator $\mathcal{K} = \mathcal{C}^*\mathcal{C} : L^2(\mathcal{I}) \to L^2(\mathcal{I})$ is given by

$$\big(\mathcal{K}\varphi\big)(\bar\mu) = \int_{\mathcal{I}} \langle u(\mu), u(\bar\mu)\rangle_V \varphi(\mu)\,\mathrm{d}\mu \quad \text{for } \varphi \in L^2(\mathcal{I}). \tag{6}$$

Due to Theorem 2.1 the mapping $u : \mathcal{I} \to V$, $\mu \mapsto u(\mu)$ is Lipschitz-continuous. Hence,

$$\int_{\mathcal{I}}\int_{\mathcal{I}} \big|\langle u(\mu), u(\bar\mu)\rangle_V\big|^2\,\mathrm{d}\bar\mu\mathrm{d}\mu < \infty.$$

This implies that $\mathcal{K} = \mathcal{C}^*\mathcal{C}$ is compact and, therefore, $\mathcal{R} = \mathcal{C}\mathcal{C}^*$ is compact as well. From the Hilbert-Schmidt theorem it follows that there exists a complete orthonormal basis $\{\psi_i\}_{i\in\mathbb{N}}$ for V and a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of non-negative real numbers so that

$$\mathcal{R}\psi_i = \lambda_i\psi_i, \quad \lambda_1 \ge \lambda_2 \ge \dots, \quad \text{and } \lambda_i \to 0 \text{ as } i \to \infty.$$

The spectra of $\mathcal{R}$ are pure point spectra except for possibly 0. Each non-zero eigenvalue of $\mathcal{R}$ has finite multiplicity and 0 is the only possible accumulation point of the spectrum of $\mathcal{R}$, see [11, p. 185]. Let us note that

$$\int_{\mathcal{I}} \|u(\mu)\|_V^2 \,\mathrm{d}\mu = \sum_{i=1}^{\infty} \lambda_i.$$

Remark 3.1. Analogous to the theory of singular value decomposition for matrices, we find that the bounded, linear, symmetric and non-negative operator $\mathcal{K}$ (see (6)) has the same eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ as the operator $\mathcal{R}$ and the eigenfunctions

$$v_i(\mu) = \frac{1}{\sqrt{\lambda_i}} \big(\mathcal{C}^*\psi_i\big)(\mu) = \frac{1}{\sqrt{\lambda_i}} \langle \psi_i, u(\mu)\rangle_V$$

for $i \in \{j \in \mathbb{N} : \lambda_j > 0\}$ and almost all $\mu \in \mathcal{I}$. $\Diamond$

For a given $\ell \in \mathbb{N}$ we introduce the mapping

$$\mathfrak{J} : \underbrace{V \times \ldots \times V}_{\ell-\text{times}} \to \mathbb{R}$$

by

$$\mathfrak{J}(\psi_1, \ldots, \psi_\ell) = \int_{\mathcal{I}} \Big\| u(\mu) - \sum_{i=1}^{\ell} \langle u(\mu), \psi_i\rangle_V \psi_i \Big\|_V^2 \,\mathrm{d}\mu. \tag{7}$$

In the following theorem, we formulate properties of the eigenvalues and eigenfunctions of $\mathcal{R}$.

Theorem 3.2. *Let $\{\lambda_i\}_{i\in\mathbb{N}}$ and $\{\psi_i\}_{i\in\mathbb{N}}$ denote the eigenvalues and eigenfunctions, respectively, of $\mathcal{R}$ introduced in* (5)*. Then, for every $\ell \in \mathbb{N}$ the first ℓ eigenfunctions $\psi_1, \ldots, \psi_\ell \in V$ solve the minimization problem*

$$\min \mathfrak{J}(\tilde\psi_1, \ldots, \tilde\psi_\ell) \quad \text{s.t.} \quad \langle \tilde\psi_j, \tilde\psi_i\rangle_V = \delta_{ij} \quad \text{for} \quad 1 \le i, j \le \ell, \tag{8}$$

where $\mathfrak{J}$ is defined in (7)*. Moreover,*

$$\mathfrak{J}(\psi_1, \ldots, \psi_\ell) = \sum_{i=\ell+1}^{\infty} \lambda_i \quad \text{for any} \quad \ell \in \mathbb{N}. \tag{9}$$

Proof. The proof of the claim relies on the fact that the eigenvalue problem

$$\mathcal{R}\psi_i = \lambda_i \psi_i \quad \text{for } i = 1, \ldots, \ell \tag{10}$$

is the first-order necessary optimality condition for (8). For more details we refer the reader to [9, Section 3]. ■

We call a solution to (8) a *POD basis of rank* ℓ. In particular, we have [9, Section 3.3]:

$$\sum_{i=1}^{\ell} \int_{\mathcal{I}} \left|\langle u(\mu), \psi_i \rangle_V\right|^2 \mathrm{d}\mu \geq \sum_{i=1}^{\ell} \int_{\mathcal{I}} \left|\langle u(\mu), \chi_i \rangle_V\right|^2 \mathrm{d}\mu$$

for every $\ell \in \mathbb{N}$, where $\{\chi_i\}_{i \in \mathbb{N}}$ is an arbitrary orthonormal basis in V.

3.2. Numerical realization of the POD method

In applications the weak solution to (3) is not known for all parameters $\mu \in \mathcal{I}$, but only for a given grid in $\mathcal{I}$. For that purpose let

$$\mu_a = \mu_1 < \mu_2 < \ldots < \mu_n = \mu_b \tag{11}$$

be a grid in $\mathcal{I}$ and let $u_i = u(\mu_i)$, $1 \leq i \leq n$, denote the corresponding solutions to (3) for the grid points μ_i. We define the snapshot set $\mathcal{V}^n = \operatorname{span}\{u_1, \ldots, u_n\} \subset V$ and determine a POD basis of rank $\ell \leq n$ for $\mathcal{V}^n$ by solving

$$\min \sum_{j=1}^{n} \alpha_j \left\| u_j - \sum_{i=1}^{\ell} \langle u_j, \psi_i \rangle_V \psi_i \right\|_V^2 \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle_V = \delta_{ij},\ 1 \leq i, j \leq \ell \tag{12}$$

where the α_j's are non-negative weights. The solution to (12) is given by the solution to the eigenvalue problem

$$\mathcal{R}^n \psi_i^n = \lambda_i^n \psi_i^n, \quad i = 1, \ldots, \ell,$$

with

$$\mathcal{R}^n \psi = \sum_{j=1}^{n} \alpha_j \langle u_j, \psi \rangle_V \, u_j \quad \text{for } \psi \in V.$$

In contrast to $\mathcal{R}$ introduced in (5) the operator $\mathcal{R}^n$ and therefore its eigenvalues and eigenfunctions depend on the grid $\{\mu_j\}_{j=1}^n$. Furthermore, the image space of $\mathcal{R}^n$ has finite dimension $d^n \le n$, whereas, in general, the image space of the operator $\mathcal{R}$ is infinite-dimensional. Since $\mathcal{R}^n$ is a linear, bounded, compact, non-negative, self-adjoint operator, there exist eigenvalues $\{\lambda_i^n\}_{i=1}^{d^n}$ and orthonormal eigenfunctions $\{\psi_i^n\}_{i=1}^{\ell}$ with $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{d^n} > 0$ and

$$\sum_{j=1}^{n} \alpha_j \left\| u_j - \sum_{i=1}^{\ell} \langle u_j, \psi_i \rangle_V \psi_i \right\|_V^2 = \sum_{i=\ell+1}^{d^n} \lambda_i^n.$$

Remark 3.3 (Snapshot POD [21])**.** Let us supply $\mathbb{R}^n$ with the weighted inner product

$$\langle v, w \rangle_{\mathbb{R}^n} = \sum_{i=1}^{n} \alpha_i v_i w_i \quad \text{for } v = (v_1, \ldots, v_n)^T, w = (w_1, \ldots, w_n)^T \in \mathbb{R}^n.$$

If the α_i's are trapezoidal weights corresponding to the parameter grid $\{\mu_i\}_{i=1}^n$ then the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is a discrete version of the inner product in $L^2(\mathcal{I})$. We define the symmetric non-negative matrix $\mathcal{K}^n \in \mathbb{R}^{n \times n}$ with the elements $\langle u_i, u_j \rangle_V$, $1 \le i, j \le n$, and consider the eigenvalue problem

$$\mathcal{K}^n v_i^n = \lambda_i^n v_i^n, \ 1 \le i \le \ell \quad \text{and} \quad \langle v_i^n, v_j^n \rangle_{\mathbb{R}^n} = \delta_{ij}, \ 1 \le i, j \le \ell \le d^n \tag{13}$$

From the singular value decomposition it follows that $\mathcal{K}^n$ has the same eigenvalues $\{\lambda_i^n\}_{i=1}^{d^n}$ as the operator $\mathcal{R}^n$; compare Remark 3.1 and [14]. Furthermore, the POD basis functions are given by the formula

$$\psi_i = \frac{1}{\sqrt{\lambda_i^n}} \sum_{j=1}^{n} \alpha_j (v_i^n)_j u_j \quad \text{for } i = 1, \ldots, \ell, \tag{14}$$

where $(v_i^n)_j$ denotes the jth-component of the eigenvector $v_i^n \in \mathbb{R}^n$. In our numerical test examples, we compute the POD basis by solving (13) and using (14). $\Diamond$

3.3. POD Galerkin scheme

Let us fix $\ell \in \mathbb{N}$ and compute the first ℓ POD basis functions $\psi_1, \ldots, \psi_\ell \in V$ by solving either (10) or $\mathcal{K} v_i = \lambda v_i$ for $i = 1, \ldots, \ell$ (see Remark 3.1). Then we define the finite dimensional linear space

$$V^\ell = \text{span}\left\{\psi_1, \ldots, \psi_\ell\right\} \subset V.$$

Endowed with the topology in V it follows that V^ℓ is a Hilbert space. Next we introduce the orthogonal projection $\mathcal{P}^\ell$ of V onto V^ℓ:

$$\mathcal{P}^\ell \varphi = \sum_{i=1}^{\ell} \langle \varphi, \psi_i \rangle_V \psi_i \quad \text{for } \varphi \in V. \tag{15}$$

By (7) and (9),

$$\mathfrak{J}(\psi, \dots, \psi_\ell) = \int_{\mathcal{I}} \left\| u(\mu) - \mathcal{P}^\ell u(\mu) \right\|_V^2 \, \mathrm{d}\mu = \sum_{i=\ell+1}^{\infty} \lambda_i. \tag{16}$$

From (15) and $\mathcal{P}^\ell \psi = \psi$ for all $\psi \in V^\ell$ it follows that

$$\begin{aligned} \langle \mathcal{P}^\ell \varphi, \psi \rangle_V &= \left\langle \sum_{i=1}^{\ell} \langle \varphi, \psi_i \rangle_V \psi_i, \psi \right\rangle_V = \sum_{i=1}^{\ell} \langle \varphi, \psi_i \rangle_V \langle \psi, \psi_i \rangle_V \\ &= \left\langle \varphi, \sum_{i=1}^{\ell} \langle \psi, \psi_i \rangle_V \psi_i \right\rangle_V = \langle \varphi, \mathcal{P}^\ell \psi \rangle_V = \langle \varphi, \psi \rangle_V \end{aligned} \tag{17}$$

for all $\varphi \in V$ and all $\psi \in V^\ell$. Since the ψ_i's are orthonormal in V, we have $\|\mathcal{P}^\ell\|_{L(V)} = 1$, where $L(V)$ denotes the Banach space of all bounded linear operators from V into itself endowed with the common norm.

The POD Galerkin scheme for (3) leads to the following linear problem: for a given $\mu \in \mathcal{I}$ determine a function $u^\ell \in V^\ell$ such that

$$a(u^\ell, \psi; \mu) = \langle f, \psi \rangle_{V',V} \quad \text{for all } \psi \in V^\ell. \tag{18}$$

The proof of the existence of a unique solution u^ℓ to (18) follows by the same arguments as Theorem 2.1. Moreover, (4) holds also for u^ℓ.

4. POD error estimates

The goal of this section is to derive error estimates for the difference between the solution $u = u(\mu)$ to (3) and the POD solution $u^\ell(\mu)$ to (18) for $\mu \in \mathcal{I}$ in terms of the sum $\sum_{i=\ell+1}^{\infty} \lambda_i$, i.e., in terms of the sum over the eigenvalues corresponding to the not-modelled eigenmodes.

4.1. Continuous POD

First, we study the case where the solution to (3) is known for all $\mu \in \mathcal{I}$. We make the decomposition

$$u^\ell(\mu) - u(\mu) = u^\ell(\mu) - \mathcal{P}^\ell u(\mu) + \mathcal{P}^\ell u(\mu) - u(\mu) = \vartheta^\ell(\mu) + \varrho^\ell(\mu),$$

where $\vartheta^\ell(\mu) = u^\ell(\mu) - \mathcal{P}^\ell u(\mu)$ and $\varrho^\ell(\mu) = \mathcal{P}^\ell u(\mu) - u(\mu)$. From (16) we know

$$\int_{\mathcal{I}} \|\varrho^\ell(\mu)\|_V^2 \,\mathrm{d}\mu = \sum_{i=\ell+1}^{\infty} \lambda_i. \tag{19}$$

Next we estimate $\vartheta^\ell(\mu) \in V^\ell$. Using (3), (17) and (18) we obtain

$$\begin{aligned} a(\vartheta^\ell(\mu), \psi; \mu) &= \langle u^\ell(\mu), \psi\rangle_V + \mu \,\langle u^\ell(\mu), \psi\rangle_H - \langle u(\mu), \psi\rangle_V - \mu \,\langle \mathcal{P}^\ell u(\mu), \psi\rangle_H \\ &= \langle f, \psi\rangle_{V',V} - \langle u(\mu), \psi\rangle_V - \mu \,\langle \mathcal{P}^\ell u(\mu), \psi\rangle_H \\ &= \mu \,\langle u(\mu) - \mathcal{P}^\ell u(\mu), \psi\rangle_H = -\mu \,\langle \varrho^\ell(\mu), \psi\rangle_H. \end{aligned}$$

Choosing $\psi = \vartheta^\ell(\mu)$ and using (1) we find

$$\begin{aligned} \big(1 + c_V^2 \min\{0, \mu_a\}\big) \,\|\vartheta^\ell(\mu)\|_V^2 &\leq |\mu| \,\|\varrho^\ell(\mu)\|_H \|\vartheta^\ell(\mu)\|_H \\ &\leq c_V |\mu| \,\|\varrho^\ell(\mu)\|_H \|\vartheta^\ell(\mu)\|_V. \end{aligned} \tag{20}$$

From Young's inequality (see [1, p. 28])

$$c_V |\mu| \,\|\varrho^\ell(\mu)\|_H \|\vartheta^\ell(\mu)\|_V \leq \frac{c_V^2 \mu^2}{2\eta_a} \,\|\varrho^\ell(\mu)\|_H^2 + \frac{\eta_a}{2} \,\|\vartheta^\ell(\mu)\|_V^2,$$

where $\eta_a > 0$ has been introduced in (2). Hence, by (20)

$$\|\vartheta^\ell(\mu)\|_V^2 \leq \frac{c_V^4 \mu^2}{\eta_a} \,\|\varrho^\ell(\mu)\|_V^2\,. \tag{21}$$

Summarizing,

$$\begin{aligned}\int_{\mathcal{I}} \|u^\ell(\mu) - u(\mu)\|_V^2 \,\mathrm{d}\mu &\le 2\int_{\mathcal{I}} \|\vartheta^\ell(\mu)\|_V^2 + \|\varrho^\ell(\mu)\|_V^2 \,\mathrm{d}\mu \\ &\le 2\int_{\mathcal{I}} \left(\frac{c_V^4\mu^2}{\eta_a^2} + 1\right)\|\varrho^\ell(\mu)\|_V^2 \,\mathrm{d}\mu \\ &\le 2\left(\frac{c_V^4 \max\{\mu_a^2,\mu_b^2\}}{\eta_a^2} + 1\right)\int_{\mathcal{I}} \|\varrho^\ell(\mu)\|_V^2 \,\mathrm{d}\mu.\end{aligned}$$

Applying (19) we have proved the next theorem.

Theorem 4.1. *Suppose that* (2) *holds. For* $\mu \in \mathcal{I} = [\mu_a, \mu_b]$ *we denote by* $u(\mu)$ *and* $u^\ell(\mu)$ *the solutions to* (3) *and* (18), *respectively. Then there exists a constant* $C > 0$ *depending on* μ_a, μ_b, c_V *such that*

$$\int_{\mathcal{I}} \|u^\ell(\mu) - u(\mu)\|_V^2 \,\mathrm{d}\mu \le C \sum_{i=\ell+1}^{\infty} \lambda_i. \tag{22}$$

Remark 4.2. Let us introduce for a given $\ell \in \mathbb{N}$ the mapping

$$\tilde{\mathfrak{J}} : \underbrace{H \times \ldots \times H}_{\ell-\text{times}} \to \mathbb{R}$$

by

$$\tilde{\mathfrak{J}}(\psi_1, \ldots, \psi_\ell) = \int_{\mathcal{I}} \Big\| u(\mu) - \sum_{i=1}^{\ell} \langle u(\mu), \psi_i \rangle_H \psi_i \Big\|_H^2 \,\mathrm{d}\mu.$$

Analogous to Section 3.1, we can compute the POD basis by solving the minimization problem

$$\min \tilde{\mathfrak{J}}(\tilde\psi_1, \ldots, \tilde\psi_\ell) \quad \text{s.t.} \quad \langle \tilde\psi_j, \tilde\psi_i \rangle_V = \delta_{ij} \quad \text{for } 1 \le i, j \le \ell. \tag{23}$$

Since (17) does not hold, we obtain

$$a(\vartheta^\ell(\mu), \psi; \mu) = -a(\varrho^\ell(\mu), \psi; \mu) \quad \text{for all } \psi \in V^\ell.$$

Choosing $\psi = \vartheta^\ell(\mu)$ we find

$$\eta_a \|\vartheta^\ell(\mu)\|_V^2 \le C_1 \|\varrho^\ell(\mu)\|_V \|\vartheta^\ell(\mu)\|_V$$

with $C_1 = 1 + c_V^2 \max\{|\mu_a|, |\mu_b|\}$. Using Young's inequality there exists a constant $C_2 > 0$ such that

$$\int_{\mathcal{I}} \|\vartheta^\ell(\mu)\|_V^2 \,\mathrm{d}\mu \le C_2 \int_{\mathcal{I}} \|\varrho^\ell(\mu)\|_V^2 \,\mathrm{d}\mu.$$

In contrast to (19) we have

$$\int_{\mathcal{I}} \|\varrho^\ell(\mu)\|_H^2 \,\mathrm{d}\mu = \sum_{i=\ell+1}^{\infty} \lambda_i.$$

From Lemma 3 in [13] it follows that

$$\int_{\mathcal{I}} \|\varrho^\ell(\mu)\|_V^2 \,\mathrm{d}\mu \le \|S^\ell\|_2 \int_{\mathcal{I}} \|\varrho^\ell(\mu)\|_H^2 \,\mathrm{d}\mu = \|S^\ell\|_2 \sum_{i=\ell+1}^{\infty} \lambda_i.$$

where $S^\ell \in \mathbb{R}^{\ell\times\ell}$ denotes the stiffness matrix with the elements $\langle\psi_i, \psi_j\rangle_V$, $1 \le i, j \le \ell$, and $\|\cdot\|_2$ stands for the spectral norm of real, symmetric matrices. Therefore, if we compute the POD basis using the H-topology, the constant C in (22) depends on the spectral norm of S^ℓ. $\Diamond$

4.2. Discrete POD

Suppose that the weak solution to (3) is not known for all parameters $\mu \in \mathcal{I}$, but for the parameter grid $\{\mu_i\}_{i=1}^n$ introduced in (11). Let $u_i = u(\mu_i)$, $1 \le i \le n$, denote the corresponding solutions to (3) for the grid points μ_i. We define the snapshot set $\mathcal{V}^n = \operatorname{span}\{u_1, \dots, u_n\} \subset V$ and determine a POD basis of rank $\ell \le n$ for $\mathcal{V}^n$ by solving (12).

Proposition 4.3. *Suppose that* (2) *holds and that* $\{\mu_j\}_{j=1}^n$ *is a grid in the interval* $\mathcal{I}$ *satisfying* (11)*. For* μ_j, $1 \le j \le n$, *we denote by* $u(\mu_j)$ *and* $u^\ell(\mu_j)$ *the solutions to* (3) *and* (18)*, respectively. Then there exists a constant* $C > 0$ *depending on* μ_a, μ_b, c_V, *but independent on the grid* $\{\mu_j\}_{j=1}^n$ *such that*

$$\sum_{j=1}^{n} \alpha_j \|u^\ell(\mu_j) - u(\mu_j)\|_V^2 \le C \sum_{i=\ell+1}^{d^n} \lambda_i^n.$$

Proof. We argue as in the proof of Theorem 4.1. Instead of (19) and (21) we obtain

$$\sum_{j=1}^{n} \alpha_j \|\varrho^\ell(\mu_j)\|_V^2 = \sum_{i=\ell+1}^{d^n} \lambda_i^n$$

and

$$\|\vartheta^\ell(\mu_j)\|_V^2 \le \frac{c_V^2\mu_j^2}{\eta_a}\,\|\varrho^\ell(\mu_j)\|_V^2,$$

respectively. Now the claim follows by analogous arguments as in the proof of Theorem 4.1. ■

Next we suppose that we are given two different grids $\{\mu_j\}_{j=1}^n$ and $\{\bar\mu_k\}_{k=1}^m$ in $\mathcal{I}$ satisfying

$$\mu_a = \mu_1 < \mu_2 < \ldots < \mu_n = \mu_b, \quad \mu_a = \bar\mu_1 < \bar\mu_2 < \ldots < \bar\mu_m = \mu_b. \tag{24}$$

We set

$$\delta\mu_j = \mu_j - \mu_{j-1},\ j = 2,\ldots,n, \quad \delta\mu = \min_{2\le j\le n}\delta\mu_j, \quad \Delta\mu = \max_{2\le j\le n}\delta\mu_j,$$

$$\delta\bar\mu_k = \bar\mu_k - \bar\mu_{k-1},\ k = 2,\ldots,m, \quad \delta\bar\mu = \min_{2\le k\le m}\delta\bar\mu_k, \quad \Delta\bar\mu = \max_{2\le k\le m}\delta\bar\mu_k.$$

Moreover, let

$$\alpha_1 = \frac{\delta\mu_2}{2}, \quad \alpha_j = \frac{\delta\mu_j + \delta\mu_{j+1}}{2} \quad \text{for } 2\le j\le n-1, \quad \alpha_n = \frac{\delta\mu_n}{2},$$

$$\beta_1 = \frac{\delta\bar\mu_2}{2}, \quad \beta_k = \frac{\delta\bar\mu_k + \delta\bar\mu_{k+1}}{2} \quad \text{for } 2\le k\le m-1, \quad \beta_m = \frac{\delta\bar\mu_m}{2}.$$

The goal is to estimate

$$\sum_{k=1}^m \beta_k\|u(\bar\mu_k) - u^\ell(\bar\mu_k)\|_V^2,$$

whereas the POD basis of rank ℓ is computed by using the snapshot ensemble $\{u(\mu_j)\}_{j=1}^n$ depending on the grid $\{\mu_j\}_{j=1}^n$. Let $\bar\mu_k\in\mathcal{I}$, $k\in\{1,\ldots,m\}$, be given. Then there exists an index $j_k\in\{1,\ldots,n-1\}$ such that

$$\mu_{j_k} \le \bar\mu_k \le \mu_{j_k+1}.$$

Let us define $\sigma_m\in\{1,\ldots,m\}$ as the maximum of the occurrence of the same value j_k as k ranges over $1\le k\le m$. Notice that

$$\max\big\{|\bar\mu_k - \mu_{j_k+1}|, |\bar\mu_k - \mu_{j_k}|\big\} \le \delta\mu_{j_k+1} \le \Delta\mu.$$

For every $k \in \{1, \dots, m\}$ we decompose the error $u(\bar{\mu}_k) - u^\ell(\bar{\mu}_k)$ as follows:

$$u(\bar{\mu}_k) - u^\ell(\bar{\mu}_k) = u(\bar{\mu}_k) - u(\mu_{j_k}) + u(\mu_{j_k}) - u^\ell(\mu_{j_k}) + u^\ell(\mu_{j_k}) - u^\ell(\bar{\mu}_k). \tag{25}$$

Since μ_{j_k} belongs to the grid, where the snapshots are taken, we have already estimated the difference $u(\mu_{j_k}) - u^\ell(\mu_{j_k})$. From

$$a(u(\bar{\mu}_k), \varphi; \bar{\mu}_k) = \langle f, \varphi \rangle_{V',V} \qquad \text{for all } \varphi \in V,$$

$$a(u(\mu_{j_k}), \varphi; \mu_{j_k}) = \langle f, \varphi \rangle_{V',V} \qquad \text{for all } \varphi \in V$$

we obtain

$$\langle u(\bar{\mu}_k) - u(\mu_{j_k}), \varphi \rangle_V + \bar{\mu}_k \langle u(\bar{\mu}_k), \varphi \rangle_H - \mu_{j_k} \langle u(\mu_{j_k}), \varphi \rangle_H = 0$$

for all $\varphi \in V$. Consequently,

$$a(u(\bar{\mu}_k) - u(\mu_{j_k}), \varphi; \bar{\mu}_k) = \big(\mu_{j_k} - \bar{\mu}_k\big) \langle u(\mu_{j_k}), \varphi \rangle_H.$$

Choosing $\varphi = u(\bar{\mu}_k) - u(\mu_{j_k})$ and using (1) we deduce

$$\begin{aligned}
&\|u(\bar{\mu}_k) - u(\mu_{j_k})\|_V^2 + \bar{\mu}_k \|u(\bar{\mu}_k) - u(\mu_{j_k})\|_H^2 \\
&\le |\mu_{j_k} - \bar{\mu}_k| \, \|u(\mu_{j_k})\|_H \|u(\bar{\mu}_k) - u(\mu_{j_k})\|_H \\
&\le \delta\mu_{j_k+1} \, \|u(\mu_{j_k})\|_H \|u(\bar{\mu}_k) - u(\mu_{j_k})\|_H \\
&\le \frac{c_V^2 \delta\mu_{j_k+1}^2}{2} \|u(\mu_{j_k})\|_H^2 + \frac{1}{2} \|u(\bar{\mu}_k) - u(\mu_{j_k})\|_V^2.
\end{aligned}$$

Thus, (1), (2) and (4) yield

$$\|u(\bar{\mu}_k) - u(\mu_{j_k})\|_V^2 \le \frac{c_V^4}{\eta_a^2} \, \delta\mu_{j_k+1}^2 \, \|f\|_{V'}^2. \tag{26}$$

Analogously, we obtain

$$\|u^\ell(\bar{\mu}_k) - u^\ell(\mu_{j_k})\|_V^2 \le \frac{c_V^4}{\eta_a^2} \, \delta\mu_{j_k+1}^2 \, \|f\|_{V'}^2. \tag{27}$$

Notice that $\beta_k \le \Delta\bar{\mu}$, $1 \le k \le m$, and $\alpha_j \ge \delta\mu/2$, $1 \le j \le n$. Thus,

$$\beta_k \le \frac{2\alpha_j \Delta\bar{\mu}}{\delta\mu} \quad \text{for } 1 \le k \le m \text{ and } 1 \le j \le n.$$

We set $C_1 = 4c_V^4 \, \|f\|_{V'}^2/\eta_a^2 > 0$. Using $\sum_{k=1}^m \beta_k = \mu_b - \mu_a$, (25), (26) and (27) we obtain

$$
\begin{aligned}
&\sum_{k=1}^m \beta_k \left\| u(\bar\mu_k) - u^\ell(\bar\mu_k) \right\|_V^2 \\
&\qquad \le 2 \sum_{k=1}^m \beta_k \left\| u(\mu_{j_k}) - u^\ell(\mu_{j_k}) \right\|_V^2 + C_1 \sum_{k=1}^m \beta_k \, \delta\mu_{j_k+1}^2 \\
&\qquad \le \frac{4\sigma_m \Delta\bar\mu}{\delta\mu} \sum_{j=1}^n \alpha_j \left\| u(\mu_j) - u^\ell(\mu_j) \right\|_V^2 + C_1(\mu_b - \mu_a)\Delta\mu^2.
\end{aligned}
$$

Hence, by Corollary 4.3 we have

$$
\sum_{k=1}^m \beta_k \left\| u(\bar\mu_k) - u^\ell(\bar\mu_k) \right\|_V^2 \le \frac{4\sigma_m \Delta\bar\mu}{\delta\mu} \sum_{i=\ell+1}^{d^n} \lambda_i^n + C_2 \Delta\mu^2
$$

with $C_2 = (\mu_b - \mu_a)C_1 > 0$. Thus, we have proved the following theorem.

Theorem 4.4. *Suppose that* (2) *holds, that* $\{\mu_j\}_{j=1}^n$ *and* $\{\bar\mu_k\}_{k=1}^m$ *are two grids in the interval* $\mathcal{I}$ *satisfying* (24)*. For* $\bar\mu_k$*,* $1 \le k \le m$*, we denote by* $u(\bar\mu_k)$ *and* $u^\ell(\bar\mu_k)$ *the solutions to* (3) *and* (18)*, respectively. Then there exists a constant* $C > 0$ *depending on* μ_a*,* μ_b*,* c_V*, but independent of the grids such that*

$$
\sum_{k=1}^m \beta_k \left\| u^\ell(\bar\mu_k) - u(\bar\mu_k) \right\|_V^2 \le C \left(\frac{\sigma_m \Delta\bar\mu}{\delta\mu} \sum_{i=\ell+1}^{d^n} \lambda_i^n + \Delta\mu^2 \right).
$$

In Theorem 4.4 the eigenvalues $\{\lambda_i\}_{i=1}^{d^n}$, the eigenfunctions $\{\psi_i\}_{i=1}^{d^n}$ and σ_m depend on the discretization of $\mathcal{I}$ for the snapshots as well as for the numerical integration. We address this dependence next. If we suppose that both grids satisfy

$$
\Delta\mu = O(\delta\bar\mu) \quad \text{and} \quad \Delta\bar\mu = O(\delta\mu), \tag{28}
$$

then there exists a constant $C_3 > 0$ independent of $\{\mu_j\}_{j=1}^n$ and $\{\bar\mu_k\}_{k=1}^m$ such that

$$
\max\left(\sigma_m, \frac{\sigma_m \Delta\bar\mu}{\delta\mu} \right) \le C_3. \tag{29}
$$

Due to Theorem 2.1 the mapping $u : \mathcal{I} \to V$, $\mu \mapsto u(\mu)$ is continuous. This implies that

$$\lim_{n\to\infty} \|\mathcal{R} - \mathcal{R}^n\|_{L(V)} = 0.$$

Let us choose and fix ℓ such that

$$\lambda_\ell \neq \lambda_{\ell+1}. \tag{30}$$

We can now proceed precisely as in [14] to assert that there exists $\overline{\Delta\mu} > 0$ such that

$$\sum_{i=\ell+1}^{d^n} \lambda_i^n \leq 2 \sum_{i=\ell+1}^{\infty} \lambda_i \quad \text{for all } \Delta\mu \leq \overline{\Delta\mu} \tag{31}$$

provided, of course, that the term on the right-hand side of (31) is different from zero.

Theorem 4.5. *Suppose that* (2) *holds, that* $\{\mu_j\}_{j=1}^n$ *and* $\{\bar{\mu}_k\}_{k=1}^m$ *are two grids in the interval* $\mathcal{I}$ *satisfying* (24). *For* $\bar{\mu}_k$, $1 \leq k \leq m$, *we denote by* $u(\bar{\mu}_k)$ *and* $u^\ell(\bar{\mu}_k)$ *the solutions to* (3) *and* (18), *respectively. If* (30) *holds and* ℓ *satisfies* (31), *then there exists a constant* $C > 0$ *depending on* μ_a, μ_b, c_V, *but independent of the grids such that*

$$\sum_{k=1}^m \beta_k \left\|u^\ell(\bar{\mu}_k) - u(\bar{\mu}_k)\right\|_V^2 \leq C\left(\sum_{i=\ell+1}^{\infty} \lambda_i + \Delta\mu^2\right).$$

5. Continuous POD for semi-linear problem

Let us turn to a certain non-linear problem. Suppose that $F : V \to V'$ is a non-linear, locally Lipschitz-continuous mapping satisfying

$$\langle F(\phi) - F(\varphi), \phi - \varphi\rangle_{V',V} \geq 0 \quad \text{for all } \varphi, \psi \in V, \tag{32}$$

i.e., F is monotone. Instead of (3), we consider

$$a(u, \varphi; \mu) + \langle F(u), \varphi\rangle_{V',V} = \langle f, \varphi\rangle_{V',V} \quad \text{for all } \varphi \in V. \tag{33}$$

Example 5.1. Let us give an example for a semi-linear problem satisfying (32). We consider

$$-\Delta u + u^3 + \mu u = g \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} + u = g_R \text{ on } \Gamma. \tag{34}$$

A *weak solution to* (34) satisfies $u \in V$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi + \big(u^3 + \mu u\big)\varphi \,\mathrm{d}x + \int_\Gamma u\varphi \,\mathrm{d}s = \int_\Omega g\varphi \,\mathrm{d}x + \int_\Gamma g_R\varphi \,\mathrm{d}s \tag{35}$$

for all $\varphi \in V$. We utilize the parametrized bilinear form $a(\cdot\,,\cdot\,;\mu) : V \times V \to \mathbb{R}$ given by

$$a(\varphi, \phi; \mu) = \int_\Omega \nabla\varphi \cdot \nabla\phi \,\mathrm{d}x + \int_\Gamma \varphi\phi \,\mathrm{d}s + \mu \int_\Omega \varphi\phi \,\mathrm{d}x = \langle \varphi, \phi\rangle_V + \mu \,\langle \varphi, \phi\rangle_H$$

for all $\varphi, \phi \in V$, $\mu \in \mathcal{I}$ and the linear and continuous functional $f : V \to \mathbb{R}$ defined as

$$\langle f, \varphi\rangle_{V',V} = \int_\Omega g\varphi \,\mathrm{d}x + \int_\Gamma g_R\varphi \,\mathrm{d}s$$

for all $\varphi \in V$. Moreover, we define the non-linear operator $F : V \to V'$ by

$$\langle F(\phi), \varphi\rangle_{V',V} = \int_\Omega \phi^3\varphi \,\mathrm{d}x \quad \text{for } \phi, \varphi \in V.$$

Then, a weak solution to (34) satisfies the variational formulation (33). Recall that $\varphi \in V$ implies $\varphi \in L^6(\Omega)$. Consequently, $F(\varphi) \in H \subset V'$. Let $\phi, \varphi \in V$ and $\chi = \phi - \varphi \in V$. From

$$\begin{aligned}
\langle F(\phi) - F(\varphi), \chi\rangle_{V',V} &= \int_\Omega (\phi^3 - \varphi^3)\chi \,\mathrm{d}x \\
&= \int_\Omega \bigg(\int_0^1 3(\varphi + s\chi)^2\chi \,\mathrm{d}s\bigg)\chi \,\mathrm{d}x \\
&= 3\int_0^1 \int_\Omega (\phi + s\chi)^2\chi^2 \,\mathrm{d}x\mathrm{d}s \geq 0
\end{aligned}$$

it follows that (32) holds the existence of a solution to (35) is proved in [7].

Suppose that $u, v \in V$ are two solutions to (35). Then we have

$$a(u-v,\varphi;\mu) + \langle F(u)-F(v),\varphi\rangle_{V',V} = 0 \quad \text{for all } \varphi \in V \text{ and } \mu \in \mathcal{I}.$$

Choosing $\varphi = u - v$, using (2) and (32) we derive that $u = v$ in V. Thus, (35) has a unique solution. From

$$\langle F(\varphi),\varphi\rangle_{V',V} = \int_\Omega \varphi^3\varphi \,\mathrm{d}x \geq 0 \quad \text{for all } \varphi \in V$$

it follows that the weak solution u satisfies the same estimate (4) as in the linear case. ♢

Suppose that we have computed a POD basis $\{\psi_i\}_{i=1}^{\ell}$ of rank ℓ by utilizing the solution $u(\mu)$ to (33) for all $\mu \in \mathcal{I}$. The POD Galerkin scheme for (34) is as follows: Find $u^\ell = u^\ell(\mu)$, $\mu \in \mathcal{I}$, such that

$$a(u^\ell,\psi;\mu) + \langle F(u^\ell),\psi\rangle_{V',V} = \langle f,\psi\rangle_{V',V} \quad \text{for all } \psi \in V^\ell. \tag{36}$$

In the following theorem, an error estimate is presented. The proof is analogous to the proof of Theorem 4.1.

Theorem 5.2. *Let $F : V \to V'$ be a locally Lipschitz-continuous mapping satisfying* (32). *Suppose that for every $\mu \in \mathcal{I} = [\mu_a, \mu_b]$ there exist unique solutions to* (33) *and* (36) *denoted by $u(\mu)$ and $u^\ell(\mu)$, respectively. Then there exists a constant $C > 0$ depending on μ_a, μ_b, c_V and a Lipschitz constant for F such that*

$$\int_{\mathcal{I}} \|u^\ell(\mu) - u(\mu)\|_V^2 \,\mathrm{d}\mu \leq C \sum_{i=\ell+1}^{\infty} \lambda_i.$$

Remark 5.3. If the POD basis is computed by the strategy in Section 3.2, POD error estimates can also be derived combining the techniques in Section 4.2 and the arguments in the proof of Theorem 5.2. ♢

6. Numerical examples

This section is devoted to present two numerical test examples. All coding is done in MATLAB using routines from the Femlab package concerning finite element implementation. The programs are executed on a standard 1.7 Ghz desktop PC.

6.1. Estimating the decay of the eigenvalues

Here we apply the strategy that was proposed in [8]. From Theorem 4.1 we conclude that the error $u^\ell - u$ can be estimated in terms of the unmodeled eigenvalues, i.e.,

$$\int_{\mathcal{I}} \|u^\ell(\mu) - u(\mu)\|_V^2 \,\mathrm{d}\mu \sim \sum_{i=\ell+1}^{\infty} \lambda_i.$$

Suppose that the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ decay exponentially. Therefore, we make the ansatz

$$\lambda_i = \lambda_1 e^{-\alpha(i-1)} \quad \text{for } i \geq 1, \tag{37}$$

where we want to determine the factor $\alpha > 0$ numerically. Let X denote either the space H or the space V. Notice that

$$\frac{\int_{\mathcal{I}} \|u^\ell(\mu) - u(\mu)\|_X^2 \,\mathrm{d}\mu}{\int_{\mathcal{I}} \|u^{\ell+1}(\mu) - u(\mu)\|_X^2 \,\mathrm{d}\mu} \sim \frac{\sum\limits_{i=\ell+1}^{\infty} \lambda_i}{\sum\limits_{i=\ell+2}^{\infty} \lambda_i} = \frac{\sum\limits_{i=\ell+1}^{\infty} e^{-\alpha(i-1)}}{\sum\limits_{i=\ell+2}^{\infty} e^{-\alpha(i-1)}} = e^{\alpha}$$

Thus, we have

$$Q(\ell) = \ln \frac{\int_{\mathcal{I}} \|u^\ell(\mu) - u(\mu)\|_X^2 \,\mathrm{d}\mu}{\int_{\mathcal{I}} \|u^{\ell+1}(\mu) - u(\mu)\|_X^2 \,\mathrm{d}\mu} \sim \alpha, \tag{38}$$

and we may introduce the *experimental order of decay (EOD)* as

$$EOD := \frac{1}{\ell_{\mathsf{max}}} \sum_{k=1}^{\ell_{\mathsf{max}}} Q(k) \tag{39}$$

so that $EOD \approx \alpha$.

6.2. Test examples

We carry out two test examples which illustrate the theoretical results of Section 4. For that purpose we consider the elliptic problem

$$-2\Delta u + \beta \cdot \nabla u + qu = 1 \qquad \text{in } \Omega = (0,1) \times (0,1) \subset \mathbb{R}^2, \tag{40a}$$

$$2\,\frac{\partial u}{\partial n} + \frac{3}{2}\,u = -1 \qquad \text{on } \Gamma = \partial\Omega, \tag{40b}$$

where q is a positive scalar, $\beta = (\beta_1, \beta_2)^T$ belongs to $C(\overline{\Omega}; \mathbb{R}^2)$, n denotes the outward normal vector. We set $H = L^2(\Omega)$ and $V = H^1(\Omega)$. To write (40) in the form (3) we introduce the parametrized bilinear mapping $a(\cdot\,,\cdot;q)$ by

$$a(\varphi, \phi; q) = \int_\Omega 2\nabla\varphi \cdot \nabla\phi + \big(\beta \cdot \nabla\varphi\big)\phi + q\varphi\phi\,\mathrm{d}x + \frac{3}{2}\int_\Gamma \varphi\phi\,\mathrm{d}s \quad \text{for } \varphi, \phi \in V$$

and the bounded linear functional $f \in V'$ as

$$\langle f, \varphi\rangle_{V',V} = \int_\Omega \varphi\,\mathrm{d}x - \int_\Gamma \varphi\,\mathrm{d}s \quad \text{for } \varphi \in V.$$

Now, for a given $q \geq q_a > 0$ a weak solution $u = u(q) \in V$ to (40) satisfies

$$a(u, \varphi; q) = \langle f, \varphi\rangle_{V',V} \quad \text{for all } \varphi \in V.$$

The finite element discretization is carried out by a uniform rectangular mesh with mesh size $h = 1/40$ and piecewise linear finite elements. This yields 1681 degrees of freedom.

Run 1. In our first test example we vary the parameter q in (40). We fix $\beta = (x,y)^T$, choose the parameter interval $\mathcal{I} = [-50.5, 50.5]$ and compute solutions to (40) for parameters $q_i = -50.5 + (i-1)$, $i = 1, \ldots, 102$. Thus, we obtain 102 snapshots and compute our POD basis as described in Remark 3.3. Then we derive the POD Galerkin scheme for the elliptic problem. For $q = 40$ the finite element and the corresponding POD solutions taking $\ell = 5$ POD ansatz functions are presented in Figure 1.

It turns out that both solutions nearly coincide. In Figure 2 (left plot) we show the real and estimated decay of the first 5 eigenvalues, where the estimated decay is computed as desribed in Section 6.1 for $X = L^2(\Omega)$.

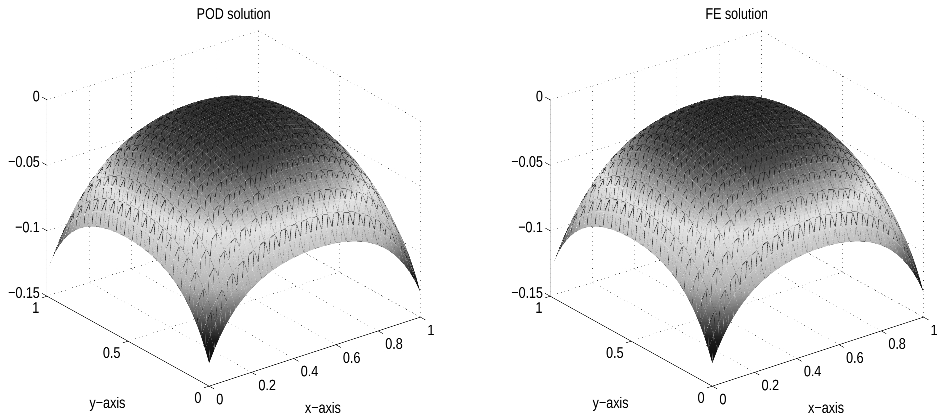

Figure 1. Run 1. POD solution (left plot) and FE solution (right plot) with $\beta = (x, y)^T$, $q = 40$, and snapshots for varying parameter $q \in [-50.5, 50.5]$.

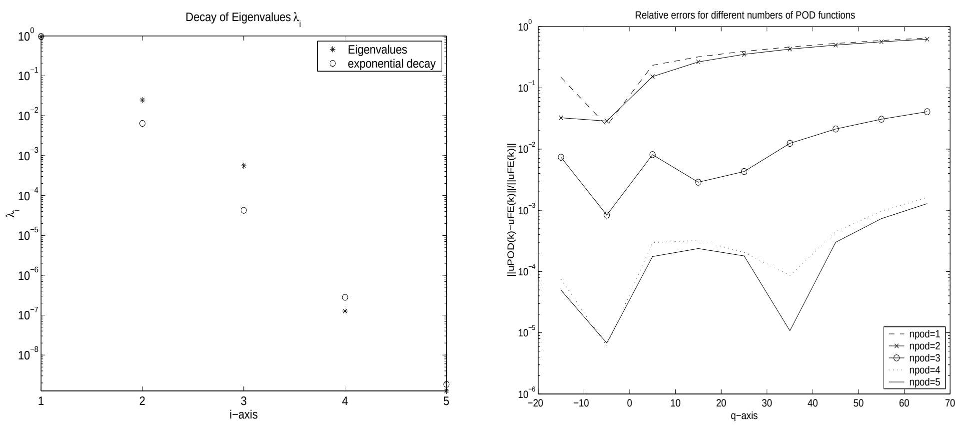

Figure 2. Run 1. Real and predicted decay of eigenvalues with snapshots for varying parameter q and $\beta = (x, y)^T$ (left plot) and relative errors for different numbers of POD functions along an interval of parameters q with $\beta = (x, y)^T$ (right plot).

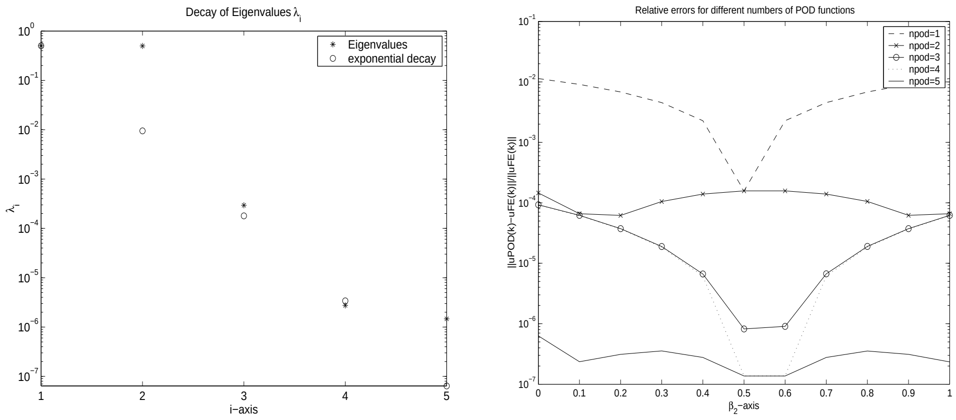

Figure 3. Run 2. Real and predicted decay of eigenvalues (left plot) and relative errors for different numbers of POD functions (right plot) with snapshots for $\beta_1 = 0.5$ and varying parameter β_2.

We observe that the experimental order of decay leads to a good estimate for the decay of the first 5 eigenvalues. Furthermore, in Figure 2 the relative error for the difference of the POD and FE solution is plotted for different values of $q \in [-20, 65]$ and for a different number of POD basis functions $1 \le \ell \le 5$. Notice that the relative L^2 error decreases with increasing ℓ. Moreover, the POD basis is also suitable for parameter values ($q \in (50.5, 65]$) that are not contained in the snapshot computation. $\diamondsuit$

Run 2. In our second example, we fix $q = 40$ and vary the parameter vector β in $[0,1] \times [0,1]$ as follows:

$$\beta = \begin{pmatrix} 0.1 + 0.15(i-1) \\ 0.1 + 0.15(j-1) \end{pmatrix} \quad \text{for } 1 \le i, j \le 7.$$

Thus, we compute 49 snapshots and compute the POD basis as described in Remark 3.3. This example does not fit into the theoretical investigations of Section 4. Compared to Run 1, we cannot expect that the experimental order of decay (EOD) leads to a good estimate at the decay of the first eigenvalues. The real and estimated decay is shown in Figure 3 (left plot). Note that the first two eigenvalues are close to 0.5 so that we do not have an exponential decay. However, the third and fourth eigenvalues can be estimated very well. The fifth eigenvalue is already small. Let us mention that if we compute the POD basis by varying β_1 and fixing β_2 (and also vice versa), we can estimate the decay of the eigenvalues in a similar manner as in Run 1. $\Diamond$

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