CONSTANT SELECTIONS
AND MINIMAX INEQUALITIES

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Abstract

In this paper, we establish two constant selection theorems for a map whose dual is upper or lower semicontinuous. As applications, matching theorems, analytic alternatives, and minimax inequalities are obtained.

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1. Introduction

Let $X$ and $Y$ be two nonempty sets. To a map $T : X \to Y$ are associated other three maps $T^c : X \to Y$, the complement of $T$, $T^\ominus : Y \to X$, the (lower) inverse of $T$ and $T^* : Y \to X$ the dual of $T$ defined by

$$T^c(x) = Y \setminus T(x), T^\ominus(y) = \{x \in X : y \in T(x)\} \quad \text{and} \quad T^*(y) = X \setminus T^\ominus(y).$$

If $y_0 \in \bigcap_{x \in X} T(x)$, then the function $t : X \to Y$ defined by $t(x) = y_0$ is a constant selection of $T$, that is $t(x) \in T(x)$, for all $x \in X$.

It is worth underlining the following straightforward fact: a map $T : X \to Y$ has a constant selection (that is, $\bigcap_{x \in X} T(x) \neq \emptyset$) if and only if $T^*(y) = \emptyset$ for at least $y \in Y$. 
The first and the main motivation of the study on the existence of a constant selection comes from the minimax theory. More precisely, if $f, g : X \times Y \to \mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$ are two functions, then one can see that $\inf_{y \in Y} \sup_{x \in X} g(x, y) \leq \sup_{x \in X} \inf_{y \in Y} f(x, y)$ if and only if the map $G_{\lambda} : X \to Y$ defined by $G_{\lambda}(x) = \{y \in Y : g(x, y) < \lambda\}$ has a constant selection for each $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$.

An additional motivation comes from mathematical economics where $T : X \to X$ represents a preference relation on a consumption set $X$. There are two possible interpretations for $T$. As a large preference relation, $y \in T(x)$ is then interpreted as $y$ is preferred or equivalent to $x$, in which case it is natural to assume that $x \in T(x)$, or as a strict preference relation $y \in T(x)$ is then interpreted as $y$ is strictly preferred to $x$, in which case it is natural to assume that $x \notin T(x)$. In the first case, if $y_0 \in \bigcap_{x \in X} T(x)$, then $y_0$ is a largest element with respect to $T$. In the other case, if $x_0 \in \bigcap_{y \in X} T^*(y)$, then $x_0$ is a maximal element with respect to $T$.

In this paper, we obtain two constant selection theorems for a map $S$ whose dual $S^*$ is upper continuous (Theorem 1), respectively lower semi-continuous (Theorem 2). Since in both theorems the map $S$ is supposed generalized KKM with respect to other map, Theorems 1 and 2 could be considered as well as KKM type theorems. As applications we obtain in Section 4 two matching theorems of Ky Fan type [10] and, in Section 5, several analytic alternatives and some very general minimax inequalities.

2. Preliminaries

If $X$ is a subset of a topological vector space we denote by $X$, $\text{co} \ X$, $\overline{\text{co}} \ X$, the closure, convex hull, closed convex hull of $X$, respectively. Given a map $T : Y \to X$, the maps $\text{co} \ T : Y \to \text{co} \ X$, $\overline{\text{co}} \ T : Y \to \overline{\text{co}} \ X$ are defined by $(\text{co} \ T)(y) = \text{co} (T(y))$, $(\overline{\text{co}} \ T)(y) = \overline{\text{co}} (T(y))$ for all $y \in Y$.

Let $T : X \to Y$ be a map. As usual the set $\{(x, y) \in X \times Y : y \in T(x)\}$ is called the graph of $T$. For $A \subset X$ and $B \subset Y$ let $T(A) = \bigcup_{x \in A} T(x)$ and $T^{-}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$. Given two maps $T : X \to Y$ and $S : Y \to Z$ the composite $S \circ T : X \to Z$ is defined by $(S \circ T)(x) = S(T(x)) = \bigcup \{S(y) : y \in T(x)\}$.

For topological spaces $X$ and $Y$ a map $T : X \to Y$ is said to be: upper semicontinuous (u.s.c.) if for any closed set $B \subset Y$ the set $T^{-}(B)$ is closed.
in $X$; lower semicontinuous (l.s.c.) if for any open set $B \subset Y$ the set $T^{-1}(B)$ is open in $X$; compact if $T(X)$ is contained in a compact subset of $Y$.

The following lemma collects known facts about u.s.c. maps [2].

**Lemma 1.**

(i) A composite of u.s.c. is u.s.c.

(ii) If $Y$ is compact and $T : X \to Y$ has closed graph, then $T$ is u.s.c. with compact values.

(iii) If $T$ is u.s.c. with compact values, then $T(K)$ is compact whenever $K \subset X$ is compact.

A nonempty topological space is called acyclic if all its reduced Čech homology groups over rationals are trivial. For nonempty sets in topological vector spaces, convex $\Rightarrow$ star-shaped $\Rightarrow$ contractible $\Rightarrow$ acyclic $\Rightarrow$ connected and not conversely.

For topological spaces $X$ and $Y$, $T : X \to Y$ is called an acyclic map whenever $T$ is u.s.c. with compact acyclic values. Let $\mathcal{V}(X,Y)$ be the class of all acyclic maps $T : X \to Y$ and $\mathcal{V}_c(X,Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces. If $Y$ is a convex subset of a topological vector space, let $\mathcal{K}(X,Y)$ be the set of all Kakutani maps $T : X \to Y$ (i.e. u.s.c. map with nonempty compact convex values). Obviously, $\mathcal{K}(X,Y) \subset \mathcal{V}(X,Y)$.

Let $X$ be a convex subset of a vector space and $Y$ be a nonempty set. If $S, T : X \to Y$ are two maps such that

$$T(coA) \subset S(A)$$

for each nonempty finite subset $A$ of $X$,

then $S$ is said to be generalized KKM w.r.t. $T$ [6].

From now all topological spaces will be assumed Hausdorff. Throughout this paper, a real locally convex Hausdorff topological vector space is abbreviated as l.c.s.

3. **Constant selections**

We proceed the main theorems of this section by some auxiliary results. The following lemma is Theorem A in [23]. More general fixed point theorems (for admissible maps in the sense of Górniewicz) can be found in [1, 7].
Lemma 2. Let \(X\) be a nonempty convex subset of a l.c.s. and \(H \in \mathcal{V}_c(X,X)\). If \(H\) is compact, then \(H\) has a fixed point \(x_0 \in X\); that is \(x_0 \in H(x_0)\).

The next lemma is well known. For instance it is a particular form of Proposition 2 in [15] and of Proposition 3.1 in [24].

Lemma 3. Let \(X\) be a convex subset of a vector space and \(Y\) be a nonempty set. If \(S,T : X \to Y\) are two maps, then the following statements are equivalent:

(i) \(S\) is a generalized KKM map w.r.t. \(T\).

(ii) \(coS^*(y) \subseteq T^*(y)\) for all \(y \in Y\).

We need also the following

Lemma 4. Let \(X\) be a topological space and \(Y\) be a nonempty convex set in a l.c.s. \(E\). Let \(T : X \to Y\) be a u.s.c. map with nonempty values such that \(coT(x)\) is compact for each \(x \in X\). Then the map \(coT\) is u.s.c.

**Proof.** Let \(V\) be a basis of open convex symmetric neighborhoods of the origin of \(E\). Let \(x_0 \in X\) arbitrarily fixed and \(G\) be an open subset of \(Y\) such that \(coT(x_0) \subset G\). We prove that for some \(V \in \mathcal{V}\)

\[(1)\quad coT(x_0) + V \subset G.\]

Otherwise, for each \(V \in \mathcal{V}\) there exists a point \(y_V \in coT(x_0)\) such that \((y_V + V) \cap (E \setminus G) \neq \emptyset\). Since \(V\) is symmetric, we infer that

\[(2)\quad y_V \in (E \setminus G) + V.\]

Since \((y_V)\) is a net in the compact \(coT(x_0)\) we may suppose that \((y_V)\) converges to a point \(y_0 \in coT(x_0)\). From (2) we get \(y_0 \in E \setminus G = E \setminus G\), hence \(y_0 \in coT(x_0) \cap (E \setminus G)\); this contradicts \(coT(x_0) \subset G\).

Let \(V \in \mathcal{V}\) for which (1) holds and \(U \in \mathcal{V}\) such that \(U \subset V\). Since \(T\) is u.s.c. and \(T(x_0)\) is contained in the open set \(coT(x_0) + U\), there exists a neighborhood \(W\) of \(x_0\) such that \(T(x) \subset coT(x_0) + U\), for all \(x \in W\). The set \(coT(x_0) + U\) is convex, hence for all \(x \in W\) we have \(coT(x) \subset coT(x_0) + U\), whence

\[coT(x) \subset coT(x_0) + U = coT(x_0) + U \subset coT(x_0) + V \subset G.\]

Thus, the map \(coT\) is u.s.c.
A particular case of Lemma 4 (when $E$ is a Banach space) appears in [2].

**Theorem 1.** Let $X$ be a nonempty convex set in a l.c.s. and $Y$ be a topological space. Suppose that either $X$ or $Y$ is compact. Let $F, S, T : X \to Y$ be three maps satisfying the following conditions:

1. $F(x) \subset T(x)$ for each $x \in X$;
2. $F \in \mathcal{V}_c(X, Y)$;
3. $S^*$ is u.s.c.;
4. $T^*$ has compact values;
5. $S$ is a generalized KKM map w.r.t. $T$.

Then $\bigcap_{x \in X} S(x) \neq \emptyset$.

**Proof.** Suppose that $\bigcap_{x \in X} S(x) = \emptyset$. This means that the map $S^*$ has nonempty values. By (1.5) and Lemma 3, for each $y \in Y$, $co S^*(y) \subset T^*(y)$. Since $T^*(y)$ is compact, $co S^*(y)$ is a compact subset of $T^*(y)$. By Lemma 4, the map $co S^*$ is u.s.c., hence $co S^* \in \mathcal{K}(Y, X) \subset \mathcal{V}(Y, X)$.

Let us consider the map $H = co S^* \circ F \in \mathcal{V}_c(X, X)$. Lemma 3 is applicable as soon as we prove that the map $H$ is compact. Clearly, this happens if $X$ is compact. When $Y$ is compact, since $co S^*$ is u.s.c. map with compact values, by Lemma 1(iii), $co S^*(Y)$ is compact. Since $H(X) \subset co S^*(Y)$, $H$ is a compact map.

By Lemma 3, $H$ has a fixed point. This implies that there exist $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in F(x_0) \subset T(x_0)$ and $x_0 \in co S^*(y_0) \subset T^*(y_0)$; a contradiction.

**Theorem 2.** Let $X$ be a nonempty metrizable convex set in a l.c.s. and $Y$ be a topological space. Suppose that either $Y$ is compact, or $Y$ is paracompact and $X$ is compact. Let $F, S, T : X \to Y$, be three maps satisfying conditions (1.1), (1.2), (1.4) and (1.5) in Theorem 1 and

1. $S^*$ is l.s.c.

Then $\bigcap_{x \in X} S(x)$ is nonempty.

**Proof.** Suppose that $\bigcap_{x \in X} S(x) = \emptyset$. Then $S^*$ is a l.s.c. map with nonempty values. By Propositions 2.3 and 2.6 in [17] the map $co S^*$ is l.s.c. As in the previous proof, for each $y \in Y$, $co S^*(y) \subset T^*(y)$ and obviously, $co S^*(y)$ is nonempty and complete. By Theorem 1.1 in Michael [18],...
there exists a u.s.c map \( R : Y \to X \) with nonempty values such that \( R(y) \subseteq \overline{\ker} S^*(y) \) for all \( y \in Y \). For each \( y \in Y \) we have

\[
\co R(y) \subseteq \overline{\ker} S^*(y) \subseteq T^*(y).
\]

Consider the map \( S_1 = R^* \). Then \( S_1^* = R \) is u.s.c. Since \( R \) has nonempty values.

\[
\bigcap_{x \in X} S_1(x) = \emptyset.
\]

By (3) and Lemma 3, \( S_1 \) is a generalized KKM map w.r.t. \( T \). Theorem 1 applied to the maps \( S_1, F, T \) leads to \( \bigcap_{x \in X} S_1(x) \neq \emptyset \). This contradicts (4) and the proof is complete.

4. Matching theorems

In this section we obtain two Ky Fan type matching theorems [10].

**Theorem 3.** Let \( X \) be a nonempty convex set in a l.c.s. and \( Y \) be a topological space. Let \( \{B_i : i \in I\} \) be a closed covering of \( X \) and \( \{x_i : i \in I\} \) a family of points of \( X \), both indexed by a finite set \( I \). If \( F \in \mathcal{N}_c(X,Y) \) and \( S, T : X \to Y \) are two maps satisfying:

1. \( S(X) = Y \);
2. \( S^* \) is u.s.c.;
3. \( T^* \) has compact values;
4. \( S^* \) is a generalized KKM map w.r.t. \( T^* \),

then, there exists a nonempty subset \( J \) of \( I \) such that

\[
F(\co \{x_i : i \in J\}) \cap T \left( \bigcap \{B_i : i \in J\} \right) \neq \emptyset.
\]

**Proof.** For each \( x \in X \) let \( I(x) = \{i \in I : x \in B_i\} \). Then \( I(x) \neq \emptyset \) for each \( x \in X \), since \( \{B_i \} \) covers \( X \). Define the map \( H : X \to X \) by

\[
H(x) = \overline{\co} \{x_i : i \in I(x)\}, \text{ for each } x \in X.
\]

It is clear that \( H(x) \) is a nonempty compact convex subset of \( X \). For each \( x \in X \), let \( U(x) = X \setminus \bigcup \{B_i : i \notin I(x)\} \). Then \( U(x) \) is an open neighborhood
of \( x \) and, if \( z \in U(x) \), then \( H(z) \subset H(x) \). This shows that \( H \) is u.s.c., hence \( H \in \mathbb{K}(X, X) \). Define the maps \( F_1 : X \to Y \), \( S_1 : X \to Y \), and \( T_1 : X \to Y \) by \( F_1 = F \circ H \), \( S_1 = S^c \), \( T_1 = T^c \). From the hypotheses it readily follows that the maps \( F_1, S_1, T_1 \) satisfy conditions (1.2), (1.3), (1.4) and (1.5) imposed on the maps \( F, S, T \) in Theorem 1 but, by (3.1), \( \bigcap S_1(x) = \emptyset \), hence the conclusion of Theorem 1 does not hold. Consequently, for some \( x_0 \in X \), \( F_1(x_0) \not\subset T_1(x_0) \) or, equivalently,

\[
F\left(\text{co} \{ x_i : i \in I(x_0) \} \right) \cap T(x_0) \neq \emptyset.
\]

Since \( x_0 \in \bigcap \{ B_i : i \in I(x_0) \} \), putting \( J = I(x_0) \) we infer that

\[
F\left(\text{co} \{ x_i : i \in J \} \right) \cap T\left( \bigcap \{ B_i : i \in J \} \right) \neq \emptyset.
\]

**Remark 1.** It is easy to check that condition (3.4) is equivalent to the following

\[
\bigcap_{x \in A} S(x) \subset \bigcap_{x \in \text{co} A} T(x), \text{ for each nonempty finite subset } A \subset X.
\]

In a similar manner, using as argument Theorem 2 instead of Theorem 1, we can prove

**Theorem 4.** Let \( X \) be a nonempty metrizable convex set in a l.c.s. and \( Y \) be a topological space. Assume that either \( Y \) is compact or \( Y \) is paracompact and \( X \) is compact. Let \( \{ B_i : i \in I \} \) be a closed covering of \( X \) and \( \{ x_i : i \in I \} \) a family of points of \( X \), both indexed by a finite set \( I \). If \( F \in \mathcal{V}_c(X, Y) \) and \( S, T : X \to Y \) are two maps satisfying conditions (3.1), (3.3), (3.4) and

\[(4.1) S^- \text{ is l.s.c.,}
\]

then, there exists a nonempty subset \( J \) of \( I \) such that

\[
F\left(\text{co} \{ x_i : i \in J \} \right) \cap T\left( \bigcap \{ B_i : i \in J \} \right) \neq \emptyset.
\]

It would be of some interest to compare Theorems 3 and 4 with other matching theorems due to Park [19, 21, 22] and Balaj [3]. Since \( T(\bigcap \{ B_i : i \in J \}) \subset \bigcap \{ T(B_i) : i \in J \} \), the conclusions of Theorems 3 and 4 are better than the conclusions of the theorems mentioned above.
5. Analytic alternatives, minimax inequalities

Let $X$ be a convex subset of a vector space, $Y$ a nonempty set and $s, t : X \times Y \to \mathbb{R} = \mathbb{R} \cup \{\pm \infty\}$ two functions. We say that $t$ is $s$-quasiconcave in $x$ if for any nonempty finite subset $A$ of $X$ we have

$$t(x, y) \geq \min_{u \in A} s(u, y) \quad \text{for any } x \in \text{co } A \text{ and all } y \in Y.$$  

It is clear that if $s(x, y) \leq t(x, y)$ for each $(x, y) \in X \times Y$ and if for each $y \in Y$ one of the functions $x \to s(x, y), x \to t(x, y)$ is quasiconcave, then $t$ is $s$-quasiconcave in $x$.

**Theorem 5.** Let $X$ be a nonempty compact convex subset of a l.c.s., $Y$ a topological compact space $f, s, t : X \times Y \to \mathbb{R}$ three functions and $\alpha, \beta, \gamma$ three real numbers such that $\alpha < \beta \leq \gamma$. Suppose that:

1. The sets $\{(x, y) \in X \times Y : f(x, y) \leq \alpha\}$ and $\{(x, y) \in X \times Y : s(x, y) \geq \gamma\}$ are closed in $X \times Y$;
2. $t(x, y) \leq f(x, y)$ for all $(x, y) \in X \times Y$;
3. For each $x \in X$ the set $\{y \in Y : f(x, y) \leq \alpha\}$ is acyclic or empty;
4. $t$ is $s$-quasiconcave in $x$;
5. For each $y \in Y$, the set $\{x \in X : t(x, y) \geq \gamma\}$ is closed in $X$.

Then, at least one of the following assertions holds:

(a) There exists $x_0 \in X$ such that $f(x_0, y) > \alpha$ for all $y \in Y$;
(b) There exists $y_0 \in Y$ such that $s(x, y_0) < \gamma$ for all $x \in X$.

**Proof.** We define the maps $F, S, T : X \to Y$ by

$$F(x) = \{y \in Y : f(x, y) \leq \alpha\}, \quad T(x) = \{y \in Y : t(x, y) < \beta\}, \quad \text{and}$$

$$S(x) = \{y \in Y : s(x, y) < \gamma\}.$$  

Since the graph of $F$ is closed (by (5.1)), and $Y$ is compact, $F$ is u.s.c. with compact values. Similarly, one obtains that $S^*$ is u.s.c. Suppose that (a) does not hold. Then $F$ has nonempty values and, by (5.3), $F \in \mathcal{V}(X, Y)$. Since $\alpha < \beta$ and $t(x, y) \leq f(x, y)$ for all $(x, y) \in X \times Y$, it follows that $F(x) \subset T(x)$ for all $x \in X$. By (5.5), for each $y \in Y$, $T^*(y)$ is a closed subset of the compact $X$, hence $T^*(y)$ is compact.
We claim that $S$ is a generalized KKM mapping w.r.t. $T$. Let $A$ be a nonempty finite subset of $X$ and $y \in T(coA)$. Then, $t(x, y) < \beta$, for some $x \in A$. By (5.4), we have

$$\gamma \geq \beta > t(x, y) \geq \min_{u \in A} s(u, y)$$

hence, for some $u \in A$, $y \in S(u) \subset S(A)$.

Therefore the maps $F, S, T$ satisfy all the requirements of Theorem 1. According to this theorem, $\bigcap_{x \in X} S(x) \neq \emptyset$. Let $y_0 \in \bigcap_{x \in X} S(x)$, then $s(x_0, y) < \gamma$ for all $x \in X$.

From Theorem 5 we derive the following minimax inequality:

**Theorem 6.** Let $X$ be a nonempty compact convex subset of a l.c.s. $Y$ be a topological compact space and $f, s, t : X \times Y \to \mathbb{R}$ three real functions which satisfy conditions (5.2), (5.4) in Theorem 5 and the following conditions:

1. $f$ is l.s.c. and $s$ is u.s.c. on $X \times Y$;
2. For each $\alpha > \sup_{x \in X} \min_{y \in Y} f(x, y)$ and any $x \in X$, $\{y \in Y : f(x, y) \leq \alpha\}$ is acyclic;
3. For each $y \in Y$, $t(x, y)$ is u.s.c. in $x$.

Then $\inf_{y \in Y} \max_{x \in X} s(x, y) \leq \sup_{x \in X} \min_{y \in Y} f(x, y)$.

**Proof.** First let us observe that if $f$ is l.s.c. on $X \times Y$, then for each $x \in X$, $f(x, \cdot)$ is also a l.s.c. function of $y$ on $Y$ and therefore its minimum $\min_{y \in Y} f(x, y)$ on the compact set $Y$ exists. Similarly, $\sup_{x \in X} s(x, y)$ can be replaced by $\max_{x \in X} s(x, y)$.

Suppose the conclusion is be false and choose three real numbers $\alpha, \beta, \gamma$ such that

$$\sup_{x \in X} \min_{y \in Y} f(x, y) < \alpha < \beta \leq \gamma < \inf_{y \in Y} \max_{x \in X} s(x, y).$$

One readily verifies that the functions $f, s, t$ satisfy all the requirements of Theorem 5. We prove that neither assertion (a) nor assertion (b) of the conclusion of Theorem 5 can take place.

If (a) happens, then

$$\sup_{x \in X} \min_{y \in Y} f(x, y) \geq \min_{y \in Y} f(x_0, y) > \alpha;$$

a contradiction.
If (b) happens, then
\[
\inf_{y \in Y} \max_{x \in X} s(x, y) \leq \max_{x \in X} s(x, y_0) < \gamma; \quad \text{a contradiction again.}
\]

Further on, versions of Theorems 5 and 6 will be obtained using as argument Theorem 2 instead of Theorem 1.

For \(X, Y\) topological spaces a function \(s : X \times Y \to \mathbb{R}\) is said to be marginally l.s.c. in \(y\) [4] if for every open subset \(U\) of \(X\) the function \(y \to \sup_{x \in U} s(x, y)\) is l.s.c. on \(Y\). Obviously, every function l.s.c. in \(y\) is marginally l.s.c. in \(y\). The example given in [4, p. 249] shows that the converse is not true.

**Theorem 7.** Let \(X\) be a nonempty metrizable convex set in a l.c.s., \(Y\) a topological compact space \(f, s, t : X \times Y \to \mathbb{R}\) three functions and \(\alpha, \beta, \gamma\) three real numbers such that \(\alpha \leq \beta \leq \gamma\). Assume that conditions (5.2), (5.3) and (5.4) in Theorem 5 are fulfilled and, moreover, the following conditions are satisfied:

1. (7.1) the set \(\{(x, y) \in X \times Y : f(x, y) \leq \alpha\}\) is closed in \(X \times Y\);
2. (7.2) for each \(y \in Y\), the set \(\{x \in X : t(x, y) \geq \gamma\}\) is compact;
3. (7.3) \(s\) is marginally l.s.c. in \(y\).

Then, at least one of the following assertions holds:

(a) There exists \(x_0 \in X\) such that \(f(x_0, y) > \alpha\) for all \(y \in Y\).

(b) There exists \(y_0 \in Y\) such that \(s(x, y_0) \leq \gamma\) for all \(x \in X\).

**Proof.** Suppose that both assertions (a) and (b) are not true Consider the maps \(F, S, T : X \to Y\) defined by

\[
F(x) = \{y \in Y : f(x, y) \leq \alpha\}, \quad T(x) = \{y \in Y : t(x, y) < \beta\},
\]

\[
S(y) = \{x \in X : s(x, y) \leq \gamma\}.
\]

Suppose (a) does not hold. As in the proof of Theorem 5, we obtain that \(F \in \mathcal{V}(X, Y)\). Let \(U\) be an open subset of \(X\). Since

\[
\{y \in Y : S^*(y) \cap U \neq \emptyset\} = \{y \in Y : \sup_{x \in U} s(x, y) > \gamma\},
\]

the set \(\{y \in Y : f(x_0, y) > \alpha\}\) is open in \(Y\) and \(\inf_{y \in Y} \max_{x \in X} s(x, y) \leq \max_{x \in X} s(x, y_0) < \gamma\); a contradiction again.

\(\blacksquare\)
by (7.3), it follows that $S^*$ is l.s.c. Other requirements of Theorem 2 are easily checked and, according to this theorem, there exists $y_0 \in \bigcap_{x \in X} S(x)$. Then, $s(x, y_0) \leq \gamma$ for all $x \in X$.

From Theorem 7 we obtain immediately the following

**Theorem 8.** Let $X$ be a nonempty metrizable convex set in a l.c.s., $Y$ a topological compact space and $f, s, t : X \times Y \to \mathbb{R}$ three functions. Suppose that $f$ is l.s.c. on $X \times Y$ and that $f, s, t$ satisfy conditions (5.2), (5.4), (6.2), (6.3) and (7.3). Then $\inf_{y \in Y} \sup_{x \in X} s(x, y) \leq \sup_{x \in X} \min_{y \in Y} f(x, y)$.

The origin of Theorems 6 and 8 goes back to the well-known Sion’s minimax theorem [25]. In our opinion, it is worth comparing our results with other two-function minimax inequalities established by Fan [8], Liu [16], Granas and Liu [11, 12] and Park [20].

We need now to recall Berge’s maximum theorem [5].

**Lemma 5.** Let $X$ and $Y$ be topological spaces, $f : X \times Y \to \mathbb{R}$ a continuous function and $F : X \to Y$ a continuous map with nonempty compact values. Then the map $G : X \to Y$ defined by $G(x) = \{y \in F(x) : f(x, y) = \max_{v \in F(x)} f(x, v)\}$ is u.s.c.

Note that if $f$ and $F$ are as in Lemma 5 then the map $G : X \to Y$ defined by $G(x) = \{y \in F(x) : f(x, y) = \min_{v \in F(x)} f(x, v)\}$ is u.s.c., too.

**Theorem 9.** Let $X$ be a nonempty compact convex set in a l.c.s. and $Y$ be a topological space. Let $f, g : X \times Y \to \mathbb{R}$ be two continuous functions and $F : X \to Y$ be a continuous map with nonempty compact values. Suppose that:

1. for each $x \in X$ the set $\{y \in F(x) : f(x, y) = \min_{v \in F(x)} f(x, v)\}$ is acyclic;
2. for each $y \in Y$ the set $\{x \in X : g(x, y) = \max_{u \in X} g(u, y)\}$ is compact;
3. $f$ is $g$-quasiconcave in $x$.

Then (a) there exists $(x_0, y_0) \in \text{graph} F$ such that

$$\max_{x \in X} g(x, y_0) \leq \min_{y \in F(x_0)} f(x_0, y);$$
and (b) the following minimax inequality holds:
\[
\inf_{y \in Y} \max_{x \in X} g(x, y) \leq \max_{x \in X} \min_{y \in F(x)} f(x, y).
\]

**Proof.** We will define three maps \( G, S, T : X \to Y \). The first two are defined by
\[
G(x) = \{ y \in F(x) : f(x, y) = \min_{v \in F(x)} f(x, v) \},
\]
\[
S(x) = \{ y \in Y : \exists u \in X \text{ such that } g(x, y) < g(u, y) \}.
\]

For the third map \( T : X \to Y \) we prefer to give its dual. This is defined by
\[
T^*(y) = \co \{ x \in X : g(x, y) = \max_{u \in X} g(u, y) \}.
\]

It is easy to check that for each \( y \in Y \),
\[
S^*(y) = \{ x \in X : g(x, y) = \max_{u \in X} g(u, y) \}.
\]

By Lemma 5 the maps \( G \) and \( S^* \) are u.s.c. By (9.1), \( G \in V(X, Y) \) and by (9.2), \( T^* \) has compact values. Since for each \( y \in Y \), \( \co S^*(y) = T^*(y) \), by Lemma 3 it follows that \( S \) is a generalized KKM map w.r.t. \( T \). Hence the maps \( G, S, T \) satisfy conditions (1.2), (1.3), (1.4) and (1.5) of Theorem 1.

Since \( X \) is compact and \( g \) is continuous, for each \( y \in Y \), \( \co S^*(y) = T^*(y) \neq \emptyset \), hence the conclusion of Theorem 1 does not hold. By Theorem 1 there exist \( x_0 \in X \) and \( y_0 \in Y \) such that \( y_0 \in G(x_0) \setminus T(x_0) \).

By \( y_0 \in G(x_0) \) it follows that
\[
f(x_0, y_0) \leq f(x_0, y) \text{ for each } y \in F(x_0).
\]

Since \( y_0 \notin T(x_0) \), we have \( x_0 \in T^*(y_0) \). Hence there exists a finite set \( \{ x_1, \ldots, x_n \} \subset X \) such that
\[
\max_{u \in X} g(u, y_0) = g(x_1, y_0) = \cdots = g(x_n, y_0) = \min_{1 \leq i \leq n} g(x_i, y_0)
\]
and \( x_0 \in \co \{ x_1, \ldots, x_n \} \). Then for each \( y \in F(x_0) \) and \( x \in X \), by (6), (5) and (9.3) we obtain
\[ g(x, y_0) \leq \max_{u \in X} g(u, y_0) = \min_{1 \leq i \leq n} g(x_i, y_0) \leq f(x_0, y_0) \leq f(x_0, y). \]

Assertion (b) follows immediately from (a) since
\[
\inf_{y \in Y} \max_{x \in X} g(x, y) \leq \max_{x \in X} g(x, y_0) \leq \min_{y \in F(x_0)} f(x_0, y) \leq \max_{x \in X} \min_{y \in F(x)} f(x, y). \]

It would be of some interest to compare Theorem 9 with earlier results of Granas and Liu [11, 12], Ha [13, 14] and Park [20, 21, 23].

The next two corollaries are particular cases of Theorem 9.

**Corollary 1.** Let \( X \) be a nonempty compact convex set in a l.c.s. and \( Y \) be a compact topological space. Let \( f, g : X \times Y \to \mathbb{R} \) be two continuous functions satisfying conditions (9.2), (9.3) and:
\[ (9.1') \text{ for each } x \in X \text{ the set } \{ y \in Y : f(x, y) = \min_{v \in Y} f(x, v) \} \text{ is acyclic.} \]

Then
\[ \min_{y \in Y} \max_{x \in X} g(x, y) \leq \max_{x \in X} \min_{y \in Y} f(x, y). \]

**Proof.** Take \( F(x) = Y \) for all \( x \in X \) and apply Theorem 9. \( \blacksquare \)

The origin of the following corollary goes back to Ky Fan’s minimax inequality [9].

**Corollary 2.** Let \( X \) be a compact convex set in a l.c.s. and \( f, g : X \times X \to \mathbb{R} \) be two continuous functions satisfying condition (9.3) and:
\[ (9.2') \text{ for each } y \in X \text{ the set } \{ x \in X : g(x, y) = \max_{u \in X} g(u, y) \} \text{ is compact.} \]

Then
\[ \min_{y \in X} \max_{x \in X} g(x, y) \leq \max_{x \in X} f(x, x). \]

**Proof.** Apply Theorem 9 with \( X = Y, F(x) = \{ x \} \). \( \blacksquare \)
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References


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