

**REPRESENTATION OF THE SET OF MILD SOLUTIONS  
TO THE RELAXED SEMILINEAR  
DIFFERENTIAL INCLUSION**

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**Abstract**

We study the relation between the solutions set to a perturbed semilinear differential inclusion with nonconvex and non-Lipschitz right-hand side in a Banach space and the solutions set to the relaxed problem corresponding to the original one. We find the conditions under which the set of solutions for the relaxed problem coincides with the intersection of closures (in the space of continuous functions) of sets of  $\delta$ -solutions to the original problem.

**Keywords:** differential inclusion, mild solution, quasi-solution, convexified and perturbed problem, relaxation theorem.

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## INTRODUCTION

The paper is concerned with the Cauchy problem for a semilinear differential inclusion of the type

$$(*) \quad x'(t) \in Ax(t) + F(t, x(t)), \quad t \in [0, d],$$

where  $A$  is the infinitesimal generator of a strongly continuous semigroup ( $C_0$ -semigroup) on a separable Banach space  $E$ , and  $F : [0, d] \times E \rightarrow E$  is a multivalued map whose images are supposed to be nonempty, compact, but not necessarily convex subsets of  $E$ . Alongside the inclusion  $(*)$  we consider the corresponding convexified inclusion and an inclusion with external perturbations, when for every pair  $(t, x(t)) \in [0, d] \times E$  instead of set  $F(t, x(t))$  we take its closed neighborhood of some radius.

We study properties of sets of mild solutions to these problems, i.e., of those that can be represented by the formula

$$(**) \quad x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds, \quad t \in [0, d],$$

where  $f(\cdot)$  is one of the selections for a multimap representing the nonlinear part of the inclusion. The relations between mild and Caratheodory type solutions for differential equations and inclusions in Banach spaces are described, e.g. in [11, 9].

Whenever the differential inclusion with some nonconvexity properties occurs, the question how its solutions set relates to that of the corresponding convexified problem arises. It is a well-known fact that if a nonconvex-valued multimap  $F$ , representing the nonlinear part of the inclusion  $(*)$  satisfies the Lipschitz condition, then the relaxation theorem holds true, i.e., the closure of the solutions set of  $(*)$  in the space of continuous functions coincides with the solutions set of the convexified problem [6, 7]. This result (sometimes referred to as the density principle) also holds under more general assumption when the Hausdorff distance between the values of  $F$  is estimated by the Kamke function (see, for example, [12]). Otherwise, such an equality may fail [13] (see also [6]).

In the present paper, we investigate the structure of the solutions (mild solutions) set for the convexified problem of  $(*)$  in a general case, i.e., without the properties mentioned above. We discuss conditions under which the solutions set for the convexified problem is equal to the intersection of

closures of the sets of so-called  $\delta$ -solutions for (\*) (also known as approximate solutions) in the space of continuous functions. Following the similar research in [4] on ordinary differential inclusions in the finite dimensional space with Caratheodory type solutions, we use here the concept of quasi-solution and the concept of modulus of continuity of a multivalued map. As a by-result we prove that the set of all quasi-solutions for (\*) coincides with the set of solutions for the convexified problem. In the last section, we show that the density principle plays a crucial role for the set of mild solutions for the problem (\*) to be stable under “small” variations of the righthand side.

1. PRELIMINARIES

In this section we will give the necessary notations and some general facts on multivalued maps (multimaps) and measure of noncompactness in the Banach space. We refer here to [1, 3, 5, 9, 14].

Let  $E$  be a topological vector space and  $\mathcal{P}(E), \mathcal{C}(E), \mathcal{K}(E), \mathcal{K}_v(E)$  be the families of all nonempty, closed, compact, compact and convex subsets of  $E$ , respectively.

**Definition 1.1.** A multivalued map (multimap)  $F : E \longrightarrow \mathcal{P}(E)$  is called:

- (a) upper semicontinuous (u.s.c.) if for every open subset  $U \subset E$  the set  $F_+^{-1}(U) = \{x \in X : F(x) \subset U\}$  is open in  $E$ ;
- (b) lower semicontinuous (l.s.c.) if for every open subset  $U \subset E$  the set  $F_-^{-1}(U) = \{x \in X : F(x) \cap U \neq \emptyset\}$  is open;
- (c) continuous if it is both u.s.c. and l.s.c.

In this paper, we consider  $E$  to be a separable Banach space.

Let function  $h^+ : \mathcal{C}(E) \times \mathcal{C}(E) \rightarrow R \cup \{\infty\}$  be defined as

$$h^+[A_1; A_2] = \sup \{ \rho[y; A_2] : y \in A_1 \},$$

where  $\rho[y; A_2] = \inf \{ \|y - z\|_E, z \in A_2 \}$  is a distance between a point and a set in  $E$ . Then function  $h : \mathcal{C}(E) \times \mathcal{C}(E) \rightarrow R \cup \{\infty\}$ ,

$$h[A_1; A_2] = \max \{ h^+[A_1; A_2], h^+[A_2; A_1] \},$$

will denote the Hausdorff distance between closed sets in  $E$ .

**Definition 1.2.** A multimap  $F : E \rightarrow \mathcal{C}(E)$  is called u.s.c. (l.s.c.) in the Hausdorff sense at point  $x \in E$  if for any sequence  $\{x_n\}_{n=1}^\infty \subset E$  which converges to  $x$  in  $E$  we have

$$\lim_{n \rightarrow \infty} h^+[F(x_n); F(x)] = 0 \quad \left( \lim_{n \rightarrow \infty} h^+[F(x); F(x_n)] = 0 \right).$$

A multimap  $F$  is called u.s.c. (l.s.c.) in the Hausdorff sense if it is u.s.c. (l.s.c.) in the Hausdorff sense for every  $x \in E$ . A multimap  $F$  is called continuous in the Hausdorff sense if it is both u.s.c. and l.s.c. in the Hausdorff sense.

**Remark 1.3.** If a multimap  $F$  has compact images in  $E$ , then continuity in the Hausdorff sense is equivalent to that in the sense of Definition 1.1.

Let  $B[u, r] \subset E$  be a closed ball in space  $E$  with center  $u$  and radius  $r > 0$ ,  $O_\epsilon X = \{y \in E : \rho[y; X] \leq \epsilon\}$  be a closed  $\epsilon$ -neighborhood of  $X$ . If  $X \subset E$ , then  $\overline{\text{co}}X$  stands for a closed convex hull of  $X$ . If  $F$  is a multimap,  $F : E \rightarrow \mathcal{K}(E)$ , then its *convex closure*  $\overline{\text{co}}F : E \rightarrow \mathcal{K}_v(E)$  is defined as follows:  $(\overline{\text{co}}F)(x) = \overline{\text{co}}(F(x))$ .

**Theorem 1.4.** *If a multimap  $F : E \rightarrow \mathcal{K}(E)$  is u.s.c. (l.s.c.), then the multimap  $\overline{\text{co}}F : E \rightarrow \mathcal{K}_v(E)$  is u.s.c. (l.s.c.).*

By  $C([0, d], E)$  we denote the Banach space of continuous functions  $x : [0, d] \rightarrow E$  with the usual sup-norm  $\|x\|_C = \sup_{t \in [0, d]} \|x(t)\|_E$  and by  $L^1([0, d], E)$  the Banach space of Bochner integrable functions  $y : [0, d] \rightarrow E$  with the norm  $\|y\|_1 = \int_0^d \|y(t)\|_E dt$ . We denote  $L^1([0, d], R_+) = L^1_+([0, d])$ . Let  $S(F)$  denote the set of all Bochner integrable selections of the multimap  $F : [0, d] \rightarrow \mathcal{K}(E)$ , i.e.,  $S(F) = \{f \in L^1([0, d], E) : f(t) \in F(t) \text{ for a.e. } t \in [0, d]\}$ . We also denote  $\|X\| = \sup\{\|x\| : x \in X\}$  for every  $X \subset E$ .

Further, we will need the following class of so-called semicompact sequences.

**Definition 1.5.** The sequence  $\{f_n\}_{n=1}^\infty \subset L^1([0, d], E)$  is called semicompact if:

- (1) it is integrably bounded, i.e., there exists a function  $\nu \in L^1_+([0, d])$  such that for a.e.  $t \in [0, d]$

$$\|f_n(t)\|_E \leq \nu(t);$$

(2) the set  $\{f_n(t)\}_{n=1}^\infty$  is relatively compact in  $E$  for a.e.  $t \in [0, d]$ .

**Theorem 1.6.** *Assume that  $\Omega \subset L^1([0, d], E)$  is integrably bounded and the sets  $\Omega(t)$  are relatively compact for a.e.  $t \in [0, d]$ . Then  $\Omega$  is weakly compact in  $L^1([0, d], E)$ .*

**Definition 1.7.** A map  $\beta : P(E) \rightarrow A$  (where  $A$  is some partially ordered set) is called a measure of noncompactness (MNC) in  $E$  if

$$\beta(\overline{\text{co}}\Omega) = \beta(\Omega)$$

for every  $\Omega \in P(E)$ .

If  $D$  is dense in  $\Omega$ , then  $\overline{\text{co}}\Omega = \overline{\text{co}}D$  and hence

$$\beta(\Omega) = \beta(D).$$

A measure of noncompactness  $\beta$  is called *real* if  $A \equiv [0, \infty]$  with natural ordering and  $\beta(\Omega) < +\infty$  for any bounded set  $\Omega \in P(E)$ . We will use one of the well-known examples of real MNC, the Hausdorff MNC, i.e., the function  $\chi : P(E) \rightarrow [0, \infty]$  defined as follows:

$$\chi(\Omega) = \inf \{ \epsilon > 0 : \Omega \text{ has a finite } \epsilon\text{-net} \}.$$

We also present here several classes of functions.

By  $K$  we denote a set of all functions  $\eta : [0, d] \times R_+ \rightarrow R_+$  with the following properties:

- (1)  $\eta(\cdot, \delta)$  is measurable for each  $\delta \in R_+$ ;
- (2) for each  $\delta \in R_+$  there exists a function  $m_\delta \in L^1_+([0, d])$  such that  $\eta(t, \tau) \leq m_\delta(t)$  for a.e.  $t \in [0, d]$  and all  $\tau \in [0, \delta]$ ;
- (3) the relations  $\lim_{\delta \rightarrow +0} \eta(t, \delta) = 0$ ,  $\eta(t, 0) = 0$  are valid for a.e.  $t \in [0, d]$ .

Further, let  $\tilde{K} \subset K$  be the set of functions  $\eta : [0, d] \times R_+ \rightarrow R_+$  such that for each  $\delta > 0$  the function  $m_\delta(t)$  is constant for a.e.  $t \in [0, d]$ ;  $P \subset K$  is the set of functions  $\eta : [0, d] \times R_+ \rightarrow R_+$  with the property: for each  $\delta > 0$  there exists an  $r(\delta) > 0$  such that  $\eta(t, \delta) \geq r(\delta)$  for a.e.  $t \in [0, d]$ ; and we set  $\tilde{P} = \tilde{K} \cap P$ .

## 2. MILD SOLUTIONS AND QUASI-SOLUTIONS

We consider the Cauchy problem for a semilinear differential inclusion

$$(2.1) \quad \begin{aligned} x'(t) &\in Ax(t) + F(t, x(t)), \quad t \in [0, d] \\ x(0) &= x_0 \in E \end{aligned}$$

under the following assumptions:

(A) the linear operator  $A : D(A) \subseteq E \rightarrow E$  is the infinitesimal generator of a  $C_0$ -semigroup  $e^{At}$ ;

the multivalued map  $F : [0, d] \times E \rightarrow \mathcal{K}(E)$  is such that:

(F1) it verifies the Caratheodory conditions, i.e.,

- (i) for every  $x \in E$  the multifunction  $F(\cdot, x) : I \rightarrow \mathcal{K}(E)$  is measurable,
- (ii) for a.e.  $t \in [0, d]$  the multimap  $F(t, \cdot) : E \rightarrow \mathcal{K}(E)$  is continuous;

(F2) there exists a function  $\alpha \in L^1_+[0, d]$  such that for every  $x \in E$

$$\|F(t, x)\| \leq \alpha(t)(1 + \|x\|_E) \quad \text{for a.e. } t \in [0, d];$$

(F3) there exists a function  $\mu \in L^1_+[0, d]$  such that

$$\chi(F(t, D)) \leq \mu(t)\chi(D) \quad \text{for a.e. } t \in [0, d]$$

for every bounded  $D \subset E$ , where  $\chi$  is the Hausdorff MNC in  $E$ .

**Definition 2.1.** A continuous function  $x \in C([0, d], E)$  is called a mild solution to the problem (2.1) on an interval  $[0, d]$  provided

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)ds, \quad t \in [0, d], \\ x(0) &= x_0 \end{aligned}$$

with  $f \in S(F(\cdot, x(\cdot)))$ .

**Remark 2.2.** Note, that under assumption (F1), for any  $x \in C([0, d], E)$  we can find at least one selection  $f \in S(F(\cdot, x(\cdot)))$  (see [3, p. 65]).

Let  $\eta \in K$ . We consider also the following problems:

$$(2.2) \quad \begin{aligned} x'(t) &\in Ax(t) + \overline{\text{co}}F(t, x(t)) , \quad t \in [0, d], \\ x(0) &= x_0 \in E; \end{aligned}$$

$$(2.3) \quad \begin{aligned} x'(t) &\in Ax(t) + O_{\eta(t, \delta)}F(t, x(t)) , \quad t \in [0, d], \\ x(0) &= x_0 \in E. \end{aligned}$$

Following the similar concepts in [4], one may refer to the inclusion in (2.3) as to an *inclusion with external perturbations*, and to each of its solutions (mild solutions) for a prescribed  $\delta > 0$  as to a  $\delta$ -*solution* (or *approximate solution*) of the inclusion (2.1). The function  $\eta$  represents the *radius of external perturbations*. It determines the error of the computation of values of the multimap  $F$ , and we consider this errors to be independent of the variable  $x \in E$ .

Let  $H, H_{\text{co}}, H_{\eta(\delta)}$  denote the sets of mild solutions to (2.1), (2.2), (2.3), respectively. It is obvious that for any function  $\eta \in K$  and any  $\delta > 0$  the inclusion  $H \subset H_{\eta(\delta)}$  holds.

**Remark 2.3.** Note that, under assumptions (A), (F1)–(F3), the set  $H$ , and hence  $H_{\eta(\delta)}$ , are nonempty subsets of  $C([0, d], E)$  [9, p. 162]. In fact, condition (F1) implies, via the Scorza- Dragoni Theorem, that  $F$  is almost continuous and we may apply existence results contained in [5, 9]. Moreover, the convex closure  $\overline{\text{co}}F$  of multimap  $F$ , according to its definition and Theorem 1.4, satisfies all the properties (F1)–(F3), which means that the set  $H_{\text{co}}$  is a nonempty compact subset of  $C([0, d], E)$  [9, p. 131].

**Definition 2.4.** We say that  $x \in C([0, d], E)$  is a quasi-solution to (2.1) if there exists a sequence  $\{f_n\}_{n=1}^\infty \subset S(F(\cdot, x(\cdot)))$  such that a sequence of functions

$$(2.4) \quad x_n(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f_n(s) ds$$

converges to  $x$  with respect to the norm of  $C([0, d], E)$ .

By  $\mathcal{H}$  we denote the set of all quasi-solutions to (2.1). It is clear, that  $H$  is a subset of  $\mathcal{H}$ , but not vice versa.

**Lemma 2.5.** *Let  $F : [0, d] \times E \rightarrow \mathcal{K}(E)$  be a multimap that satisfies (F1)–(F3), and let  $T(t) : E \rightarrow E$ ,  $t \in [0, d]$  be a strongly continuous family of linear bounded operators. Then for every interval  $[t_0, t_1] \subseteq [0, d]$  the following equality takes place:*

$$(2.5) \quad \int_{t_0}^{t_1} T(t_1 - s) \overline{\text{co}}F(s, x(s)) ds = \overline{\int_{t_0}^{t_1} T(t_1 - s) F(s, x(s)) ds}.$$

**Proof.** Since  $F(s, x(s))$  is compact for every  $s \in [0, d]$ , the set  $\overline{\text{co}}(F(s, x(s)))$  is compact as well, and this makes  $T(t_1 - s)(\overline{\text{co}}F(s, x(s)))$  compact for every  $s \in [t_0, t_1]$ . So, we get equalities  $T(t_1 - s)\overline{\text{co}}F(s, x(s)) = \overline{\text{co}}(T(t_1 - s)F(s, x(s)))$  and

$$(2.6) \quad \int_{t_0}^{t_1} \overline{\text{co}}(T(t_1 - s)F(s, x(s))) ds = \int_{t_0}^{t_1} T(t_1 - s) \overline{\text{co}}(F(s, x(s))) ds.$$

Using the result in [14] (see also [2]) we have that

$$(2.7) \quad \int_{t_0}^{t_1} \overline{\text{co}}(T(t_1 - s)F(s, x(s))) ds = \overline{\int_{t_0}^{t_1} T(t_1 - s)F(s, x(s)) ds},$$

and from (2.6) and (2.7) we obtain relation (2.5). ■

**Lemma 2.6** [7]. *Let  $U : [0, d] \rightarrow \mathcal{C}(E)$  be a measurable, integrably bounded multimap, and  $T(t) : E \rightarrow E$ ,  $t \in [0, d]$  be a strongly continuous family of linear bounded operators. Let  $[0, d] \ni s \rightarrow g(s) \in T(d - s)U(s)$  be a measurable selection. Then there exists a measurable selection  $u \in S(U)$  such that  $g(s) = T(d - s)u(s)$  almost everywhere on  $[0, d]$ .*

**Lemma 2.7.** *Under assumptions (A), (F1)–(F3) we have that*

$$H_{\text{co}} = \mathcal{H}.$$

**Proof.** Suppose  $x \in H_{\text{co}}$ . From the definition of a mild solution we have:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s) ds,$$

where  $f(s) \in \overline{\text{co}}F(s, x(s))$  for a.e.  $s \in [0, d]$ . Denote by  $I_j$  the interval  $[(j-1)/n)d, (j/n)d]$ ,  $j = 1, \dots, n$  and set  $d_j = (j/n)d$ . Then from Lemmas



2.5 and 2.6 it follows, that for every  $j = 1, 2, \dots$  there exists a sequence of measurable selections  $\{f_{jn}\}_{n=1}^\infty \subset S(F(\cdot, x(\cdot)))$  such that

$$\left\| \int_{I_j} e^{A(d_j-s)} f_{jn}(s) ds - \int_{I_j} e^{A(d_j-s)} f(s) ds \right\| \leq \frac{1}{n^2 M},$$

where  $M = \sup_{t \in [0, d]} \|e^{At}\|$ .

Let  $\{f_n\}_{n=1}^\infty \subset S(F(\cdot, x(\cdot)))$  be the sequence of selections such that  $f_n(t) = f_{jn}(t)$  on  $I_j$ . We define the sequence

$$x_n(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} f_n(s) ds$$

and show that  $x_n$  converges uniformly to  $x$  as  $n \rightarrow \infty$ .

Take  $\epsilon > 0$ , and let  $\delta(\epsilon) > 0$  be a number such that for every measurable set  $e \subset [0, d]$ , satisfying inequality  $\mu(e) < \delta(\epsilon)$ , the relation  $\int_e \|y(s)\|_E ds < \frac{\epsilon}{3M}$  holds for every  $y \in S(\overline{\text{co}}F(\cdot, x(\cdot)))$ . Let  $N(\epsilon)$  be a number such that for every  $n \geq N(\epsilon)$  we have  $\frac{1}{n} < \frac{\epsilon}{3}$  and  $\mu(I_j) < \delta(\epsilon)$  for every  $j = 1, \dots, n$ . Let  $t \in [0, d]$ , then there exists  $j(t) \in \{1, \dots, n\}$  for which  $t \in [d_{j(t)}, d_{j(t)+1}]$ . It means, that either  $t \in \{d_{j(t)}, d_{j(t)+1}\}$  or  $t \in (d_{j(t)}, d_{j(t)+1})$ . We assume that  $t \in (d_{j(t)}, d_{j(t)+1})$ . Then for every  $n \geq N(\epsilon)$  the following relations hold:

$$\begin{aligned} \|x_n(t) - x(t)\|_E &= \left\| \int_0^t e^{A(t-s)} f_n(s) ds - \int_0^t e^{A(t-s)} f(s) ds \right\| \\ &\leq \left\| \sum_{i=1}^{j(t)} e^{A(t-d_i)} \int_{I_i} e^{A(d_i-s)} (f_{in}(s) - f(s)) ds \right\| \\ &+ \int_{d_{j(t)}}^t \|e^{A(t-s)} (f_n(s) - f(s))\| ds \leq M \sum_{i=1}^{j(t)} \left\| \int_{I_i} e^{A(d_i-s)} (f_{in}(s) - f(s)) ds \right\| \\ &+ M \left( \int_{I_{j(t)+1}} \|f_n(s)\| ds + \int_{I_{j(t)+1}} \|f(s)\| ds \right) < \frac{\epsilon}{3} + \frac{2\epsilon}{3}. \end{aligned}$$

So,  $x_n$  converges to  $x$  uniformly as  $n \rightarrow \infty$ , and, hence,  $\|x_n - x\|_C \rightarrow 0$ , i.e.  $x$  is a quasi-solution to (2.1).

Suppose now that  $x \in \mathcal{H}$ . Then there exists a sequence of measurable selections  $\{f_n\}_{n=1}^\infty \subset S(F(\cdot, x(\cdot)))$  such that the sequence

$$(2.8) \quad x_n(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f_n(s) ds$$

converges to  $x$  in  $C([0, d], E)$ . From (F2) it follows that the sequence  $\{f_n\}_{n=1}^\infty$  is integrably bounded. Moreover,  $\{f_n(t)\} \subset F(t, x(t))$  for a.e.  $t \in [0, d]$ . This means that the set  $\{f_n(t)\}_{n=1}^\infty$  is relatively compact for a.e.  $t \in [0, d]$ , hence  $\{f_n\}_{n=1}^\infty$  is semicompact and, according to Theorem 1.6, it is weakly compact in  $L^1([0, d], E)$ . So, without loss of generality, we can assume that  $f_n \rightarrow f$  weakly. Since  $f_n \in S(F(\cdot, x(\cdot))) \subset S(\overline{\text{co}}F(\cdot, x(\cdot)))$  for every  $n$ , then, according to [9, p. 124],  $f \in S(\overline{\text{co}}F(\cdot, x(\cdot)))$ . From the properties of Cauchy operator [9, p. 124], passing to a limit in (2.8), we get

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s) ds,$$

where  $f \in S(\overline{\text{co}}F(\cdot, x(\cdot)))$ , which means that  $x \in H_{\text{co}}$ . ■

### 3. REPRESENTATION OF THE SET OF MILD SOLUTIONS TO CONVEXIFIED PROBLEM

**Definition 3.1.** Let  $\psi \in K$ . We say that the function  $\eta \in K$  uniformly estimates the modulus of continuity of the multimap  $F$  from above with respect to the radius  $\psi$  on the set  $V \subset E$  if

$$(3.1) \quad \sup_{y \in B[x, \psi(t, \delta)]} h[F(t, x); F(t, y)] \leq \eta(t, \delta)$$

for a.e.  $t \in [0, d]$ , all  $x \in V$ , and  $\delta \in R_+$ .

**Lemma 3.2.** *Let the multimap  $F$  satisfy assumptions (F1), (F2) and  $V \in \mathcal{K}(E)$ . Then for every function  $\psi \in \tilde{K}$  there exists a function  $\eta \in K$  which uniformly estimates the modulus of continuity of the map  $F$  from above with respect to the radius  $\psi$  on the set  $V$ .*

**Proof.** Let us take  $\psi \in \tilde{K}$  and  $V \in \mathcal{K}(E)$ . We introduce a function  $\lambda(\psi, V) : [0, d] \times R_+ \rightarrow R_+$  by the formula

$$(3.2) \quad \lambda(\psi, V)(t, \delta) = \sup_{x \in V} \left\{ \sup_{y \in B[x, \psi(t, \delta)]} h[F(t, x); F(t, y)] \right\}.$$

This function belongs to the class  $K$  (the proof repeats the one in [4]), and, obviously, estimates the modulus of continuity of the map  $F$  on the set  $V$ , so it can be taken as the function  $\eta$ . ■

**Lemma 3.3.** *Let the multimap  $F$  be continuous by both arguments and satisfy assumption (F2), and let  $V \in \mathcal{K}(E)$ . Then for every function  $\eta \in P$  there exists a function  $\psi \in \tilde{P}$  with respect to which function  $\eta$  uniformly estimates the modulus of continuity of the map  $F$  from above on the set  $V$ .*

**Proof.** Indeed, let the set  $V$  be a compact subset of the space  $E$  and  $\delta > 0$ . Since  $\eta \in P$ , there exists a number  $r(\delta) > 0$ , such that  $r(\delta) \leq \eta(t, \delta)$  for a.e.  $t \in [0, d]$ . Further, since  $F$  is continuous by both arguments, it follows that it is uniformly continuous on  $[0, d] \times O_1V$ . Then, for  $r(\delta) \in (0, \infty)$  we can find a number  $\beta(r(\delta)) > 0$ , such that for every pair  $t, s \in [0, d]$  and every pair  $x, y \in O_1V$ , satisfying conditions  $|t - s| < \beta(r(\delta))$  and  $\|x - y\|_E < \beta(r(\delta))$ , the inequality  $h[F(t, x); F(s, y)] < r(\delta)$  takes place. We define the radius of continuity  $\psi$  as follows:

$$\psi(t, \delta) = \min\{r(\delta), \beta(r(\delta)), 1\}.$$

Then, for every  $(t, x) \in [0, d] \times V$  we have the relation

$$\sup_{y \in B[x, \psi(t, \delta)]} h[F(t, x); F(t, y)] < r(\delta) (\leq \eta(t, \delta))$$

which completes the proof. ■

Let us denote set  $V(H_{co})$  as follows:

$$V(H_{co}) = \{y \in E : \exists(t, x) \in [0, d] \times H_{co}, \quad x(t) = y\}.$$

**Remark 3.4.** Note, that since the set  $H_{co}$  is compact in the space  $C([0, d], E)$ , the set  $V(H_{co})$  is compact in  $E$ .

**Theorem 3.5.** *Let assumptions (A), (F1)–(F3) hold, and let  $\psi \in \tilde{P}$ . Then for every function  $\eta$  uniformly estimating the modulus of continuity of  $F$  from above with respect to the radius  $\psi$  on the set  $V(H_{\text{co}})$  the following equality holds true:*

$$(3.3) \quad H_{\text{co}} = \bigcap_{\delta > 0} \overline{H_{\eta(\delta)}}.$$

**Proof.** First, let us show that  $H_{\text{co}} \subset \bigcap_{\delta > 0} \overline{H_{\eta(\delta)}}$ . Let  $x \in H_{\text{co}}$  and  $\delta > 0$ . We claim that for every  $\tau > 0$  there exists a  $y \in H_{\eta(\delta)}$  such that  $\|x - y\|_C < \tau$ . According to the definition of a mild solution,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s) ds,$$

where  $f(s) \in \overline{\text{co}}F(s, x(s))$  for a.e.  $s \in [0, d]$ . Then from Lemma 2.7,  $x$  is a quasi-solution to the problem (2.1), i.e., the sequence  $\{x_n\}_{n=1}^\infty$  defined as

$$(3.4) \quad x_n(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f_n(s) ds,$$

where  $f_n \in S(F(\cdot, x(\cdot)))$ , converges to  $x$  in the space  $C([0, d], E)$ . Since  $\psi \in \tilde{P}$ , there exists a number  $n(\delta)$  such that  $x_n(t) \in B[x(t), \psi(t, \delta)]$  for all  $n \geq n(\delta)$  and a.e.  $t \in [0, d]$ . From the estimate (3.1) we get that for all  $n \geq n(\delta)$

$$\rho[f_n(t); F(t, x_n(t))] \leq h[F(t, x(t)); F(t, x_n(t))] \leq \eta(t, \delta),$$

hence  $f_n(t) \in O_{\eta(t, \delta)}F(t, x_n(t))$  and for every  $n \geq n(\delta)$  we have that  $x_n$  is a mild solution to the problem (2.3). Let  $n_0$  be a number such that  $\|x - x_{n_0}\|_C < \tau$  and  $n_0 \geq n(\delta)$ . Then  $y = x_{n_0}$  is a desired solution to the inclusion (2.3). Therefore  $H_{\text{co}} \subset \bigcap_{\delta > 0} \overline{H_{\eta(\delta)}}$ .

Now let  $x \in \bigcap_{\delta > 0} \overline{H_{\eta(\delta)}}$ . It means, that for every  $n = 1, 2, \dots$  there exists an  $x_n \in H_{\eta(\frac{1}{n})}$  such that  $\|x - x_n\|_C < \frac{1}{n}$ , and

$$x_n(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f_n(s) ds$$

with

$$(3.5) \quad f_n(s) \in O_{\eta(t, \frac{1}{n})}F(s, x_n(s))$$

for a.e.  $s \in [0, d]$ . Determine the sequence  $\{g_n\}_{n=1}^\infty \subset S(F(\cdot, x(\cdot)))$  as follows:

$$\rho[f_n(s); F(s, x(s))] = \|f_n(s) - g_n(s)\|_E$$

for a.e.  $s \in [0, d]$ , i.e.  $\{g_n\}_{n=1}^\infty \subset S(F(\cdot, x(\cdot)))$ . According to [14] such a sequence does exist. Then, from the inclusion (3.5), it follows that for a.e.  $s \in [0, d]$

$$\begin{aligned} (3.6) \quad \|f_n(s) - g_n(s)\|_E &\leq h \left[ O_{\eta(t, \frac{1}{n})} F(s, x_n(s)); F(s, x(s)) \right] \\ &\leq \eta \left( s, \frac{1}{n} \right) + h[F(s, x_n(s)); F(s, x(s))], \end{aligned}$$

and, according to the properties of function  $\eta$  and condition (F1), we have  $\|f_n(s) - g_n(s)\|_E \rightarrow 0$  as  $n \rightarrow \infty$  for a.e.  $s \in [0, d]$ .

Next, we consider the sequence  $\{z_n\}_{i=1}^\infty$  defined as

$$z_n(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}g_n(s) ds$$

and show that  $\|z_n - x\|_C \rightarrow 0$ . Indeed, from the properties of the integral multioperator [9] we get:

$$\|z_n - x\|_C \leq \|z_n - x_n\|_C + \|x_n - x\|_C \leq M\|f_n - g_n\|_1 + \|x_n - x\|_C \rightarrow 0$$

as  $n \rightarrow \infty$ , where  $M = \sup_{t \in [0, d]} \|e^{At}\|$ .

Then  $x$  is a quasisolution and from Lemma 2.7 we have  $x \in H_{co}$ . ■

**Theorem 3.6.** *Let assumptions (A), (F2), (F3) hold,  $F$  be continuous by both arguments, and let the set  $V(H_{co})$  be defined as in Theorem 3.5. Then for every function  $\eta \in P$  the equality (3.3) holds.*

**Proof.** The proof follows immediately from Lemma 3.3 and Theorem 3.5. ■

#### 4. SOLUTIONS SET STABILITY WITH RESPECT TO EXTERNAL PERTURBATIONS

In this brief section, we make some comments on the question of stability of the set of mild solutions to the problem (2.1). Note that stability in the sense

represented below is a specific issue of the theory of differential inclusions since it has to do with the multivaluedness of the right-hand side of the inclusion. The question as such cannot occur in the theory of differential equations.

Let  $\eta \in K$  and consider the right-hand side of inclusion (2.3). Its nonlinear part can be represented by a multimap  $\tilde{F} : [0, d] \times E \times R_+ \rightarrow \mathcal{K}(E)$  defined as

$$\tilde{F}(t, x, \delta) = O_{\eta(t, \delta)} F(t, x)$$

and satisfying for every  $x \in E$  and a.e.  $t \in [0, d]$  the property

$$(4.1) \quad \lim_{\delta \rightarrow +0} h[\tilde{F}(t, x, \delta); F(t, x)] = 0.$$

This means that all the multimaps  $\tilde{F}$  defined as above and depending on the function  $\eta \in K$  and parameter  $\delta$  are “close” (in the sense of equality (4.1)) to a multimap  $F$  representing the right-hand side nonlinearity of inclusion (2.1). And this leads to a natural question: under which conditions does the equality

$$(4.2) \quad \overline{H} = \bigcap_{\delta > 0} \overline{H_{\eta(\delta)}}$$

hold for every function  $\eta \in K$ ? In other words, when do the “small” changes of the right-hand side of the differential inclusion (2.1) “insignificantly” change the set of its solutions?

**Definition 4.1.** We say that differential inclusion (2.1) is stable under external perturbations from class  $K$  (or  $P$ ), if for any function  $\eta \in K$  (or  $P$ ) equality (4.2) holds.

We remind that if the relation

$$(4.3) \quad \overline{H} = H_{co}$$

takes place, we say that the density principle holds for the problem (2.1).

**Theorem 4.2.** *Let assumptions (A), (F1)–(F3) be satisfied. Then differential inclusion (2.1) is stable under external perturbations from class  $K$  if and only if the density principle for (2.1) holds.*

**Proof.** Let equality (4.2) hold true for every function  $\eta \in K$ . Then, according to Theorem 3.5, there is a function  $\lambda \in K$  for which we have the relation

$$(4.4) \quad H_{\text{co}} = \bigcap_{\delta > 0} \overline{H_{\lambda(\delta)}}.$$

So, from (4.2) and (4.4) we get equality (4.3).

Now let the density principle hold for inclusion (2.1) and let  $\eta \in K$ . Then from the definition of the set  $H_{\eta(\delta)}$  we have  $H \subset H_{\eta(\delta)}$  and hence  $\overline{H} \subset \overline{H_{\eta(\delta)}}$  for every  $\delta > 0$ . It gives us

$$(4.5) \quad \overline{H} \subset \bigcap_{\delta > 0} \overline{H_{\eta(\delta)}}.$$

From the second part of the proof of Theorem 3.5 it follows that for every  $\eta \in K$

$$(4.6) \quad H_{\text{co}} \supset \bigcap_{\delta > 0} \overline{H_{\eta(\delta)}},$$

and from equality (4.3) we get an inclusion opposite to (4.5). ■

**Remark 4.3.** Note that according to Theorem 3.6, the density principle also represents the necessary and sufficient condition for an inclusion to be stable under external perturbations from a more narrow class of functions, i.e., set  $P$ .

**Remark 4.4.** The density principle always holds for a differential equation, since its right-hand side is a convex set (a single point). It means that the differential equation (as a particular case of a differential inclusion) is always stable under external perturbations in the sense mentioned above.

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