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A FIBERING METHOD APPROACH TO A SYSTEM OF QUASILINEAR EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract

We provide an existence result for a system of quasilinear equations subject to nonlinear boundary conditions on a bounded domain by using the fibering method.

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1. Introduction

In this paper, we study the following quasilinear system

$$(1) \qquad \begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega \\ \Delta_q v = |v|^{q-2}v & \text{in } \Omega\,, \end{cases}$$

subject to the nonlinear boundary conditions

$$(2)\qquad \begin{cases} |\nabla u|^{p-2}\nabla u\cdot\eta = \lambda a(x)|u|^{p-2}u + c(x)|u|^{\alpha-1}u|v|^{\beta+1} & \text{on } \partial\Omega \\ |\nabla v|^{q-2}\nabla v\cdot\eta = \mu b(x)|v|^{q-2}v + c(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{on } \partial\Omega \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$, $\alpha, \beta, \lambda, \mu, p, q$ are real numbers, Δ_p and Δ_q are the p- and q-Laplace operators, $a(.)$ and $b(.)$ are nonnegative weights and $c(.)$ is an integrable function. The system (1)–(2) is related to the eigenvalue problem for the p-Laplacian

$$(3)\qquad \begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\nabla u\cdot\eta = \lambda a(x)|u|^{p-2}u & \text{on } \partial\Omega\,, \end{cases}$$

which, in the case $p = 2$, is known as the *Steklov* problem, [1]. Our main tool in the study of this system is the fibering method which was introduced by Pohozaev, [4, 5], for a single equation and developed for systems by Bozhkov and Mitidieri [3].

The main result of this work is the following theorem. For the related definitions and hypotheses we refer to Section 2.

Theorem 1. *Assume that hypotheses* (H)–H(c) *are satisfied,* $0 \leq \lambda < \lambda_1$ *and* $0 \leq \mu < \mu_1$. *Then the system* (1)–(2) *admits at least one weak solution* $(u^*, v^*) \in E$ *with* $u^*, v^* > 0$ *on* Ω.

2. Notation and preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $p > 1$, $q > 1$. We assume that the Sobolev spaces $X = W^{1,p}(\Omega)$ and $Y = W^{1,q}(\Omega)$ are supplied with the norms

$$||u||_{1,p} = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx\right)^{\frac{1}{p}},\ ||v||_{1,q} = \left(\int_\Omega |\nabla v|^q dx + \int_\Omega |v|^q dx\right)^{\frac{1}{q}},$$

respectively. We denote $E = X \times Y$ and for $(u, v) \in E$,

$$||(u, v)|| = ||u||_{1,p} + ||v||_{1,q}.$$

We state the hypotheses that we shall use throughout this paper.

(H) $\alpha, \beta, p > 1$, $q > 1$ are positive real numbers such that $\alpha + 1 < p$, $\beta + 1 < q$ and $\delta := pq - q(\alpha + 1) - p(\beta + 1) > 0$.

(Ha) $a(.) \in L^s(\partial\Omega)$ with $a \geq 0$ on $\partial\Omega$ and $m\{x \in \partial\Omega : a(x) > 0\} > 0$, where $s > (N-1)/(p-1)$ if $1 < p \leq N$ and $s \geq 1$ if $p \geq N$.

(Hb) $b(.) \in L^t(\partial\Omega)$ with $b \geq 0$ on $\partial\Omega$ and $m\{x \in \partial\Omega : b(x) > 0\} > 0$, where $t > (N-1)/(q-1)$ if $1 < q \leq N$ and $t \geq 1$ if $q \geq N$.

(Hc) $\frac{c(.)}{a(.)^{\frac{\alpha+1}{p}} b(.)^{\frac{\beta+1}{q}}} \in L^{pq/\delta}(\partial\Omega)$ and $c^+ \neq 0$ on $\partial\Omega$.

Remark 2. Since $\delta > 0$, we have that

$$\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} < 1.$$

The proof of the following Lemma can be found in ([2]).

Lemma 3. *Suppose that $a(.)$ satisfies* (Ha). *Then*

(i) *there exists a real number λ_1, the first eigenvalue of* (3), *such that*

$$\frac{1}{\lambda_1} = \sup_{C \in C_1} \min_{u \in C} \frac{\int_{\partial\Omega} a|u|^p dx}{\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx}, \tag{4}$$

where $C_1 = \{C \subset W^{1,p}(\Omega) : C$ is compact, symmetric and $\gamma(C) \geq 1\}$, $\gamma(.)$ being the genus function.

(ii) *λ_1 is simple and isolated.*

(iii) *if u is an eigenfunction corresponding to λ_1 then $u \in C^{1,\alpha}(\Omega)$ and does not change sign in Ω.*

Remark 4. As a consequence of (4) we have

$$\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx \geq \lambda_1 \int_{\partial\Omega} a|u|^p dx \tag{5}$$

for every $u \in X$.

Consider the functionalI $\Phi(\cdot,\cdot)$ defined on E by

$$\Phi(u,v)=\frac{\alpha+1}{p}\left(\int_\Omega|\nabla u|^p dx+\int_\Omega|u|^p dx-\lambda\int_{\partial\Omega}a|u|^p dx\right) \tag{6}$$
$$+\frac{\beta+1}{q}\left(\int_\Omega|\nabla v|^q dx+\int_\Omega|v|^q dx-\mu\int_{\partial\Omega}b|v|^q dx\right)-\int_{\partial\Omega}c|u|^{\alpha+1}|v|^{\beta+1}dx$$

Then $\Phi(\cdot,\cdot)$ is of class C^1 and its critical points give rise to weak solutions of (1)–(2).

3. The fibering method

In this Section, we give a brief description of the fibering method. Let $L : E \to R$ be a C^1-functional. We express $(z,w)\in E$ in the form

$$z=ru,\quad w=sv \tag{7}$$

where $r,s\in\mathbb{R}$ and $u,v\in E$.If $(z,w)=(ru,sv)$ is a critical point of $L(\cdot,\cdot)$ then

$$\frac{\partial L(ru,sv)}{\partial r}=0\quad\text{and}\quad\frac{\partial L(ru,sv)}{\partial s}=0. \tag{8}$$

Assume that (8) can be uniquely solved for $r=r(u)$ and $s=s(v)$ and that the functions $r(\cdot)$ and $s(\cdot)$ are continuously differentiable. In addition, let $H : E\to R$ and $F : E\to R$ be continuously differentiable functions such that if

$$H(u,v)=c_1,\qquad F(u,v)=c_2, \tag{9}$$

for some $c_1,c_2\in R$, then their Gateaux derivatives H' and F' at $(u,v)\in E$ satisfy

$$\det\begin{pmatrix} H'(u,v)(u,0) & F'(u,v)(u,0)\\ H'(u,v)(0,v) & F'(u,v)(0,v)\end{pmatrix}\neq 0. \tag{10}$$

We have the following:

Proposition 5 ([3, Lemma 4]). *Let $\widehat{L}(u,v) := L(r(u)u,s(v)v)$ and assume that* (10) *holds f or u,v satisfying* (9). *If (u,v) is a conditional critical point of $\widehat{L}(\cdot,\cdot)$ with the conditions* (9), *then $(r(u)u,s(v)v)$ is a critical point of $L(\cdot,\cdot)$.*

4. Proof of Theorem 1

We apply the fibering method to the functional Φ defined in (6). Combining (7) and (6) we get that

$$
\begin{aligned}
\Phi(ru,sv) \;=\; & \frac{\alpha+1}{p}|r|^p\left(\int_\Omega |\nabla u|^p dx+\int_\Omega |u|^p dx-\lambda\int_{\partial\Omega} a|u|^p dx\right)\\
&+\frac{\beta+1}{q}|s|^q\left(\int_\Omega |\nabla v|^q dx+\int_\Omega |v|^q dx-\mu\int_{\partial\Omega} b|v|^q dx\right)\\
&-|r|^{\alpha+1}|s|^{\beta+1}\int_{\partial\Omega} c|u|^{\alpha+1}|v|^{\beta+1}dx.
\end{aligned}
\tag{11}
$$

If $(z,w)=(ru,sv)$ is a critical point of $\Phi(\cdot,\cdot)$, then by (8)

$$
\begin{aligned}
0=&|r|^{p-2}r\left(\int_\Omega |\nabla u|^p dx+\int_\Omega |u|^p dx-\lambda\int_{\partial\Omega} a|u|^p dx\right)\\
&-|r|^{\alpha-1}r|s|^{\beta+1}\int_{\partial\Omega} c|u|^{\alpha+1}|v|^{\beta+1}dx
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
0=&|s|^{q-2}s\left(\int_\Omega |\nabla v|^q dx+\int_\Omega |v|^q dx-\mu\int_{\partial\Omega} b|v|^q dx\right)\\
&-|r|^{\alpha+1}|s|^{\beta-1}s\int_{\partial\Omega} c|u|^{\alpha+1}|v|^{\beta+1}dx.
\end{aligned}
\tag{13}
$$

Let

$$
H(u):=\left(\int_\Omega |\nabla u|^p dx+\int_\Omega |u|^p dx-\lambda\int_{\partial\Omega} a|u|^p dx\right), \tag{14}
$$

$$
F(v):=\left(\int_\Omega |\nabla v|^q dx+\int_\Omega |v|^q dx-\mu\int_{\partial\Omega} b|v|^q dx\right) \tag{15}
$$

and

$$
C(u,v):=\int_{\partial\Omega} c|u|^{\alpha+1}|v|^{\beta+1}dx. \tag{16}
$$

The equations (12) and (13) in view of (14)–(16), become

$$\begin{cases} |r|^{p-\alpha-1}H(u) - |s|^{\beta+1}C(u,v) = 0 \\ |s|^{q-\beta-1}F(v) - |r|^{\alpha+1}C(u,v) = 0. \end{cases} \tag{17}$$

It is clear that $H(u)$, $F(v)$ and $C(u,v)$ must have the same sign. In view of Remark 4, $H(u)$, $F(v) > 0$, thus $C(u,v) > 0$.

It is not difficult to see that the solution to (17) is

$$\begin{cases} |r| = C(u,v)^{\frac{q}{\delta}} F(v)^{-\frac{\beta+1}{\delta}} H(u)^{\frac{\beta+1-q}{\delta}} \\ |s| = C(u,v)^{\frac{p}{\delta}} H(u)^{-\frac{\alpha+1}{\delta}} F(v)^{\frac{\alpha+1-p}{\delta}} \end{cases} \tag{18}$$

where δ is defined in (H). For $(u,v) \in E$ with $u \neq 0$, $v \neq 0$, let $r = r(u) > 0$ and $s = s(v) > 0$ be the positive solutions of (18). Combining (11) and (18), we conclude that

$$\widehat{\Phi}(u,v) := \Phi(ru, sv) = \zeta C(u,v)^{\frac{pq}{\delta}} H(u)^{-\frac{q(\alpha+1)}{\delta}} F(v)^{-\frac{p(\beta+1)}{\delta}} \tag{19}$$

where $\zeta = \frac{\alpha+1}{p} + \frac{\beta+1}{q}$. Clearly, any solution to the problem

$$\max C(u,v)$$

with the restrictions

$$H(u) = 1 \quad \text{and} \quad F(v) = 1,$$

is a conditional critical point of $\widehat{\Phi}$, providing this way a solution to (1)–(2) by Proposition 5, since (10) is satisfied on

$$G := \{(u,v) : H(u) = 1 \quad \text{and} \quad F(v) = 1\}. \tag{20}$$

We show next that $C : E \to R$ is compact. To this end let $\{(u_n, v_n)\}$, $n \in N$, be a bounded sequence in E. Without loss of generality we may assume that $u_n \to u$ weakly in X and $v_n \to v$ weakly in Y. Due to the compactness of the embeddings $X \subseteq L^p(a, \partial\Omega)$ and $Y \subseteq L^q(b, \partial\Omega)$ we have that $u_n \to u$ and $v_n \to v$ strongly in $L^p(a, \partial\Omega)$ and $L^q(b, \partial\Omega)$ respectively.

By (16),

$$|C(u_n,v_n)-C(u,v)| = \left|\int_{\partial\Omega} c|u_n|^{\alpha+1}|v_n|^{\beta+1}dx - \int_{\partial\Omega} c|u|^{\alpha+1}|v|^{\beta+1}dx\right|$$

$$\leq \int_{\partial\Omega} |c||u_n|^{\alpha+1}\left||v_n|^{\beta+1}-|v|^{\beta+1}\right|dx + \int_{\partial\Omega}|c||v|^{\beta+1}\left||u_n|^{\alpha+1}-|u|^{\alpha+1}\right|dx.$$

By the Lebesgue dominated convergence theorem,

$$\int_{\partial\Omega} |c||u_n|^{\alpha+1}\left||v_n|^{\beta+1}-|v|^{\beta+1}\right|dx$$

$$= \int_{\partial\Omega} \frac{|c|}{a^{\frac{\alpha+1}{p}} b^{\frac{\beta+1}{q}}} a^{\frac{\alpha+1}{p}}\left|u_n^{\alpha+1}\right| b^{\frac{\beta+1}{q}}\left||v_n|^{\beta+1}-|v|^{\beta+1}\right|dx$$

$$\leq \left\|\frac{c}{a^{\frac{\alpha+1}{p}} b^{\frac{\beta+1}{q}}}\right\|_{\frac{pq}{\delta}} \|u_n\|_{a,p}^{\alpha+1}\left(\int_{\partial\Omega} b\left||v_n|^{\beta+1}-|v|^{\beta+1}\right|^{\frac{q}{\beta+1}}dx\right)^{\frac{\beta+1}{q}} \to 0.$$

Similarly,

$$\int_{\partial\Omega}|c||v|^{\beta+1}\left||u_n|^{\alpha+1}-|u|^{\alpha+1}\right|dx \to 0.$$

Thus C is compact.

If $(u,v)\in G$, then

$$\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx = \lambda\int_{\partial\Omega} a|u|^p dx + 1$$

$$\leq \frac{\lambda}{\lambda_1}\left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx\right) + 1,$$

so

$$||u||_{1,p}^p = \int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx \leq \frac{\lambda_1}{\lambda_1-\lambda}.$$

Similarly, $||v||_{1,q} \leq \frac{c\mu_1}{\mu_1-\mu}$. Consequently, $C(\cdot,\cdot)$ is bounded on G. Let $\{(u_n,v_n)\}_{n\in N}$ be a maximizing sequence in G, that is, $|C(u_n,v_n)|\to M := \sup\{|C(u,v)| : (u,v)\in G\} > 0$. We may assume that $u_n \to u_0$ weakly in X

and $v_n \to v_0$ weakly in Y. Since C is compact $|C(u_0, v_0)| = M$. It remains to show that $(u_0, v_0) \in G$. Note that, by (20), $H(u_0) \leq 1$ and $F(v_0) \leq 1$. If one or both of these two inequalities were strict, there would exist $t_1, t_2 \in R$ with $\max\{t_1, t_2\} > 1$, such that $H(t_1 u_0) = 1$ and $F(t_2 v_0) = c$. But then $(t_1 u_0, t_2 v_0) \in G$ and $|C(t_1 u_0, t_2 v_0)| > M$, a contradiction. Thus (u_0, v_0) is a conditional critical point of $\widehat{\Phi}$. Since $(|u_0|, |v_0|)$ is also a conditional critical point of $\widehat{\Phi}$, we may assume that $u_0 \geq 0$ and $v_0 \geq 0$ in Ω. Proposition 5, guarantees that $(u^*, v^*) := (r(u_0)u_0, s(v_0)v_0)$ is a critical point of $\Phi(.)$. By standard arguments, we can show that $u^*, v^* \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, and so $u^*, v^* > 0$ in Ω due to the Harnack inequality.

Remark 6.

(i) If $u^*, v^* \in C^{1,\alpha}(\overline{\Omega})$, then $u^*, v^* > 0$ on $\overline{\Omega}$. Indeed, if we assume that $u^*(x_0) = 0$ for some $x_0 \in \partial\Omega$ then by Theorem 5 in [6], $\nabla u^*(x_0) \cdot \eta(x_0) < 0$, contradicting (2). Thus $u^* > 0$ on $\overline{\Omega}$. Similarly $v^* > 0$ on $\overline{\Omega}$.

(ii) The fibering method is a powerful tool in proving that an equation or a system admits a solution. However, at least in the case where the system (8) has a unique solution, it cannot provide more than one solution. This is a consequence of the fact that the maximum or the minimum of L in Proposition 5 is independent of the choice of H and F.

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