

THE METHOD OF UPPER AND LOWER SOLUTIONS
FOR PERTURBED n^{th} ORDER DIFFERENTIAL
INCLUSIONS

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Abstract

In this paper, an existence theorem for n^{th} order perturbed differential inclusion is proved under the mixed Lipschitz and Carathéodory conditions. The existence of extremal solutions is also obtained under certain monotonicity conditions on the multi-functions involved in the inclusion. Our results extend the existence results of Dhage *et al.* [7, 8] and Agarwal *et al.* [1].

Keywords: differential inclusion, method of upper and lower solutions, existence theorem.

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1. INTRODUCTION

Let \mathbb{R} denote the real line and let $J = [0, a]$ be a closed and bounded interval in \mathbb{R} . Consider the initial value problem of n^{th} order perturbed differential inclusion (in short PDI)

$$(1) \quad \begin{cases} x^{(n)}(t) \in F(t, x(t)) + G(t, x(t)) \text{ a.e. } t \in J, \\ x^{(i)}(0) = x_i \in \mathbb{R} \end{cases}$$

where $F, G : J \times \mathbb{R} \rightarrow P(\mathbb{R})$, $i \in \{0, 1, \dots, n-1\}$ and $P(\mathbb{R})$ is the set of all nonempty subsets of \mathbb{R} .

By a solution of problem (1) we mean a function $x \in AC^{n-1}(J, \mathbb{R})$ whose n^{th} derivative $x^{(n)}$ exists and is a member of $L^1(J, \mathbb{R})$ in $F(t, x)$, i.e., there exists a $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, x(t)) + G(t, x(t))$ a.e. $t \in J$, and $x^{(n)}(t) = v(t)$, $t \in J$ and $x^{(i)}(0) = x_i \in \mathbb{R}$, $i \in \{0, 1, \dots, n-1\}$, where $AC^{n-1}(J, \mathbb{R})$ is the space of all continuous real-valued functions whose $(n-1)$ derivatives exist and are absolutely continuous on J .

The method of upper and lower solutions has been successfully applied to the problems of nonlinear differential equations and inclusions. We refer to Heikkilä and Lakshmikantham [10], Halidias and Papageorgiou [9], and Benchohra [3]. In this paper, we apply the multi-valued version of Krasnoselskii's fixed point theorem due to Dhage [5] to IVP (1) for proving the existence of solutions between the given lower and upper solutions, using the Carathéodory condition on F . A few details of differential inclusions and multi-valued analysis appear in Aubin and Cellina [2] and Hu and Papageorgiou [11] respectively.

2. PRELIMINARIES

Throughout this paper, X will be a Banach space and let $P(X)$ denote the set of all nonempty subsets of X . By $P_{bd,cl}(X)$ and $P_{cp,cv}(X)$ we will denote the classes of all nonempty, bounded, closed and respectively compact, convex subsets of X . For $x \in X$ and $Y, Z \in P_{bd,cl}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$ and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$.

The function $H : P_{bd,cl}(X) \times P_{bd,cl}(X) \rightarrow \mathbb{R}^+$ defined by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

is called the Pompeiu-Hausdorff metric on X . Note that $\|Y\| = H(Y, \{0\})$.

$T : X \rightarrow P(X)$ is called a multi-valued mapping on X . A point $x_0 \in X$ is called a fixed point of the multi-valued operator $T : X \rightarrow P(X)$ if $x_0 \in T(x_0)$. The fixed point set of T will be denoted by $Fix(T)$.

Definition 2.1. Let $T : X \rightarrow P_{bd,cl}(X)$ be a multi-valued operator. Then T is called a multi-valued contraction if there exists a constant $k \in (0, 1)$ such that for each $x, y \in X$ we have

$$H(T(x), T(y)) \leq k\|x - y\|.$$

The constant k is called a contraction constant of T .

We apply in the sequel the following form of a fixed point theorem given by Dhage [4].

Theorem 2.2. Let X be a Banach space and $A : X \rightarrow P_{cl,cv,bd}(X)$, $B : X \rightarrow P_{cp,cv}(X)$ two multi-valued operators satisfying:

- (i) A is a contraction with a contraction constant k
- (ii) B is u.s.c. and compact.

Then either

- (i) the operator inclusion $\lambda x \in Ax + Bx$ has a solution for $\lambda = 1$
or
- (ii) the set $\mathcal{E} = \{u \in X \mid \lambda x \in Ax + Bx, \lambda > 1\}$ is unbounded.

We also need the following definitions.

Definition 2.3. Let J be an interval of the real axis. A multi-valued map $F : J \rightarrow P_{cp,cv}(\mathbb{R})$ is said to be measurable if for every $y \in X$, the function $t \rightarrow d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.

Definition 2.4. A multi-valued map $F : J \times \mathbb{R} \rightarrow P(\mathbb{R})$ is said to be L^1 -Carathéodory if

- (i) $t \rightarrow F(t, x)$ is measurable for each $x \in \mathbb{R}$,
- (ii) $x \rightarrow F(t, x)$ is upper semi-continuous for almost all $t \in J$, and

(iii) for each real number $k > 0$, there exists a function $h_k \in L^1(J, \mathbb{R})$ such that

$$\|F(t, x)\| = \sup\{|u| : u \in F(t, x)\} \leq h_k(t), \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq k$.

Denote

$$S_F^1(x) = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \text{ a.e. } t \in J\}.$$

Then we have the following lemmas due to Lasota and Opial [12].

Lemma 2.1. *If $\dim(E) < \infty$ and $F : J \times E \rightarrow P(E)$ is L^1 -Carathéodory, then $S_F^1(x) \neq \emptyset$ for each $x \in E$.*

Lemma 2.2. *Let X be a Banach space, F an L^1 -Carathéodory multi-valued map with $S_F^1 \neq \emptyset$ and $K : L^1(J, \mathbb{R}) \rightarrow C(J, E)$ be a linear continuous mapping. Then the operator*

$$\mathcal{K} \circ S_F^1 : C(J, E) \longrightarrow KC(E)$$

has a closed graph in $C(J, E) \times C(J, E)$.

We define the partial ordering \leq in $W^{n,1}(J, \mathbb{R})$ (the Sobolev class of functions $x : J \rightarrow \mathbb{R}$ for which $x^{(n-1)}$ are absolutely continuous and $x^{(n)} \in L^1(J, \mathbb{R})$) as follows. Let $x, y \in W^{n,1}(J, \mathbb{R})$. Then we define

$$x \leq y \Leftrightarrow x(t) \leq y(t), \quad \text{for all } t \in J.$$

If $a, b \in W^{n,1}(J, \mathbb{R})$ and $a \leq b$, then we define an order interval $[a, b]$ in $W^{n,1}(J, \mathbb{R})$ by

$$[a, b] = \{x \in W^{n,1}(J, \mathbb{R}) : a \leq x \leq b\}.$$

The following definition appears in Dhage *et al.* [1].

Definition 2.5. A function $\alpha \in W^{n,1}(J, \mathbb{R})$ is called a lower solution of problem (1) if for all $v_1 \in L^1(J, \mathbb{R})$ with $v_1(t) \in F(t, \alpha(t))$ and $v_2 \in L^1(J, \mathbb{R})$

with $v_2(t) \in G(t, \alpha(t))$ a.e. $t \in J$ we have that $\alpha^{(n)}(t) \leq v_1(t) + v_2(t)$ a.e. $t \in J$ and $\alpha^{(i)}(0) \leq x_i, i = 0, 1, \dots, n-1$. Similarly, a function $\beta \in W^{n,1}(J, \mathbb{R})$ is called an upper solution of problem (1) if for all $v_1 \in L^1(J, \mathbb{R})$ with $v_1(t) \in F(t, \alpha(t))$ and $v_2 \in L^1(J, \mathbb{R})$ with $v_2(t) \in G(t, \alpha(t))$ a.e. $t \in J$ we have that $\alpha^{(n)}(t) \geq v_1(t) + v_2(t)$ a.e. $t \in J$ and $\alpha^{(i)}(0) \geq x_i, i = \{0, 1, \dots, n-1\}$.

Now we are ready to prove our main existence result for the IVP (1) in the following section under suitable conditions on the multi-function F and G .

3. EXISTENCE RESULT

We consider the following set of assumptions in the sequel.

- (H₁) The multi-function $t \rightarrow F(t, x)$ is measurable and integrably bounded for each $x \in \mathbb{R}$.
- (H₂) There exists a function $k \in L^1(J, \mathbb{R})$ such that the multi-function $F : J \times \mathbb{R} \rightarrow P_{cl,cv,bd}(\mathbb{R})$ satisfies

$$H(F(t, x), F(t, y)) \leq k(t)|x - y| \text{ a.e. } t \in J$$

for all $x, y \in \mathbb{R}$.

- (H₃) The multi $G(t, x)$ has compact and convex values for each $(t, x) \in J \times \mathbb{R}$.
- (H₄) $G(t, x)$ is L^1 -Carathéodory.
- (H₅) There exists a function $\phi \in L^1(J, \mathbb{R})$ with $\phi(t) > 0$ a.e. $t \in J$ and a nondecreasing function $\psi : \mathbb{R}^+ \rightarrow (0, \infty)$ such that

$$\|G(t, x)\| \leq \phi(t)\psi(|x|) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$.

- (H₆) Problem (1) has a lower solution α and an upper solution β with $\alpha \leq \beta$.

We use the following lemma in the sequel.

Lemma 3.1. *Suppose that hypothesis (H₃) holds. Then for any $a \in F(t, x)$,*

$$|a| \leq k(t)|x| + \|F(t, 0)\|, \quad t \in J$$

for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$ be arbitrary. Then by the triangle inequality

$$\begin{aligned} \|F(t, x)\| &= H(F(t, x), 0) \\ &\leq H(F(t, x), F(t, 0)) + H(F(t, 0), 0) \\ &\leq H(F(t, x), F(t, 0)) + \|F(t, 0)\|, \end{aligned}$$

for all $t \in J$. Hence for any $a \in F(t, x)$,

$$\begin{aligned} |a| &\leq \|F(t, x)\| \\ &\leq H(F(t, x), F(t, 0)) + \|F(t, 0)\| \\ &\leq k(t)|x| + \|F(t, 0)\|, \end{aligned}$$

for all $t \in J$. The proof of the lemma is complete. ■

Theorem 3.1. Assume that (H_1) – (H_6) hold. Suppose that

$$(2) \quad \int_{C_1}^{\infty} \frac{ds}{\psi(s)} ds > C_2 \|\phi\|_{L^1}$$

where

$$C_1 = \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} (\|\alpha\| + \|\beta\|) (\|k\|_{L^1} + L), \quad L = \int_0^1 \|F(s, 0)\| ds$$

and $C_2 = \frac{a^{n-1}}{(n-1)!}$. Further, if $\frac{a^{n-1}}{(n-1)!} \|k\|_{L^1} < 1$, then the IVP (1) has at least one solution x such that

$$\alpha(t) \leq x(t) \leq \beta(t), \quad \text{for all } t \in J.$$

Proof. First, we transform the IVP (1) into a fixed point inclusion in a suitable Banach space. Consider the following problem

$$(3) \quad \begin{cases} x^{(n)}(t) \in F(t, \tau x(t)) + G(t, \tau x(t)) \quad \text{a.e. } t \in J, \\ x^{(i)}(0) = x_i \in \mathbb{R} \end{cases}$$

for all $i \in \{0, 1, \dots, n-1\}$, where $\tau : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ is the truncation operator defined by

$$(4) \quad (\tau x)(t) = \begin{cases} \alpha(t), & \text{if } x(t) < \alpha(t) \\ x(t), & \text{if } \alpha(t) \leq x(t) \leq \beta(t) \\ \beta(t), & \text{if } \beta(t) < x(t). \end{cases}$$

The problem of existence of a solution of problem (1) reduces to finding the solution of the integral inclusion

$$(5) \quad x(t) \in \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s, \tau x(s)) ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} G(s, \tau x(s)) ds, \quad t \in J.$$

We study the integral inclusion (5) in the space $C(J, \mathbb{R})$ of all continuous real-valued functions on J with a supremum norm $\|\cdot\|$. Define two multi-valued maps $A, B : C(J, \mathbb{R}) \rightarrow P_f(C(J, \mathbb{R}))$ by

$$(6) \quad Ax = \left\{ u \in C(J, \mathbb{R}) : u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds, \quad v \in \overline{S_F^1}(\tau x) \right\}$$

$$(7) \quad Bx = \left\{ u \in C(J, \mathbb{R}) : u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds, \quad v \in \overline{S_G^1}(\tau x) \right\}$$

where

$$\overline{S_F^1}(\tau x) = \{v \in S_F^1(\tau x) : v(t) \geq \alpha(t) \text{ a.e. } t \in A_1 \text{ and } v(t) \leq \beta(t), \text{ a.e. } t \in A_2\}$$

$$\overline{S_G^1}(\tau x) = \{v \in S_G^1(\tau x) : v(t) \geq \alpha(t) \text{ a.e. } t \in A_1 \text{ and } v(t) \leq \beta(t), \text{ a.e. } t \in A_2\}$$

and

$$A_1 = \{t \in J : x(t) < \alpha(t) \leq \beta(t)\},$$

$$A_2 = \{t \in J : \alpha(t) \leq \beta(t) < x(t)\},$$

$$A_3 = \{t \in J : \alpha(t) \leq x(t) \leq \beta(t)\}.$$

By Lemma 2.1, $S_F^1(\tau x) \neq \emptyset$ for each $x \in C(J, \mathbb{R})$ which further yields that $\overline{S_F^1}(\tau x) \neq \emptyset$ for each $x \in C(J, \mathbb{R})$. Indeed, if $v \in S_F^1(x)$ then the function $w \in L^1(J, \mathbb{R})$ defined by

$$w = \alpha\chi_{A_1} + \beta\chi_{A_2} + v\chi_{A_3},$$

is in $\overline{S_F^1}(\tau x)$ by virtue of decomposability of w . Similarly, $\overline{S_G^1}(\tau x) \neq \emptyset$ for each $x \in C(J, \mathbb{R})$.

We shall show that the operators A and B satisfy all the conditions of Theorem 2.2.

Step I. First, we show that Ax is a closed convex and bounded subset of X for each $x \in X$. This follows easily if we show that the values of Nemytzki operator are closed in $L^1(J, \mathbb{R})$. Let $\{w_n\}$ be a sequence in $L^1(J, \mathbb{R})$ converging to a point w . Then $w_n \rightarrow w$ in measure and so, there exists a subsequence S of positive integers with w_n converging a.e. to w as $n \rightarrow \infty$ through S . Now since (H_1) holds, the values of S_F^1 are closed in $L^1(J, \mathbb{R})$. Thus for each $x \in X$ we have that Ax is a non-empty and closed subset of X .

First, we prove that $A(x)$ are convex subsets of $C(J, \mathbb{R})$ for all $x \in C(J, \mathbb{R})$. Let $u_1, u_2 \in A(x)$. Then there exist v_1 and v_2 in $\overline{S_F^1}(\tau x)$ such that

$$u_j(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_j(s) ds, \quad j = 1, 2.$$

Since $F(t, x)$ has convex values, one has for $0 \leq k \leq 1$

$$[kv_1 + (1-k)v_2](t) \in S_F^1(\tau x)(t), \quad \forall t \in J.$$

As a result we have

$$[ku_1 + (1-k)u_2](t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} [kv_1(s) + (1-k)v_2(s)] ds.$$

Therefore $[ku_1 + (1-k)u_2] \in Ax$ and consequently Ax has convex values in $C(J, \mathbb{R})$. Thus $A : X \rightarrow P_{cl,cv,bd}(X)$. Again from hypothesis (H_1) it follows that Ax is a bounded subset of X for each $x \in X$. Thus we have $A : X \rightarrow P_{cl,cv,bd}(X)$.

Step II. Next we show that B has compact values on X . Now the operator B is equivalent to the composition $\mathcal{L} \circ S_F^1$ of two operators on $L^1(J, \mathbb{R})$, where $\mathcal{L} : L^1(J, \mathbb{R}) \rightarrow X$ is defined by

$$\mathcal{L}v(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds$$

To show B that has compact values, it is enough that the composition operator $\mathcal{L} \circ S_G^1$ has compact values on X . It can be shown as in the Step IV below that $(\mathcal{L} \circ S_F^1)(x)$ is a compact subset of X for each $x \in X$. Further as in the case of operator A it can be shown that B has convex values on X . Thus we have $B : X \rightarrow P_{cp,cv}(X)$.

Step III. Next we show that A is a multi-valued contraction on X . Let $x, y \in X$ and $u_1 \in A(x)$. Then $u_1 \in X$ and $u_1(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_1(s) ds$ for some $v_1 \in S_F^1(x)$. Since $H(F(t, x(t)), F(t, y(t))) \leq k(t)\|x(t) - y(t)\|$, one obtains that there exists $w \in F(t, y(t))$ such that $\|v_1(t) - w\| \leq k(t)\|x(t) - y(t)\|$. Thus the multi-valued operator U defined by $U(t) = S_F^1(y)(t) \cap K(t)$, where and

$$K(t) = \{w \mid \|v_1(t) - w\| \leq k(t)\|x(t) - y(t)\|\}$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U (which exists by Kuratowski-Ryll-Nardzewski's selection theorem. See [3]). Then $v_2 \in F(t, y(t))$ and $\|v_1(t) - v_2(t)\| \leq k(t)\|x(t) - y(t)\|$ a.e. on J .

Define $u_2(t) = q(t) + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_2(s) ds$. It follows that $u_2 \in A(y)$ and

$$\begin{aligned} \|u_1(t) - u_2(t)\| &\leq \left| \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_1(s) ds - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_2(s) ds \right| \\ &\leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |v_1(s) - v_2(s)| ds \\ &\leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} k(s) |x(s) - y(s)| ds \\ &\leq \frac{a^{n-1}}{(n-1)!} \|k\|_{L^1} \|x - y\|. \end{aligned}$$

Taking the supremum over t , we obtain

$$\|u_1 - u_2\| \leq \frac{a^{n-1}}{(n-1)!} \|k\|_{L^1} \|x - y\|.$$

From this and the analogous inequality obtained by interchanging the roles of x and y we get that

$$H(A(x), A(y)) \leq \frac{a^{n-1}}{(n-1)!} \|k\|_{L^1} \|x - y\|,$$

for all $x, y \in X$. This shows that A is a multi-valued contraction on X since $\frac{a^{n-1}}{(n-1)!} \|k\|_{L^1} < 1$.

Step IV. Next we show that the multi-valued operator B is completely continuous on X . To finish, first we show that B maps bounded sets into uniformly bounded sets in $C(J, \mathbb{R})$. To see this, let S be a bounded set in $C(J, \mathbb{R})$. Then there exists a real number $r > 0$ such that $\|x\| \leq r, \forall x \in S$.

Now for each $u \in Bx$, there exists a $v \in \overline{S_G^1}(\tau x)$ such that

$$u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds.$$

Then for each $t \in J$,

$$\begin{aligned} |u(t)| &\leq \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \int_0^t \frac{a^{n-1}}{(n-1)!} |v(s)| ds \\ &\leq \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \int_0^t \frac{a^{n-1}}{(n-1)!} h_r(s) ds \\ &= \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \|h_r\|_{L^1}. \end{aligned}$$

This further implies that

$$\|u\| \leq \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \|h_r\|_{L^1}$$

for all $u \in Bx \subset B(S)$. Hence $B(S)$ is bounded.

Step V. Next we show that B maps bounded sets into equicontinuous sets. Let S be a bounded set as in step II, and $u \in Bx$ for some $x \in S$. Then there exists $v \in \overline{S_G^1}(\tau x)$ such that

$$u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds.$$

Then for any $t_1, t_2 \in J$ we have

$$\begin{aligned} & |u(t_1) - u(t_2)| \\ & \leq \left| \sum_{i=0}^{n-1} \frac{x_i t_1^i}{i!} - \sum_{i=0}^{n-1} \frac{x_i t_2^i}{i!} \right| \\ & \quad + \left| \int_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!} v(s) ds - \int_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s) ds \right| \\ & \leq |q(t_1) - q(t_2)| \\ & \quad + \left| \int_0^{t_1} \frac{(t_1-s)^{n-1}}{(n-1)!} v(s) ds - \int_0^{t_1} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s) ds \right| \\ & \quad + \left| \int_0^{t_1} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s) ds - \int_0^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} v(s) ds \right| \\ & \leq |q(t_1) - q(t_2)| \\ & \quad + \int_0^{t_1} \left| \frac{(t_1-s)^{n-1}}{(n-1)!} - \frac{(t_2-s)^{n-1}}{(n-1)!} \right| |v(s)| ds \\ & \quad + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{n-1}}{(n-1)!} |v(s)| ds \right| \\ & \leq |q(t_1) - q(t_2)| \\ & \quad + \frac{1}{(n-1)!} \left(\int_0^{t_1} |(t_1-s)^{n-1} - (t_2-s)^{n-1}|^2 ds \right)^{1/2} \left(\int_0^{t_2} h_r^2(s) ds \right)^{1/2} \\ & \quad + |p(t_1) - p(t_2)| \end{aligned}$$

$$\begin{aligned}
&\leq |q(t_1) - q(t_2)| \\
&\quad + \frac{1}{(n-1)!} \left(\int_0^a |(t_1 - s)^{n-1} - (t_2 - s)^{n-1}|^2 ds \right)^{1/2} \left(\int_0^a h_r^2(s) ds \right)^{1/2} \\
&\quad + |p(t_1) - p(t_2)|
\end{aligned}$$

where

$$q(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} \quad \text{and} \quad p(t) = \int_0^t \frac{(a-s)^{n-1}}{(n-1)!} h_r(s) ds.$$

Now the functions p and q are continuous on the compact interval J , hence they are uniformly continuous on J . Hence we have

$$|u(t_1) - u(t_2)| \rightarrow 0 \quad \text{as} \quad t_1 \rightarrow t_2.$$

As a result $\bigcup B(S)$ is an equicontinuous set in $C(J, \mathbb{R})$. Now an application of Arzelá-Ascoli theorem yields that B is totally bounded on $C(J, \mathbb{R})$.

Step VI. Next we prove that B has a closed graph. Let $\{x_n\} \subset C(J, \mathbb{R})$ be a sequence such that $x_n \rightarrow x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Bx_n$ for each $n \in \mathbb{N}$ such that $\frac{y_n}{n} \rightarrow y_*$. We just show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S_F^1(\tau x_n)$ such that

$$y_n(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s) ds.$$

Consider the linear and continuous operator $\mathcal{K} : L^1(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ defined by

$$\mathcal{K}v(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s) ds.$$

Now

$$\begin{aligned}
\left| y_n(t) - \sum_{i=0}^{n-1} \frac{|x_i| t^i}{i!} - y_*(t) - \sum_{i=0}^{n-1} \frac{|x_i| t^i}{i!} \right| &\leq |y_n(t) - y_*(t)| \\
&\leq \|y_n - y_*\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\end{aligned}$$

From Lemma 2.2 it follows that $(\mathcal{K} \circ \overline{S_G^1})$ is a closed graph operator and from the definition of K one has

$$y_n(t) - \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} \in (\mathcal{K} \circ \overline{S_F^1}(\tau x_n)).$$

As $x_n \rightarrow x_*$ and $y_n \rightarrow y_*$, there is a $v_* \in \overline{S_F^1}(\tau x_*)$ such that

$$y_* = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_*(s) ds.$$

Hence the multi B is an upper semi-continuous operator on $C(J, \mathbb{R})$.

Step VII. Finally, we show that the set

$$\mathcal{E} = \{x \in X : \lambda x \in Ax + Bx \text{ for some } \lambda > 1\}$$

is bounded.

Let $u \in \mathcal{E}$ be any element. Then there exists $v_1 \in \overline{S_F^1}(\tau x)$ and $v_2 \in \overline{S_G^1}(\tau x)$ such that

$$\begin{aligned} u(t) &= \lambda^{-1} \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \lambda^{-1} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_1(s) ds \\ &\quad + \lambda^{-1} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_2(s) ds. \end{aligned}$$

Then

$$\begin{aligned} |u(t)| &\leq \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |v_1(s)| ds \\ &\quad + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |v_2(s)| ds \\ &\leq \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} (k(s) \|\tau u\| + \|F(s, 0)\|) ds \\ &\quad + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \phi(s) \psi(|\tau(u(s))|) ds. \end{aligned}$$

Since $\tau x \in [\alpha, \beta], \forall x \in C(J, \mathbb{R})$, we have

$$\|\tau x\| \leq \|\alpha\| + \|\beta\| := D.$$

Therefore

$$\begin{aligned} |u(t)| &\leq \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} \int_0^t k(s) D ds + \frac{a^{n-1}}{(n-1)!} \int_0^t \|F(s, 0)\| ds \\ &\quad + \frac{a^{n-1}}{(n-1)!} \int_0^t \phi(s) \psi(|\tau(u(s))|) ds. \end{aligned}$$

Without loss of generality we may assume that $|\tau u(t)| \leq |u(t)|$ for all $t \in J$. Then from the above inequality we obtain:

$$\begin{aligned} |u(t)| &\leq \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} D \int_0^1 k(s) ds + \frac{a^{n-1}}{(n-1)!} \int_0^1 \|F(s, 0)\| ds \\ &\quad + \frac{a^{n-1}}{(n-1)!} \int_0^t \phi(s) \psi(|u(s)|) ds \leq C_1 + C_2 \int_0^t \phi(s) \psi(|u(s)|) ds, \end{aligned}$$

where

$$C_1 = \sum_{i=0}^{n-1} \frac{|x_i| a^i}{i!} + \frac{a^{n-1}}{(n-1)!} (\|\alpha\| + \|\beta\|) (\|k\|_{L^1} + L) \text{ and } C_2 = \frac{a^{n-1}}{(n-1)!}.$$

Let

$$w(t) = C_1 + C_2 \int_0^t \phi(s) \psi(|u(s)|) ds.$$

Then we have $|u(t)| \leq w(t)$ for all $t \in J$. Differentiating w.r.t. t , we obtain

$$w'(t) = C_2 \phi(t) \psi(|u(t)|), \text{ a.e. } t \in J, w(0) = C_1.$$

This further implies that

$$w'(t) \leq C_2 \phi(t) \psi(w(t)), \text{ a.e. } t \in J, w(0) = C_1,$$

that is,

$$\frac{w'(t)}{\psi(w(t))} \leq C_2 \phi(t) \text{ a.e. } t \in J, \quad w(0) = C_1.$$

Integrating from 0 to t we get

$$\int_0^t \frac{w'(s)}{\psi(w(s))} ds \leq C_2 \int_0^t \phi(s) ds.$$

By the change of variable,

$$\int_{C_1}^{w(t)} \frac{ds}{\psi(s)} \leq C_2 \|\phi\|_{L^1} < \int_{C_1}^{\infty} \frac{ds}{\psi(s)}.$$

Now an application of the mean-value theorem yields that there exists a point M such that

$$|u(t)| \leq w(t) \leq M \text{ for all } t \in J,$$

and so the set \mathcal{E} is bounded in $C(J, \mathbb{R})$. As a result the conclusion (ii) of Theorem 2.2 does not hold. Hence the conclusion (i) holds and consequently IVP (3) has a solution u on J . Next we show that u is also a solution of the IVP (1) on J . First, we show that $u \in [\alpha, \beta]$. Suppose not. Then either $\alpha \not\leq u$ or $u \not\leq \beta$ on some subinterval J' of J . If $u \not\leq \alpha$, then there exist $t_0, t_1 \in J, t_0 < t_1$ such that $u(t_0) = \alpha(t_0)$ and $\alpha(t) > u(t)$ for all $t \in (t_0, t_1) \subset J$. From the definition of the operator τ it follows that

$$u^{(n)}(t) \in F(t, \alpha(t)) + G(t, \alpha(t)) \text{ a.e. } t \in J.$$

Then there exists a $v_1(t) \in F(t, \alpha(t))$ and $v_2(t) \in G(t, \alpha(t)), \forall t \in J$ with

$$u^{(n)}(t) = v_1(t) + v_2(t) \text{ a.e. } t \in J.$$

Integrating from t_0 to t n times yields

$$u(t) - \sum_{i=0}^{n-1} \frac{u_i(0)(t-t_0)^i}{i!} = \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} v_1(s) ds + \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} v_2(s) ds.$$

Since α is a lower solution of IVP (1), we have

$$\begin{aligned}
u(t) &= \sum_{i=0}^{n-1} \frac{u_i(0)(t-t_0)^i}{i!} + \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} v_1(s) ds \\
&\quad + \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} v_2(s) ds \\
&\geq \sum_{i=0}^{n-1} \frac{\alpha_i(0)(t-t_0)^i}{i!} + \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} \alpha(s) ds \\
&\quad + \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} \alpha(s) ds \\
&= \alpha(t)
\end{aligned}$$

for all $t \in (t_0, t_1)$. This is a contradiction. Similarly, if $u \not\leq \beta$ on some subinterval of J , then we also get a contradiction. Hence $\alpha \leq u \leq \beta$ on J . As a result the IVP (3) has a solution u in $[\alpha, \beta]$. Finally, since $\tau x = x, \forall x \in [\alpha, \beta]$, u is a required solution of the IVP (1) on J . This completes the proof. ■

4. EXISTENCE OF EXTREMAL SOLUTIONS

In this section, we shall prove the existence of maximal and minimal solutions of the FDI (1) under suitable monotonicity conditions on the multifunctions involved in it. We equip the space $C(J, \mathbb{R})$ with the order relation \leq defined by the cone K in $C(J, \mathbb{R})$ defined by

$$(8) \quad K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \forall t \in J\}.$$

Thus “ $x \leq y$ ” if and only if $y - x \in K$. This order relation is equivalent to the order relation defined in the previous section. It is known that the cone K is normal in $C(J, \mathbb{R})$. The details of cones and their properties may be found in Heikkilä and Lakshmikantham [10].

Let $A, B \in P_{cl}(C(J, \mathbb{R}))$. Then by $A \leq B$ we mean $a \leq b$ for all $a \in A$ and $b \in B$. Thus $a \leq B$ implies that $a \leq b$ for all $b \in B$ in particular, if $A \leq A$, then it follows that A is a singleton set.

Definition 4.1. Let X be an ordered Banach space. A mapping $T : X \rightarrow P_{cl}(X)$ is called isotone increasing if $x, y \in X$ with $x < y$, then we have that $Tx \leq Ty$.

We use the following fixed point theorem in the proof of main existence result of this section.

Theorem 4.2 (Dhage [5]). *Let $[a, b]$ be an order interval in a Banach space and let $A, B : [a, b] \rightarrow P_{cl}(X)$ be two multi-valued operators satisfying*

- (a) A is a multi-valued contraction.
- (b) B is completely continuous,
- (c) A and B are isotone increasing, and
- (d) $Ax + Bx \subset [a, b] \forall x \in [a, b]$.

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has the least fixed point x_ and the greatest fixed point x^* in $[a, b]$. Moreover, $x_* = \lim_n x_n$ and $x^* = \lim_n y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a, b]$ defined by*

$$x_{n+1} \in Ax_n + Bx_n, \quad x_0 = a \quad \text{and} \quad y_{n+1} \in Ay_n + By_n, \quad y_0 = b.$$

We need the following definition in the sequel.

Definition 4.3. A solution x_M of the FDI (1) is said to be maximal if x is any other solution of the FDI (1) on J , then we have $x(t) \leq x_M(t)$ for all $t \in J$. Similarly, a minimal solution of the FDI (1) can be defined.

We consider the following assumptions in the sequel.

- (H₇) The multi-functions $F(t, x)$ and $G(t, x)$ are nondecreasing in x almost everywhere for $t \in I$, i.e. if $x < y$, then $F(t, x) \leq F(t, y)$ for all $x, y \in \mathbb{R}$.

Theorem 4.4. *Assume that hypotheses (H₁)–(H₄) and (H₆)–(H₇) hold. Then the FDI (1) has minimal and maximal solutions on J .*

Proof. Let $X = C(J, \mathbb{R})$ and consider the order interval $[\alpha, \beta]$ in X which is well defined in view of hypothesis (H₇). Define two operators $A, B : [\alpha, \beta] \rightarrow P_{cl}(X)$ by (4) and (5) respectively. It can be shown as in the proof of Theorem 3.1 that A and B define the multi-valued operators

$A : [\alpha, \beta] \rightarrow P_{cl,cv,bd}(X)$ and $B : [\alpha, \beta] \rightarrow P_{cp,cv}(X)$. It can be shown similarly that A a multi-valued contraction while B is completely continuous on $[\alpha, \beta]$. We shall show that A and B are isotone increasing on $[\alpha, \beta]$. Let $x \in [\alpha, \beta]$ be such that $x \leq y, x \neq y$. Then by (H₇), we have

$$\begin{aligned} Ax(t) &= \left\{ u(t) : u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds, v \in S_F^1(x) \right\} \\ &\leq \left\{ u(t) : u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds, v \in S_F^1(y) \right\} \\ &= Ay(t) \end{aligned}$$

for all $t \in J$. Hence $Ax \leq Ay$. Similarly, by (H₇), we have

$$\begin{aligned} Bx(t) &= \left\{ u(t) : u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds, v \in S_F^1(x) \right\} \\ &\leq \left\{ u(t) : u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds, v \in S_F^1(y) \right\} \\ &= By(t) \end{aligned}$$

for all $t \in J$. Hence $Bx \leq By$. Thus A and B are isotone increasing on $[a, b]$. Finally, let $x \in [\alpha, \beta]$ be any element. Then by (H₆),

$$a \leq Aa + Ba \leq Ax + Bx \leq Ab + Bb \leq b$$

which shows that $Ax + Bx \in [\alpha, \beta]$ for all $x \in [\alpha, \beta]$. Thus the multi-valued operator A and B satisfy all the conditions of Theorem 4.2 to yield that the operator inclusion and consequently the FDI (1) has maximal and minimal solutions on J . This completes the proof. \blacksquare

5. CONCLUSION

Note that when $F(t, x) \equiv 0$ in Theorem 3.1 we obtain the existence theorem for the IVP

$$(9) \quad \begin{cases} x^{(n)}(t) \in G(t, x(t)) \text{ a.e. } t \in J, \\ x^{(i)}(0) = x_i \in \mathbb{R} \end{cases}$$

proved in Dhage *et al.* [7]. Again when $F(t, x) \equiv 0$ and $n = 2$, Theorem 3.1 reduces to the existence result proved in Benchohra [3]. We also mention that our existence result for extremal solutions includes the results of Agarwal *et al.* [1] and Dhage and Kang [6], where a much stronger condition is used (i.e., the continuity of the multi-function).

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