Discussiones Mathematicae
Differential Inclusions, Control and Optimization 25 (2005) 47–58

SECOND-ORDER NECESSARY CONDITIONS FOR DISCRETE INCLUSIONS WITH END POINT CONSTRAINTS

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Abstract

We study an optimization problem given by a discrete inclusion with end point constraints. An approach concerning second-order optimality conditions is proposed.

Keywords: tangent cone, discrete inclusion, necessary optimality conditions.

2000 Mathematics Subject Classification: 49J30, 93C30.

1. Introduction

Consider the problem

$$\text{minimize} \quad g(x_N) \tag{1.1}$$

over the solutions of the discrete inclusion

$$x_i \in F_i(x_{i-1}), \quad i = 1, 2, \ldots, N, \quad x_0 \in X_0, \tag{1.2}$$

with end point constraints of the form

$$x_N \in X_N, \tag{1.3}$$

where $F_i : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$, $i = 1, 2, \ldots, N$, $X_0, X_N \subset \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}$ are given.

There are several papers devoted to first-order necessary optimality conditions for this problem ([5, 6, 7] etc.). The aim of the present paper is to develop an approach to second-order necessary optimality conditions for the problem (1.1)–(1.3). The general idea is to consider our problem as the problem of minimizing the terminal payoff on the intersection of the (known) target set with an (unknown) reachable set and to use a general result of the nonsmooth analysis (e.g. [1]). This general (abstract) optimality condition was formulated for the first time by Zheng ([8]), but this result (Theorem 2.2 below) is, in fact, an obvious consequence of Theorems 6.3.1, 6.6.2, 4.7.4, Proposition 6.2.4 and Corollary 4.3.5 in [1].

In order to apply the general abstract optimality conditions (namely, Theorem 2.2 below) we must check a certain constraint qualification, so we are naturally led to study first and second order approximations of the reachable set along optimal solutions.

One of the first results concerning second-order conditions of optimality using second-order directions is due to Ben-Tal and Zowe ([2]). We note that Theorem 2.2 below may be interpreted as an alternative to Theorem 2.1 in [2].

Let us mention that this idea has been already used in [3, 4, 8] to obtain second-order necessary optimality conditions for problems given by differential inclusions and hyperbolic differential inclusions.

The paper is organized as follows: in Section 2 we present the notations and definitions to be used in the sequel, while in Section 3 we present our main results.

2. Preliminaries

Since the reachable set that appears in optimization problems is, generally, neither a differentiable manifold, nor a convex set, its infinitesimal properties may be characterized only by tangent cones in a generalized sense, extending the classical concepts of tangent cones in Differential Geometry and Convex Analysis, respectively. From a rather large number of tangent cones in the literature (e.g. [1]) we use only the following concepts.

Let $X \subset \mathbf{R}^n$ and $x \in cl(X)$ (the closure of X).

Definition 2.1. (a) the *quasitangent* (*intermediate*) *cone* to X at x is defined by

$$Q_x X = \{v \in \mathbf{R}^n; \quad \forall s_m \to 0+,\ \exists v_m \to v :\ x + s_m v_m \in X\}$$

(b) the *second-order quasitangent set* to X at x relative to $v \in Q_x X$ is defined by

$$Q^2_{(x,v)} X = \Big\{ w \in \mathbf{R}^n; \quad \forall s_m \to 0+,\ \exists w_m \to w:\ x + s_m v + s_m^2 w_m \in X \Big\}$$

(c) *Clarke's tangent cone* to X at x is defined by

$$C_x X = \Big\{ v \in \mathbf{R}^n; \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X: \frac{y_m - x_m}{s_m} \to v \Big\}.$$

For equivalent definitions and for several properties of these cones we refer to [1]. We recall that in contrast with $Q_x X$, Clarke's tangent cone $C_x X$ is convex and one has $C_x X \subset Q_x X$.

We denote by C^+ the positive dual cone of $C \subset \mathbf{R}^n$, namely

$$C^+ = \{ q \in \mathbf{R}^n;\ < q, v > \geq 0,\ \forall\, v \in C \}.$$

The negative dual cone of $C \subset \mathbf{R}^n$ is $C^- = -C^+$.

As it was often remarked, the geometric interpretation of the classical (Fréchet) derivative, suggests the possibility of the introduction of generalized differentiability concepts corresponding to each type of tangent cone (to the graph, to the epigraph or to the subgraph of the function) but, of course, not all these concepts are equally important. In what follows, for a mapping $g(.) : X \subset \mathbf{R}^n \to \mathbf{R}$, which is not differentiable, we shall use only the first and second order uniform lower Dini derivative. We refer to [1] for the main properties of such derivatives.

$$D_\uparrow g(x; v) = \liminf_{(v', \theta) \to (v, 0+)} \frac{g(x + \theta v') - g(x)}{\theta},$$

$$D^2_\uparrow g(x, v; w) = \liminf_{(w', \theta) \to (w, 0+)} \frac{g(x + \theta v + \theta^2 w') - g(x) - \theta D_\uparrow g(x; v)}{\theta^2}.$$

When $g(.)$ is of class C^2 one has

$$D_\uparrow g(x, v) = g'(x)^T v, \quad D^2_\uparrow g(x, v; w) = g'(x)^T z + \frac{1}{2} v^T g''(x) v.$$

The key tool in the proof of our main result is the following abstract optimality condition.

Theorem 2.2 ([8]). *Let $g : \mathbf{R}^n \to \mathbf{R}$ be Lipschitzean in some open set containing z, let S_1, S_2 be nonempty sets of $\mathbf{R}^n$ containing z. If z solves the following minimization problem*

$$\textit{minimize } g(x) \quad \textit{over all } \ x \in S_1 \cap S_2$$

and also satisfies the constraint qualification

$$(CQ) \qquad (C_z S_1)^- \cap (C_z S_2)^+ = \{0\},$$

then we have the first-order necessary condition

$$(NC1) \qquad D_{\uparrow} g(z; v) \geq 0 \quad \forall v \in Q_z S_1 \cap Q_z S_2.$$

Furthermore, if equality holds for some v_0, then we have the second-order necessary condition

$$(NC2) \qquad D^2_{\uparrow} g(z, v_0; w) \geq 0 \quad \forall w \in Q^2_{(z,v_0)} S_1 \cap Q^2_{(z,v_0)} S_2.$$

Correspondingly, to each type of tangent cone, say $\tau_x X$, one may introduce (e.g. [1]) a *set-valued directional derivative* of a multifunction $G(.) : X \subset \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ (in particular of a single-valued mapping) at a point $(x, u) \in Graph(G)$ as follows

$$\tau_u G(x, \xi) = \{\nu \in \mathbf{R}^n; \ (\xi, \nu) \in \tau_{(x,u)} Graph(G)\}, \ \nu \in \tau_x X.$$

This first-order derivative may be characterized, equivalently, by

$$graph \tau_u G(x, .) = \tau_{(x,u)}(graph G(.)).$$

If the set-valued map $G(.)$ is Lipschitz, i.e. there exists $L > 0$ such that

$$G(x_1) \subset G(x_2) + L\|x_1 - x_2\| B \quad \forall x_1, x_2 \in X,$$

where B denotes the closed unit ball in $\mathbf{R}^n$, then the first order quasitangent derivative is given by (e.g. [1])

$$Q_u G(x; \xi) = \left\{ \nu \in \mathbf{R}^n; \quad \lim_{\theta \to 0+} \frac{1}{\theta} d(u + \theta \nu', G(x + \theta \xi)) = 0 \right\}.$$

Similarly, one may define (e.g. [1]) second-order directional deivatives of the set-valued map $G(\cdot)$. For example, the second-order quasitangent derivative of G at (x,u) relative to $(y,v) \in Q_{(x,u)}(graph(G(\cdot))$ is the set-valued map $Q^2_{(u,v)}G(x,y,\cdot)$ defined by

$$graphQ^2_{(u,v)}G(x,y;\cdot) = Q^2_{((x,u),(y,v))}(graphG(\cdot)).$$

We recall that a set-valued map $A(\cdot) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is called a *closed (respectively, convex) process* if $graph(A(\cdot))$ is a closed (respectively, convex) cone.

For the basic properties of convex processes we refer to [1], but we shall use here only the above definition.

If $G(\cdot) : X \subset \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is a given set-valued map and $(x,u) \in Graph(G)$ as a closed convex process one may take the Clarke directional derivative of $G(\cdot)$ at (x,u), $A(\cdot) = C_uG(x,\cdot)$.

The adjoint process $A^* : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ of the closed convex process A is defined by (e.g. [7])

$$A^*(p) = \left\{q \in \mathbf{R}^n; < q, v >\leq< p, v' > \quad \forall\, (v,v') \in graphA(\cdot)\right\}.$$

Denote by S_F the solution set of inclusion (1.2), i.e.

$$S_F := \{x = (x_0, x_1, \ldots, x_N); \quad x \text{ is a solution of (1.2)}\}.$$

and by $R^N_F := \{x_N; \quad x \in S_F\}$ the reachable set of inclusion (1.2).

In what follows, we consider $\overline{x} = (\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_N) \in S_F$ a solution to problem (1.1)–(1.3) and we shall assume the following hypothesis.

Hypothesis 2.3. (i) $X_0, X_N \subset \mathbf{R}^n$ *are closed sets.*
(ii) *There exists* $L > 0$ *such that* $F_i(\cdot)$ *is Lipschitz with the Lipschitz constant* L, $\forall i \in \{1, \ldots, N\}$.

Hypothesis 2.4. *There exists* $A_i : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$, $i = 1, 2, \ldots, N$ *a family of closed convex processes such that*

$$A_i(v) \subset Q_{\overline{x}_i}F_i(\overline{x}_{i-1}; v) \quad \forall v \in \mathbf{R}^n, \forall i \in \{1, \ldots, N\}.$$

Let $A_0 \subset Q_{\overline{x}_0} X_0$ be a closed convex cone. To the problem (1.2) we associate the linearized problem

$$w_i \in A_i(w_{i-1}), \quad i = 1, 2, \ldots, N, \quad w_0 \in A_0. \tag{2.1}$$

Denote by S_A the solution set of inclusion (2.1) and by R_A^N the reachable set of inclusion (2.1).

The next lemma, due to Tuan and Ishizuka, characterizes the positive dual of the solution set S_A of the problem (2.1).

Lemma 2.5 ([7])**.** *Assume that Hypotheses* 2.3 *and* 2.4 *are verified. Then, one has*

$$S_A^+ = \Big\{ w = (w_0, w_1, \ldots, w_N); \exists p = (p_0, p_1, \ldots, p_N) \in \mathbf{R}^{(N+1)n} \textit{ such that}$$
$$p_0 \in A_0^+, \; p_0 \in A_1^*(p_1) + w_0, p_1 \in A_2^*(p_2) + w_1, \ldots, p_{N-1} \in A_N^*(p_N)$$
$$+ w_{N-1}, p_N = w_N \Big\}.$$

Lemma 2.5 allows the characterization of the positive dual of the reachable set R_A^N.

Lemma 2.6. *Assume that Hypotheses* 2.3 *and* 2.4 *are verified. Then, one has*

$$\begin{aligned} (R_A^N)^+ \subseteq \Big\{ & w_N; \exists p = (p_0, p_1, \ldots, p_N) \in \mathbf{R}^{(N+1)n} \textit{ such that } p_0 \in A_0^+, \\ & p_0 \in A_1^*(p_1), p_1 \in A_2^*(p_2), \ldots, p_{N-1} \in A_N^*(p_N), p_N = w_N \Big\}. \end{aligned} \tag{2.2}$$

Proof. For $w = (w_0, w_1, \ldots, w_N) \in S_A$ we define $\gamma(w) = w_N$. Therefore, $R_A^N = \gamma(S_A)$; hence $S_A = \gamma^{-1}(R_A^N)$ and thus $S_A^+ = (\gamma^{-1}(R_A^N))^+ = \gamma^*((R_A^N)^+)$, or, equivalently $(R_A^N)^+ = (\gamma^*)^{-1}(S_A^+)$.

Take $w \in (R_A^N)^+$; it follows $\gamma^*(w) \in S_A^+$, i.e. there exists $\tilde{w} \in S_A^+$ such that $\gamma^*(w) = \tilde{w}$. Then,

$$< \gamma^*(w), x >=< \tilde{w}, x > \quad \forall x = (x_0, x_1, \ldots, x_N) \in \mathbf{R}^{(N+1)n},$$

or, equivalently

$$(2.3) \qquad < \tilde{w}, x >=< w, \gamma(x) >=< w, x_N > \quad \forall x = (x_0, x_1, \ldots, x_N) \in \mathbf{R}^{(N+1)n}.$$

If we take $x_0 = x_1 = \ldots = x_{N-1} = 0$, $x_N \in \mathbf{R}^n$ arbitrarly, then $w = \tilde{w}_N$. From (2.3) it follows that $\tilde{w}_0 = \ldots = \tilde{w}_{N-1} = 0$, i.e. (2.2) is satisfied

Remark 2.7. Hypothesis 2.4, that appears in Lemmas 2.5 and 2.6, is satisfied if we take

$$A_i(v) = C_{\overline{x}_i} F_i(\overline{x}_{i-1}; v) \quad \forall v \in \mathbf{R}^n, \forall i \in \{1, \ldots, N\}.$$

3. THE MAIN RESULTS

We prove first an approximation of the reachable set R_F^N at $\overline{x}_N$.

Theorem 3.1. *Assume that Hypothesis* 2.3 *is satisfied and denote by* R_Q^N *the reachable set of the discrete inclusion.*

$$(3.1) \qquad w_i \in Q_{\overline{x}_i} F_i(\overline{x}_{i-1}, w_{i-1}), \quad i = \overline{1, N}, \quad w_0 \in Q_{\overline{x}_0} X_0.$$

Then

$$R_Q^N \subset Q_{\overline{x}_N} R_F^N.$$

Proof. Let $w \in R_Q^N$ and $s_k \to 0+$. It follows that there exists $(w_0, w_1, \ldots, w_N)$ solution to (3.1) such that $w = w_N$.

In particular, $w_0 \in Q_{\overline{x}_0} X_0$ and therefore there exists $w_0^k \to w_0$ such that $\overline{x}_0 + s_k w_0^k \in X_0$.

On the other hand, $w_1 \in Q_{\overline{x}_1} F_1(\overline{x}_0, w_0)$ and by the definition of the quasitangent derivative of F_1 we have that there exist $(\tilde{w}_1^k, \tilde{w}_0^k) \to (w_1, w_0)$ such that

$$\overline{x}_1 + s_k \tilde{w}_1^k \in F_1(\overline{x}_0 + s_k \tilde{w}_0^k) \quad \forall k \in \mathbf{N}.$$

Using the Lipschitz property of the set-valued map $F_1(\cdot)$ one may write

$$\overline{x}_1 + s_k \tilde{w}_1^k \in F_1(\overline{x}_0 + s_k w_0^k) + s_k L \|\tilde{w}_0^k - w_0^k\| B \quad \forall k \in \mathbf{N},$$

where B denotes the unit ball in $\mathbf{R}^n$. Thus, there exists $b_k^1 \in B$ such that

$$\overline{x}_1 + s_k(\tilde{w}_1^k - L\|\tilde{w}_0^k - w_0^k\|b_k^1) \in F_1(\overline{x}_0 + s_k w_0^k) \quad \forall k \in \mathbf{N}$$

and if we define $w_1^k := \tilde{w}_1^k - L\|\tilde{w}_0^k - w_0^k\|b_k^1$ we have $w_1^k \to w_1$ and

$$\overline{x}_1 + s_k w_1^k \in F_1(\overline{x}_0 + s_k w_0^k) \quad \forall k \in \mathbf{N}.$$

By repeating this construction for $p = 2, \ldots, N$ we find that there exists $w_p^k \in \mathbf{R}^n$ such that $w_p^k \to w_p$, $p = 0, 1, \ldots, N$ and

$$\overline{x}_p + s_k w_p^k \in F_p(\overline{x}_{p-1} + s_k w_{p-1}^k) \quad \forall k \in N, \quad p = 1, \ldots, N.$$

In particular, for $s_k \to 0+$ there exists $w_N^k \to w_N$ such that $\overline{x}_N + s_k w_N^k \in R_F^N$, i.e. $w = w_N \in Q_{\overline{x}_N} R_F^N$ and the proof is complete.

Another first-order approximation of the reachable set R_F^N at $\overline{x}_N$ can be obtained in terms of the variational inclusion defined by the Clarke derivative of the set valued map.

Theorem 3.2. *Assume that Hypothesis* 2.3 *is satisfied and denote by* R_C^N *the reachable set of the discrete inclusion.*

$$w_i \in C_{\overline{x}_i} F_i(\overline{x}_{i-1}, w_{i-1}), \quad i = \overline{1, N}, \quad w_0 \in C_{\overline{x}_0} X_0. \tag{3.2}$$

Then

$$R_C^N \subset C_{\overline{x}_N} R_F^N.$$

The proof of Theorem 3.2 can be done using the same arguments employed to prove Theorem 3.1.

In order to apply Theorem 2.2 to our problem (1.1)–(1.4) we need to know the second-order quasitangent set to the reachable set R_F^N at $\overline{x}_N$.

Theorem 3.3. *Assume that Hypothesis* 2.3 *is satisfied, let* $\overline{y} = (\overline{y}_0, \overline{y}_1, \ldots, \overline{y}_N)$ *satisfy* (3.1) *and let* R_Q^2 *denote the reachable set of the discrete inclusion*

$$v_i \in Q^2_{(\overline{x}_i, \overline{y}_i)} F_i(\overline{x}_{i-1}, \overline{y}_{i-1}; v_{i-1}), \quad i = \overline{1, N}, \quad w_0 \in Q^2_{(\overline{x}_0, \overline{y}_0)} X_0. \tag{3.3}$$

Then

$$R_Q^2 \subset Q^2_{(\overline{x}_N,\overline{y}_N)} R_F^N.$$

Proof. Let $v \in R_Q^2$ and $t_k \to 0+$. It follows that there exists $(v_0, v_1, \dots, v_N)$ solution to (3.3) such that $v = v_N$.

In particular, $v_0 \in Q^2_{(\overline{x}_0,\overline{y}_0)} X_0$ and therefore there exists $v_0^k \to v_0$ such that $\overline{x}_0 + t_k \overline{y}_0 + t_k^2 v_0^k \in X_0$.

On the other hand, $v_1 \in Q^2_{(\overline{x}_1,\overline{y}_1)} F_1(\overline{x}_0, \overline{y}_0; v_0)$ and by the definition of the second-order quasitangent derivative of F_1 we have that there exist $(\tilde{v}_1^k, \tilde{v}_0^k) \to (v_1, v_0)$ such that

$$(\overline{x}_0, \overline{x}_1) + t_k(\overline{y}_0, \overline{y}_1) + t_k^2(\tilde{v}_0^k, \tilde{v}_1^k) \in graph F_1(.) \quad \forall k \in \mathbf{N}.$$

Using the Lipschitz property of the set-valued map $F_1(\cdot)$ one may write

$$\overline{x}_1 + t_k \overline{y}_1 + t_k^2 \tilde{v}_1^k \in F_1(\overline{x}_0 + t_k \overline{y}_0 + t_k^2 \tilde{v}_0^k)$$

$$\subset F_1(\overline{x}_0 + t_k \overline{y}_0 + t_k^2 v_0^k) + L\|v_0^k - \tilde{v}_0^k\| B.$$

Thus, there exists $b_k^1 \in B$ such that

$$\overline{x}_1 + t_k \overline{y}_1 + t_k^2(\tilde{v}_1^k - L\|v_0^k - \tilde{v}_0^k\| b_k^1) \in F_1(\overline{x}_0 + t_k \overline{y}_0 + t_k^2 \tilde{v}_0^k)$$

and if we define $v_1^k := \tilde{v}_1^k - L\|\tilde{v}_0^k - v_0^k\| b_k^1$ we have $v_1^k \to v_1$ and

$$\overline{x}_1 + t_k \overline{y}_1 + t_k^2 v_1^k \in F_1(\overline{x}_0 + t_k \overline{y}_0 + t_k^2 v_0^k).$$

By repeating this construction for $p = 2, \dots, N$ we find that there exists $v_p^k \in \mathbf{R}^n$ such that $v_p^k \to v_p$, $p = 0, 1, \dots, N$ and

$$\overline{x}_p + t_k \overline{y}_p + t_k^2 v_p^k \in F_p(\overline{x}_{p-1} + t_k \overline{y}_{p-1} + t_k^2 v_{p-1}^k).$$

In particular, for $t_k \to 0+$ there exists $\overline{x}_N + t_k \overline{y}_N + t_k^2 v_N^k \in R_F^N$, i.e. $v = v_N \in Q^2_{(\overline{x}_N,\overline{y}_N)} R_F^N$ and the proof is complete.

We are know able to prove our main result.

Theorem 3.4. *Assume that Hypothesis* 2.3 *is satisfied, let* $g(\cdot) : \mathbf{R}^n \to \mathbf{R}$ *be a locally Lipschitz function, let* $C_0 \subset Q_{\overline{x}_0} X_0$ *be a closed convex cone, let* $\overline{x} = (\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_N) \in S_F$ *be an optimal solution to problem* (1.1)–(1.3) *and assume that the following constraint qualification is satisfied*

$$\begin{aligned} &\Big\{-w_N; \exists p = (p_0, p_1, \ldots, p_N) \in \mathbf{R}^{(N+1)n} \text{ such that } p_0 \in C_0^+, \\ &p_0 \in (C_{\overline{x}_1} F_1(\overline{x}_0, .))^* p_1, \ldots, p_{N-1} \in (C_{\overline{x}_N} F_N(\overline{x}_{N-1}, .))^* p_N, p_N = w_N\Big\} \\ &\cap (C_{\overline{x}_N} X_N)^+ = \{0\}. \end{aligned} \tag{3.4}$$

Then we have the first-order necessary condition

$$D_\uparrow g(\overline{x}_N; y_N) \geq 0 \quad \forall y_N \in R_Q^N \cap Q_{\overline{x}_N} X_N. \tag{3.5}$$

Furthermore, if equality holds for some $\overline{y}_N$, *then we have the second-order necessary condition*

$$D_\uparrow^2 g(\overline{x}_N, \overline{y}_N; w_N) \geq 0 \quad \forall w_N \in R_Q^2 \cap Q^2_{(\overline{x}_N, \overline{y}_N)} X_N. \tag{3.6}$$

Proof. According to Theorem 3.2, $R_C^N \subset C_{\overline{x}_N} R_F^N$, hence

$$(C_{\overline{x}_N} R_F^N)^+ \subset (R_C^N)^+. \tag{3.7}$$

From Lemma 2.6, applied with $A_i(v) = C_{\overline{x}_i} F_i(\overline{x}_{i-1}; v)$ $\forall v \in \mathbf{R}^n$, $\forall i \in \{1, \ldots, N\}$ we find that

$$\begin{aligned} (R_C^N)^+ \subset \Big\{&w_N; \exists p = (p_0, p_1, \ldots, p_N) \in \mathbf{R}^{(N+1)n} \text{ such that } p_0 \in C_0^+, \\ &p_0 \in (C_{\overline{x}_1} F_1(\overline{x}_0, .))^* p_1, \ldots, p_{N-1} \in (C_{\overline{x}_N} F_N(\overline{x}_{N-1}, .))^* p_N, p_N = w_N\Big\}. \end{aligned} \tag{3.8}$$

Therefore from (3.4), (3.7) and (3.8) we deduce that

$$(C_{\overline{x}_N} R_F^N)^- \cap (C_{\overline{x}_N} X_N)^+ = \{0\}. \tag{3.9}$$

From Theorem 3.1 we have

$$R_Q^N \subset Q_{\overline{x}_N} R_F^N \tag{3.10}$$

and from Theorem 3.3 we have

$$R_Q^2 \subset Q^2_{(\overline{x}_N,\overline{y}_N)} R_F^N. \tag{3.11}$$

If $\overline{x} = (\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_N) \in S_F$ is an optimal solution to problem (1.1)–(1.3), then we have

$$g(\overline{x}_N) = \min\{g(z); \quad z \in R_F^N \cap X_N\}.$$

So, we apply Theorem 2.2 with $S_1 = R_F^N$ and $S_2 = X_N$.

Condition (3.9) assures that the constraint qualification (CQ) is satisfied. Hence from (3.10) and (NC1) we obtain (3.5) and from (3.11) and (NC2) we obtain (3.6).

Acknowledgments

The author wishes to thank an anonymous referee for his helpful comments which improved the paper.

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Received 13 July 2004