

## ON A LINEAR FUNCTIONAL EQUATION WITH A MEAN-TYPE MAPPING HAVING NO FIXED POINTS

KATARZYNA SAJBURA

*Faculty of Mathematics*

*Computer Science and Econometrics*

*University of Zielona Góra*

*Prof. Z. Szafrana 4a, 65–516 Zielona Góra, Poland*

**e-mail:** K.Sajbura@wmie.uz.zgora.pl

### Abstract

Our aim is to study continuous solutions  $\varphi$  of the classical linear iterative equation

$$\varphi(f(x, y)) = g(x, y)\varphi(x, y) + h(x, y),$$

where the given function  $f$  is defined as a pair of means. We are interested in the case when  $f$  has no fixed points. It turns out that in such a case continuous solutions of (1) depend on an arbitrary function.

**Keywords and phrases:** linear functional equation, iteration, mean, continuous solution, solution depending on an arbitrary function.

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### INTRODUCTION

The first section is auxiliary and yields some results dealing with level sets of means. One of the main tools here is Theorem A proved in [4]. The second one is a result of J. Matkowski [2, 3] (see Theorem B). It describes the limit behaviour of the sequence of iterates of a map built as a pair of means. In particular, it turns out that under very weak assumptions on the mean this sequence converges to a fixed point of the map. In addition, its limit function has some important invariance property. Both fixed point and

invariance properties seem to be crucial for the paper. The linear functional equation

$$(1) \quad \varphi(f(x, y)) = g(x, y)\varphi(x, y) + h(x, y)$$

is considered in the second section.

### 1. LEVEL SETS OF MEANS

If  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ , we will write

$$(x_1, y_1) \leq (x_2, y_2) : \iff x_1 \geq x_2 \quad \text{and} \quad y_1 \leq y_2$$

and

$$(x_1, y_1) < (x_2, y_2) : \iff (x_1, y_1) \leq (x_2, y_2) \quad \text{and} \quad (x_1, y_1) \neq (x_2, y_2).$$

Of course,  $\leq$  is a partial ordering in  $\mathbb{R}^2$ .

Let  $I, J$  be real intervals and let  $f : I \times J \rightarrow \mathbb{R}$  be a function. For every  $a \in \mathbb{R}$  define the *level set*  $\Gamma_f(a)$  by

$$\Gamma_f(a) = \{(x, y) \in I \times J : f(x, y) = a\}.$$

From now on we will impose a slightly stronger conditions on  $f$ . Namely, we will assume that  $J = I$  and  $f : I \times I \rightarrow \mathbb{R}$ , as usual continuous and increasing with respect to each variable, satisfies also the condition

$$f(x, x) = x \quad \text{for } x \in I.$$

Observe that this is equivalent to the following:  $f : I \times I \rightarrow \mathbb{R}$  is continuous, increasing with respect to each variable and

$$(2) \quad \min\{x, y\} \leq f(x, y) \leq \max\{x, y\}$$

for every  $x, y \in I$ . Functions satisfying (2) for all  $x, y \in I$  are called *means* on  $I$ . So, in other words, our hypothesis now reads as follows.

(H)  $I$  is a real interval and  $f$  is a continuous mean on  $I$ , increasing with respect to each variable.

Assuming (H) put

$$\Delta := \{(x, y) \in I \times I : x \geq y\}, \quad \Delta^0 := \{(x, y) \in I \times I : x > y\}$$

and, for every  $a \in \mathbb{R}$ ,

$$\Delta_f^-(a) := \{(x, y) \in \Delta : f(x, y) < a\},$$

$$\Delta_f^+(a) := \{(x, y) \in \Delta : f(x, y) > a\}.$$

Observe that for every  $a \in \mathbb{R}$

$$\Delta = \Gamma_f(a) \cap \Delta \cup \Delta_f^-(a) \cup \Delta_f^+(a),$$

$$(3) \quad \Delta_f^-(a) = \bigcup_{s \in \mathbb{R}, s < a} \Gamma_f(s) \cap \Delta, \quad \Delta_f^+(a) = \bigcup_{s \in \mathbb{R}, s > a} \Gamma_f(s) \cap \Delta.$$

In [4, Theorem 2] we characterized those level sets which are arcs, i.e. homeomorphic images of a real interval, and proved what follows (see also [4, Theorem 1]).

**Theorem A.** *Assume (H). Let  $a \in I$ . Then the following conditions are pairwise equivalent:*

- (i)  $\Gamma_f(a) \cap \Delta$  is an arc,
- (ii) the ordering  $\leq$  is linear on  $\Gamma_f(a) \cap \Delta$ ,
- (iii)  $\text{int}(\Gamma_f(a) \cap \Delta) = \emptyset$ .

Before stating our next results we will prove some auxiliary results.

**Lemma 1.** *Assume (H). Let  $a \in I$ . If  $\gamma$  is a homeomorphism mapping an interval with 0 as the left endpoint onto  $\Gamma_f(a) \cap \Delta$  and  $\gamma(0) = (a, a)$ , then  $\gamma$  is strictly decreasing.*

**Proof.** Suppose that  $\gamma = (\alpha, \beta)$  is not strictly decreasing. Then, by virtue of Theorem A, there exist  $s, t \in X$  such that  $0 \leq s < t$  and

$$\gamma(s) \leq \gamma(t) \leq \gamma(0).$$

Since

$$\alpha(s) \geq \alpha(t) \geq \alpha(0) \quad \text{and} \quad \beta(s) \leq \beta(t) \leq \beta(0)$$

we can find  $u, v \in [0, s]$  with  $\alpha(u) = \alpha(t)$  and  $\beta(v) = \beta(t)$ . If  $\beta(u) = \beta(t)$ , then  $\gamma(u) = \gamma(t)$ , whence  $u = t$ , which is impossible. Therefore  $\beta(u) \neq \beta(t)$ . Similarly, one can show that  $\alpha(v) \neq \alpha(t)$ .

Assume, for instance, that  $u \leq v$ . Since  $\gamma(u), \gamma(t) \in \Gamma_f(a)$  and  $f$  is increasing with respect to the second variable, we have either

$$\begin{aligned} a &= f(\gamma(u)) = f(\alpha(u), \beta(u)) = f(\alpha(t), \beta(u)) \\ &\leq f(\alpha(t), y) \leq f(\alpha(t), \beta(t)) = f(\gamma(t)) = a, \end{aligned}$$

or

$$\begin{aligned} a &= f(\gamma(u)) = f(\alpha(u), \beta(u)) = f(\alpha(t), \beta(u)) \\ &\geq f(\alpha(t), y) \geq f(\alpha(t), \beta(t)) = f(\gamma(t)) = a \end{aligned}$$

for every  $y$  between  $\beta(u)$  and  $\beta(t)$  depending on whether  $\beta(u) \leq \beta(t)$  or  $\beta(u) \geq \beta(t)$ . Consequently, the segment  $X_u$  with the endpoints  $\gamma(u), \gamma(t)$  is contained in  $\Gamma_f(a) \cap \Delta$ . By  $\alpha(v) \neq \alpha(t)$  we have  $\gamma(v) \notin X_u$ , that is  $v \notin \gamma^{-1}(X_u)$ . On the other hand,  $\gamma^{-1}(X_u)$  contains the segment  $[u, t]$ . Thus  $v < u$  contrary to the assumed condition  $u \leq v$ . ■

Assuming (H) let  $\xi \in I$ . Denote by  $\mathcal{C}_f^+(\xi)$  the set of all curves (that is homeomorphic images of an interval)  $C$  with the endpoint  $(\xi, \xi)$  satisfying the conditions  $C \setminus \{(\xi, \xi)\} \subset \Delta_f^+(\xi) \cap \Delta^0$  and  $\#C \cap \Gamma_f(a) = 1$  for every  $a \in (\xi, \infty) \cap I$ .

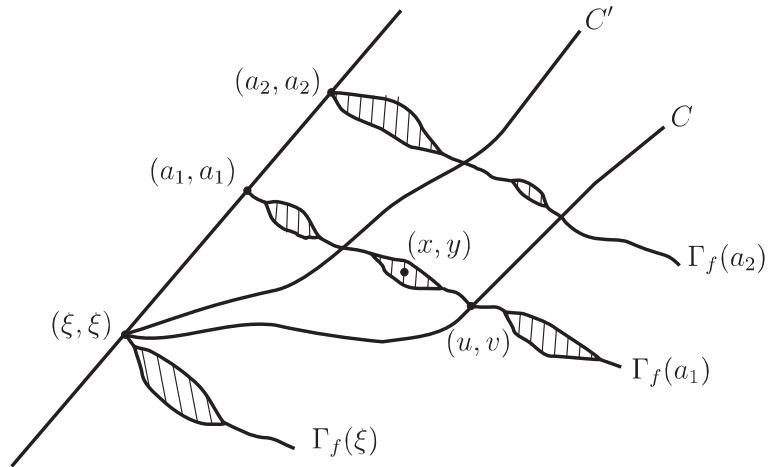


Figure 1

If  $C \in \mathcal{C}_f^+(\xi)$  and  $(x, y) \in \Delta_f^+(\xi)$  then

$$C \leq (x, y) : \iff (u, v) \leq (x, y)$$

$$[C < (x, y) : \iff (u, v) < (x, y) \text{ (cf. Figure 1)}]$$

and

$$(x, y) \leq C : \iff (x, y) \leq (u, v) \quad [(x, y) < C : \iff (x, y) < (u, v)],$$

where  $a := f(x, y)$  and  $C \cap \Gamma_f(a) = \{(u, v)\}$ . Similarly, we define the set  $\mathcal{C}_f^-(\xi)$  and the relations  $\leq$  and  $<$  for curves  $C \in \mathcal{C}_f^-(\xi)$  and points  $(x, y) \in \Delta_f^-(\xi)$ .

**Lemma 2.** *Assume (H). Let  $\xi \in I$ ,  $C \in \mathcal{C}_f^+(\xi)$  and let  $\gamma$  be a homeomorphism mapping an interval  $X$  onto  $C$ . Then  $f \circ \gamma$  is a homeomorphism mapping  $X$  onto  $[\xi, \infty) \cap I$ . In particular,  $f|_C$  is a homeomorphism of  $C$  and  $[\xi, \infty) \cap I$ .*

**Proof.** The function  $f \circ \gamma$  is continuous and, by the definition of  $\mathcal{C}_f^+(\xi)$ , maps  $X$  onto  $[\xi, \infty) \cap I$ . Moreover, if  $t_1, t_2 \in X$  and  $(f \circ \gamma)(t_1) = (f \circ \gamma)(t_2)$ , then, since  $C \in \mathcal{C}_f^+(\xi)$ , we have  $\gamma(t_1) = \gamma(t_2)$ , whence  $t_1 = t_2$ . Thus the function  $f \circ \gamma$  is one-to-one. ■

From Theorem A it follows that the ordering  $\leq$  is, in general, not linear. However, we will show that if  $\xi \in I$  and  $C \in \mathcal{C}_f^+(\xi)$ , then for every  $a \in [\xi, \infty) \cap I$  the (unique) element of  $C \cap \Gamma_f(a)$  is comparable with each point of  $\Gamma_f(a)$ .

**Theorem 1.** *Assume (H). Let  $a \in I$  and  $(u, v) \in \Gamma_f(a)$ . If there are  $\xi \in I$  and  $C \in \mathcal{C}_f^-(\xi) \cup \mathcal{C}_f^+(\xi)$  such that  $(u, v) \in C$ , then*

$$\text{either } (u, v) \leq (x, y), \text{ or } (x, y) \leq (u, v)$$

for every  $(x, y) \in \Gamma_f(a) \cap \Delta$ .

**Proof.** Fix a point  $(x, y) \in \Gamma_f(a) \cap \Delta$  and choose  $\xi \in I$  and  $C \in \mathcal{C}_f^+(\xi)$  such that  $(u, v) \in C$ . If  $(u, v) = (\xi, \xi)$ , then  $(u, v) = (a, a)$  and, consequently,  $(x, y) \leq (u, v)$ . So we may assume that  $(u, v) \neq (\xi, \xi)$ . Then  $(u, v) \in \Delta_f^+(\xi) \cap \Delta^0$ . Suppose that the points  $(u, v)$  and  $(x, y)$  are not comparable. Then either  $u < x$  and  $v < y$  or  $x < u$  and  $y < v$ . Consider, for instance, the

first possibility. Since  $(u, v)$  is an element of  $\Delta_f^+(\xi) \cap \Delta^0 \cap (-\infty, x) \times (-\infty, y)$  which is open in the space  $\Delta_f^+(\xi) \cap \Delta^0$  and  $f(u, v) = a \in I$ , it follows from the homeomorphicity of  $f|_C$  (cf. Lemma 2) that  $f(u_0, v_0) > a$  for a point  $(u_0, v_0) \in \Delta_f^+(\xi) \cap \Delta^0 \cap (-\infty, x) \times (-\infty, y)$  (in fact, we may also require that  $(u_0, v_0) \in C$ ). On the other hand, since  $f$  is increasing with respect to each variable, we have

$$f(u_0, v_0) \leq f(x, y) = a,$$

which is impossible. ■

**Corollary 1.** *Assume (H). If  $\xi \in I$  and  $C \in \mathcal{C}_f^+(\xi)$ , then*

$$\text{either } C \leq (x, y), \text{ or } (x, y) \leq C$$

for every  $(x, y) \in \Delta_f^+(\xi)$ .

A mean  $f : I \times I \rightarrow \mathbb{R}$  is called *strict* if

$$\min\{x, y\} < f(x, y) < \max\{x, y\}$$

for every  $x, y \in I$ ,  $x \neq y$ , and *homogeneous* if

$$f(tx, ty) = tf(x, y)$$

for every  $x, y \in I$  and  $t \in \mathbb{R}$  with  $tx, ty \in I$ .

Fix continuous strict means  $f_1 : I \times I \rightarrow \mathbb{R}$  and  $f_2 : I \times I \rightarrow \mathbb{R}$  and let  $f = (f_1, f_2)$ . A basic tool, which we are going to use, is the following result proved by J. Matkowski in [2] and [3].

**Theorem B.** *There exists a unique continuous mean  $m : I \times I \rightarrow I$  such that  $m \circ f = m$ . Moreover, the sequence  $(f^n : n \in \mathbb{N})$  of iterates of  $f$  converges to  $(m, m)$ :*

$$\lim_{n \rightarrow \infty} f^n = (m, m)$$

and the mean  $m$  is strict.

In the sequel, we will always use  $m$  for denoting the unique  $f$ -invariant mean introduced by Theorem B.

We will use the following assumption:

(H<sub>1</sub>)  $I$  is a real interval and  $f = (f_1, f_2)$ , where  $f_1$  and  $f_2$  are continuous strict means on  $I$ , increasing with respect to each variable.

**Remark 1.** If  $(H_1)$  is satisfied, then the  $f$ -invariant mean  $m$  is increasing with respect to each variable.

**Proof.** It is enough to observe that since  $f_1, f_2$  have the desired property, so have both coordinates of each  $f^n$ , whence, by Theorem B, also  $m$ . ■

**Corollary 2.** Assume  $(H_1)$ . Then

$$f(\Gamma_m(t)) \subset \Gamma_m(t)$$

and, whenever  $f(\Delta) \subset \Delta$ , we have

$$f(\Delta_m^-(t)) \subset \Delta_m^-(t), \quad f(\Delta_m^+(t)) \subset \Delta_m^+(t)$$

for every  $t \in I$ . If  $f(\Delta) \subset \Delta$  and, in addition,  $f|_\Delta$  is one-to-one, then

$$f(\Delta^0) \subset \Delta^0, \quad (f|_\Delta)^{-1}(\Delta^0) \subset \Delta^0.$$

**Proof.** If  $t \in I$  and  $(x, y) \in \Gamma_m(t)$ , then, in view of Theorem B,

$$m(f(x, y)) = m(x, y) = t,$$

whence  $f(x, y) \in \Gamma_m(t)$ . Consequently,  $f(\Gamma_m(t)) \subset \Gamma_m(t)$  for every  $t \in I$ , which, by virtue of (3), implies the second assertion.

Assume additionally that  $f|_\Delta$  is one-to-one. If  $f(x, y) \in \Delta \setminus \Delta^0$  for a pair  $(x, y) \in \Delta$ , then there exists a  $u \in I$  such that  $f(x, y) = (u, u)$ . Therefore, by  $f(u, u) = (u, u)$  we have  $(x, y) = (u, u) \in \Delta \setminus \Delta^0$ . This means that  $f(\Delta^0) \subset \Delta^0$ . Similarly, one can prove that  $(f|_\Delta)^{-1}(\Delta^0) \subset \Delta^0$ . ■

**Lemma 3.** Assume  $(H_1)$ . Then

$$(x, y) < f(x, y)$$

for every  $(x, y) \in \Delta^0$ .

**Proof.** It is enough to observe that if  $(x, y) \in \Delta^0$ , then  $x > y$ , whence  $x > f_1(x, y)$  and  $y < f_2(x, y)$ , that is  $(x, y) < f(x, y)$ . ■

**Lemma 4.** *Assume  $(H_1)$  and let  $f$  map  $\Delta$  onto its subset homeomorphically. If  $\xi \in I$  and  $C \in \mathcal{C}_m^+(\xi)$ , then  $f(C) \in \mathcal{C}_m^+(\xi)$ .*

**Proof.** Fix  $\xi \in I$  and  $C \in \mathcal{C}_m^+(\xi)$  and let  $\gamma$  be a homeomorphism mapping an interval onto  $C$ . Then  $f|_C \circ \gamma$  maps homeomorphically that interval onto  $f(C)$ . Since  $(\xi, \xi) \in C$  we have

$$(\xi, \xi) = f(\xi, \xi) \in f(C).$$

By Corollary 2

$$f(C) \setminus \{(\xi, \xi)\} = f(C \setminus \{(\xi, \xi)\}) \subset f(\Delta_m^+(\xi) \cap \Delta^0) \subset \Delta_m^+(\xi) \cap \Delta^0.$$

Let  $a \in (\xi, \infty) \cap I$ . By virtue of Corollary 2 we have  $f(\Gamma_m(a)) \subset \Gamma_m(a)$ , whence

$$(x, y) \in C \cap \Gamma_m(a) \quad \text{iff} \quad f(x, y) \in f(C) \cap \Gamma_m(a)$$

for every  $(x, y) \in \Delta_m^+(\xi)$ . Therefore, since  $C \cap \Gamma_m(a)$  is a singleton, so is  $f(C) \cap \Gamma_m(a)$ . Consequently,  $f(C) \in \mathcal{C}_m^+(\xi)$ . ■

**Lemma 5.** *Assume  $(H_1)$  and let  $f$  map  $\Delta$  onto its subset homeomorphically. Let  $a \in I$  and  $(x_1, y_1), (x_2, y_2) \in \Gamma_m(a) \cap \Delta^0$ . If  $(x_1, y_1) \leq (x_2, y_2)$  and there is a  $\xi \in I$  such that at least one of the points  $(x_1, y_1)$  and  $(x_2, y_2)$  lies on a curve from  $\mathcal{C}_m^+(\xi)$ , then  $(x_2, y_2) < f^n(x_1, y_1)$  for an  $n \in \mathbb{N}$ .*

**Proof.** Suppose that this is not the case. Then, by virtue of Theorem 1, Corollary 2 and Lemma 4,

$$f^n(x_1, y_1) \leq (x_2, y_2) \quad \text{for } n \in \mathbb{N}.$$

Using Theorem B we infer that

$$m(x_1, y_1) \geq x_2 \quad \text{and} \quad m(x_1, y_1) \leq y_2,$$

whence, since  $(x_2, y_2) \in \Delta^0$ , we obtain  $x_2 \leq m(x_1, y_1) \leq y_2 < x_2$ , which is impossible. ■



**Lemma 6.** *Assume  $(H_1)$ . Let  $\xi \in I$  and  $C \in \mathcal{C}_m^+(\xi)$ . If  $I$  is open, then*

$$\{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C < (x, y)\} \text{ and } \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : (x, y) < C\}$$

*are open subsets of  $\mathbb{R}^2$ .*

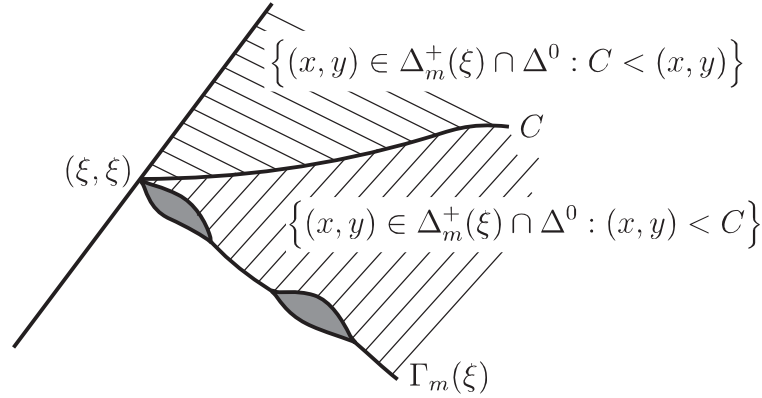


Figure 2

**Proof.** Fix a point  $(x_0, y_0) \in \Delta_m^+(\xi) \cap \Delta^0$  with  $C < (x_0, y_0)$  and choose a homeomorphism  $\gamma$  (cf. Lemma 2) mapping  $[\xi, \infty) \cap I$  onto  $C$  and fulfilling  $\gamma(\xi) = (\xi, \xi)$ . Put  $a_0 := m(x_0, y_0)$ . We can find  $(u_0, v_0) \in \Delta^0$  and  $t_0 \in [\xi, \infty) \cap I$  such that  $C \cap \Gamma_m(a_0) = \{(u_0, v_0)\}$  and  $(u_0, v_0) = \gamma(t_0)$ . Then  $(u_0, v_0) < (x_0, y_0)$  that is either  $u_0 \geq x_0$  and  $v_0 < y_0$ , or  $u_0 > x_0$  and  $v_0 \leq y_0$ . Assume, for instance, the first possibility. Denote by  $\gamma(t)_2$  the second coordinate of  $\gamma(t)$ . Since  $\gamma(t_0)_2 = v_0 < y_0$ , we have  $\gamma(t_0)_2 < \frac{v_0 + y_0}{2}$  and there exists a neighbourhood  $W$  (in the space  $[\xi, \infty) \cap I$ ) of  $t_0$  such that

$$(4) \quad \gamma(t)_2 < \frac{v_0 + y_0}{2} \quad \text{for } t \in W.$$

By virtue of Lemma 2 the set  $m(\gamma(W))$  is a neighbourhood of  $a_0$ . Therefore, due to the continuity of  $m$ , the set

$$U := \left\{ (x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : m(x, y) \in m(\gamma(W)) \text{ and } y > \frac{v_0 + y_0}{2} \right\}$$

is a neighbourhood of  $(x_0, y_0)$ . Take an arbitrary  $(x, y) \in U$  and choose a  $t \in W$  with  $m(x, y) = m(\gamma(t))$ . By Theorem 1 we have

$$\gamma(t) \leq (x, y) \quad \text{or} \quad (x, y) \leq \gamma(t).$$

It follows from (4) that

$$y > \frac{v_0 + y_0}{2} > \gamma(t)_2,$$

so the second possibility does not occur and, consequently,  $\gamma(t) < (x, y)$ . Thus  $C < (x, y)$ . This completes the proof of the openness of  $\{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C < (x, y)\}$ . ■

Now consider the following assumptions (H<sub>2</sub>), stronger than (H<sub>1</sub>):

(H<sub>2</sub>)  $I$  is a real interval and  $f = (f_1, f_2)$ , where  $f_1$  and  $f_2$  are continuous strict means on  $I$ , increasing with respect to each variable.  $f$  maps  $\Delta$  onto its subset homeomorphically and for every  $a \in I$  the function  $f$  strictly increases on  $\Gamma_m(a) \cap \Delta$ :

$$\text{if } (x_1, y_1) < (x_2, y_2) \quad \text{then} \quad f(x_1, y_1) < f(x_2, y_2)$$

for every  $(x_1, y_1), (x_2, y_2) \in \Gamma_m(a) \cap \Delta$ , and

$$(5) \quad \begin{aligned} & \Gamma_m(a) \cap \Delta^0 \\ &= \bigcup_{n=0}^{\infty} f^n(\Gamma_m(a) \cap A_C^0) \cap \Delta^0 \cup \bigcup_{n=1}^{\infty} (f^n|_{\Delta})^{-1}(\Gamma_m(a) \cap A_C^0) \cap \Delta^0 \end{aligned}$$

whenever  $C \in \mathcal{C}_m^+(\xi)$ ,  $\xi \in I$ ,  $\xi < a$  and

$$A_C^0 := \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C \leq (x, y) \quad \text{and} \quad (x, y) < f(C)\}.$$

**Remark 2.** If (H<sub>1</sub>) is satisfied,  $f$  maps  $\Delta$  onto its subset homeomorphically and  $\Gamma_m(a) \cap \Delta$  is an arc and  $f$  strictly increases on it for every  $a \in I$ , then (H<sub>2</sub>) is satisfied.

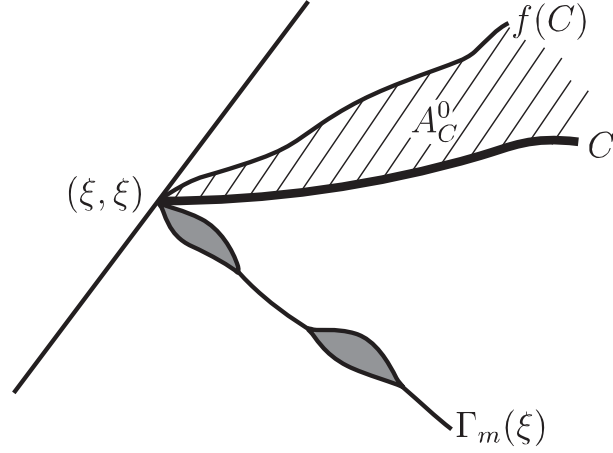


Figure 3

**Proof.** Fix an  $a \in I$ . Assume that  $\Gamma_m(a) \cap \Delta$  is an arc. Then there exists a homeomorphism  $\gamma : X \rightarrow \Gamma_m(a) \cap \Delta$ , where  $X$  is an interval with 0 as the left endpoint and  $\gamma(0) = (a, a)$ . Since (cf. Corollary 2)  $f(\Gamma_m(a)) \subset \Gamma_m(a)$ , we can define  $F : X \rightarrow X$  by  $F := \gamma^{-1} \circ f \circ \gamma$ . Note that  $F$  is continuous and

$$F^n = \gamma^{-1} \circ f^n \circ \gamma$$

for every  $n \in \mathbb{N}$ . Thus, by virtue of Theorem B,

$$\begin{aligned} \lim_{n \rightarrow \infty} F^n(t) &= \lim_{n \rightarrow \infty} \gamma^{-1} \left( f^n(\gamma(t)) \right) = \gamma^{-1} \left( m(\gamma(t)), m(\gamma(t)) \right) \\ &= \gamma^{-1}((a, a)) = 0, \end{aligned}$$

that is

$$\lim_{n \rightarrow \infty} F^n(t) = 0$$

for every  $t \in X$  which, due to the continuity of  $F$ , gives

$$(6) \quad F(t) < t \quad \text{for } t \in X \setminus \{0\}.$$

Moreover, it follows from (H<sub>2</sub>) and Lemma 1 that  $F$  is strictly increasing.

Since  $C \in \mathcal{C}_m^+(\xi)$  we can find a  $t_0 \in X \setminus \{0\}$  such that  $\gamma(t_0) \in C$ . For every  $n \in \mathbb{N}$  define  $t_n := F^n(t_0)$ .

We know that

$$(7) \quad \lim_{n \rightarrow \infty} t_n = 0$$

and  $t_{n+1} < t_n$  for every  $n \in \mathbb{N}$ . In particular,

$$(8) \quad \bigcup_{n=0}^{\infty} F^n((t_1, t_0]) = (0, t_0].$$

We will show that

$$\bigcup_{n=1}^{\infty} (F^n)^{-1}((t_1, t_0]) = X \setminus (0, t_0].$$

Since, by (6),

$$(F^n)^{-1}((t_1, t_0]) \subset X \setminus (0, t_0]$$

for every  $n \in \mathbb{N}$ , we have

$$\bigcup_{n=1}^{\infty} (F^n)^{-1}((t_1, t_0]) \subset X \setminus (0, t_0].$$

Let  $t \in X$ ,  $t > t_0$ . Then, in view of (7), there exists a  $k \in \mathbb{N}$  with  $F^k(t) \leq t_0$ . On account of (8) we can find an  $n \in \mathbb{N}_0$  such that  $F^k(t) \in F^n((t_1, t_0])$ . If  $n \geq k$ , then

$$t \leq F^{n-k}(t_0) \leq t_0$$

contrary to the choice of  $t$ . Therefore  $n < k$  and, consequently,  $t \in (F^{k-n})^{-1}((t_1, t_0])$ , whence

$$t \in \bigcup_{n=1}^{\infty} (F^n)^{-1}((t_1, t_0]).$$

We have just shown that

$$\bigcup_{n=0}^{\infty} F^n((t_1, t_0]) \cup \bigcup_{n=1}^{\infty} (F^n)^{-1}((t_1, t_0]) = X \setminus \{0\},$$

that is

$$\bigcup_{n=0}^{\infty} f^n(\gamma((t_1, t_0])) \cup \bigcup_{n=1}^{\infty} (f^n|_{\Delta})^{-1}(\gamma((t_1, t_0])) = \gamma(X \setminus \{0\}).$$

Since  $\gamma(t_0) \in C$ , we have also

$$\gamma(t_1) = \gamma(F(t_0)) = f(\gamma(t_0)) \in f(C)$$

and, by virtue of Lemma 1,  $C \leq \gamma(t) < f(C)$  for every  $t \in (t_1, t_0]$ , i.e.  $\gamma((t_1, t_0]) \subset \Gamma_m(a) \cap A_C^0$ . Hence, using the monotonicity of  $\gamma$  again, we get  $\gamma((t_1, t_0]) = \Gamma_m(a) \cap A_C^0$ . Consequently, we come to (5), which completes the proof. ■

Making use of Remarks 1 and 2 and [4, Corollaries 1 and 2] we obtain what follows.

**Remark 3.** If  $(H_1)$  is satisfied,  $f$  maps  $\Delta$  onto its subset homeomorphically and strictly increases on  $\Gamma_m(a) \cap \Delta$  for every  $a \in I$ , and  $m$  is either strictly increasing with respect to at least one of the variables, or homogeneous, then  $(H_2)$  is satisfied.

**Example 1.** Let  $I = (0, \infty)$  and  $f_1, f_2$  be the arithmetic and harmonic means, respectively:

$$f_1(x, y) = \frac{x + y}{2} \quad \text{and} \quad f_2(x, y) = \frac{2xy}{x + y} \quad \text{for } x, y \in I.$$

Then the  $f$ -invariant mean  $m$  is the geometric one:

$$m(x, y) = \sqrt{xy} \quad \text{for } x, y \in I.$$

Moreover,  $f$  maps  $\Delta$  homeomorphically onto itself.

Let  $a \in I$ . Then

$$\Gamma_m(a) = \{(x, y) \in (0, \infty)^2 : xy = a^2\}.$$

Therefore, if  $(x_1, y_1), (x_2, y_2) \in \Gamma_m(a) \cap \Delta$  and  $(x_1, y_1) \leq (x_2, y_2)$ , then  $x_1 y_1 = x_2 y_2 = a^2$ ,  $x_1 \geq x_2 \geq a$  and  $y_1 \leq y_2 \leq a$ , whence  $x_1 + \frac{a^2}{x_1} \geq x_2 + \frac{a^2}{x_2}$  and  $\frac{a^2}{y_1} + y_1 \geq \frac{a^2}{y_2} + y_2$ , so

$$f_1(x_1, y_1) = \frac{x_1 + \frac{a^2}{x_1}}{2} \geq \frac{x_2 + \frac{a^2}{x_2}}{2} = f_1(x_2, y_2)$$

and

$$f_2(x_1, y_1) = \frac{2a^2}{\frac{a^2}{y_1} + y_1} \leq \frac{2a^2}{\frac{a^2}{y_2} + y_2} = f_2(x_2, y_2),$$

consequently  $f(x_1, y_1) \leq f(x_2, y_2)$ . If, in addition,  $(x_1, y_1) \neq (x_2, y_2)$ , then either  $x_1 > x_2$ , or  $y_1 < y_2$  and, consequently,  $f(x_1, y_1) < f(x_2, y_2)$ . Therefore  $f$  is strictly increasing on the hiperbola  $\Gamma_m(a) \cap \Delta$ . According to Remark 2 this means that assumptions (H<sub>2</sub>) are fulfilled.

**Theorem 2.** Assume (H<sub>2</sub>). Let  $\xi \in I$  and  $C \in \mathcal{C}_m^+(\xi)$ . Then

- (i) the sets  $f^n(A_C^0)$ ,  $n \in \mathbb{N}_0$ , and  $(f^n |_\Delta)^{-1}(A_C^0)$ ,  $n \in \mathbb{N}$ , are pairwise disjoint;
- (ii) for every  $n \in \mathbb{N}_0$

$$f^n(A_C^0) = \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : f^n(C) \leq (x, y) < f^{n+1}(C)\};$$

- (iii)

$$\{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C \leq (x, y)\} = \bigcup_{n=0}^{\infty} f^n(A_C^0)$$

and

$$\{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : (x, y) < C\} = \bigcup_{n=1}^{\infty} (f^n |_\Delta)^{-1}(A_C^0);$$

- (iv)  $\Delta_m^+(\xi) \cap \Delta^0 =$

$$= \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C \leq (x, y)\} \cup \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : (x, y) < C\}.$$

**Proof.** (i) Suppose the contrary. Then there exist a point  $(x, y) \in A_C^0$  and a  $k \in \mathbb{N}$  with  $f^k(x, y) \in A_C^0$ . In particular,  $f^k(x, y) < f(C)$ , so  $f^k(x, y) < f(u, v)$ , where  $\{(u, v)\} = C \cap \Gamma_m(a)$  with  $a := m(x, y)$ . Moreover, since  $C \leq (x, y)$  we have  $(u, v) \leq (x, y)$ , whence  $f(u, v) \leq f(x, y)$ , and, consequently,  $f^k(x, y) < f(x, y)$ , which contradicts Lemma 3.

(iv) follows from Corollary 1.

(ii) and (iii) Fix an  $(x, y) \in \Delta_m^+(\xi) \cap \Delta^0$  and let  $(u, v)$  be the unique common point of  $C$  and  $\Gamma_m(a)$ , where  $a = m(x, y)$ .

Fix an  $n \in \mathbb{N}_0$  and assume that

$$f^n(C) \leq (x, y) < f^{n+1}(C),$$

that is

$$(9) \quad f^n(u, v) \leq (x, y) < f^{n+1}(u, v).$$

By (5) we can find a  $k \in \mathbb{N}_0$  such that either  $f^k(x, y) \in \Gamma_m(a) \cap A_C^0$ , or  $(x, y) \in f^k(\Gamma_m(a) \cap A_C^0)$ . In the first case we have

$$(u, v) \leq f^k(x, y) < f(u, v).$$

Then, by Lemma 3, inequality (9) and the monotonicity of  $f$  we get

$$(u, v) \leq f^{n+k}(u, v) \leq f^k(x, y) < f(u, v) \leq f^{n+k+1}(u, v),$$

which means that  $n + k = 0$ , that is  $k = n = 0$ . Therefore  $(x, y) \in A_C^0$ . So we may assume the second possibility, i.e. that  $f^k(s, t) = (x, y)$  for a pair  $(s, t) \in \Gamma_m(a) \cap A_C^0$ . Then, by the monotonicity of  $f$ , we have

$$(10) \quad f^k(u, v) \leq (x, y) < f^{k+1}(u, v).$$

If  $k < n$ , then, by virtue of Lemma 3, (9) and (10),

$$f^k(u, v) < f^n(u, v) \leq (x, y) < f^{k+1}(u, v) \leq f^n(u, v),$$

which is impossible. If  $n < k$ , then arguing similarly we come to

$$f^n(u, v) < f^k(u, v) \leq (x, y) < f^{n+1}(u, v) \leq f^k(u, v),$$

which is again impossible. Therefore  $k = n$  and, consequently,  $(x, y) \in f^n(A_C^0)$ . Thus we have proved that

$$(11) \quad \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : f^n(C) \leq (x, y) < f^{n+1}(C)\} \subset f^n(A_C^0)$$

for  $n \in \mathbb{N}_0$ .

Now assume that  $(x, y) < C$ , that is  $(x, y) < (u, v)$ . By virtue of Lemma 5 there exists a  $k \in \mathbb{N}$  such that  $(u, v) < f^k(x, y)$ . Therefore  $C < f^k(x, y)$ . Making use of the previous part of the proof we can find an  $n \in \mathbb{N}_0$  with  $f^k(x, y) \in f^n(A_C^0)$ . If  $k \leq n$  then, since  $f^{n-k}|_\Delta$  is one-to-one, we would have  $(x, y) = f^{n-k}(x_1, y_1)$  for some  $(x_1, y_1) \in A_C^0 \cap \Gamma_m(a)$ , whence, by virtue of Lemma 3,

$$(u, v) \leq (x_1, y_1) \leq (x, y),$$

which is impossible. Therefore  $k > n$  and, consequently,

$$(x, y) \in (f^{k-n}|_\Delta)^{-1}(A_C^0).$$

This proves that

$$(12) \quad \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : (x, y) < C\} \subset \bigcup_{n=1}^{\infty} (f^n|_\Delta)^{-1}(A_C^0).$$

It follows from (11) that

$$\{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C \leq (x, y)\} \subset \bigcup_{n=0}^{\infty} f^n(A_C^0).$$

This, (12), (iv), the inclusions  $A_C^0 \subset \Delta_m^+(\xi) \cap \Delta^0$  and (cf. Corollary 2)

$$f^n(A_C^0) \subset \Delta_m^+(\xi) \cap \Delta^0 \quad \text{and} \quad (f^n|_\Delta)^{-1}(A_C^0) \subset \Delta_m^+(\xi) \cap \Delta^0$$

held for every  $n \in \mathbb{N}$ , give the remaining inclusions of (ii) and (iii).  $\blacksquare$

## 2. LINEAR FUNCTIONAL EQUATION

Let us start with the following observation.

Assume that (H<sub>1</sub>) is satisfied,  $f$  maps  $\Delta$  onto its subset homeomorphically and  $\Gamma_m(a) \cap \Delta$  is an arc and  $f$  strictly increases on it for every  $a \in I$ . Let  $\xi \in I$  and  $\varphi : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$  be a solution to equation (1) where  $g$  and  $h$  are given real functions defined on  $\Delta_m^+(\xi) \cap \Delta^0$ . We proceed as in the proof of Remark 2. For every  $a \in I$  let  $\gamma_a$  be a homeomorphism mapping an interval  $X$  with 0 as the left endpoint onto  $\Gamma_m(a) \cap \Delta^0$ . Then, putting

$$F_a := \gamma_a^{-1} \circ f \circ \gamma_a, \quad G_a := g \circ \gamma_a, \quad H_a := h \circ \gamma_a,$$



we see that  $\phi_a := \varphi \circ \gamma_a$  is a solution to the equation

$$\phi(F_a(t)) = G_a(t)\phi(t) + H_a(t),$$

defined on the interval  $X$  as well as the given functions  $F_a, G_a, H_a$ . This allows us to use the classical theory of linear functional equations (cf., for instance, [1; Chap. II]) and to get some information on  $\varphi$  on every level set  $\Gamma_m(a) \cap \Delta^0$ ,  $a \in I$ . However, in many situations, mainly if we are interested in continuous  $\varphi$ , this information is not satisfactory. Moreover, this attempt requires that the level sets of  $m$  are arcs which, generally, need not be the case. So in our further study we proceed in a different way.

Assume (H<sub>2</sub>). Given a  $\xi \in I$  and a function  $g : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$  define functions  $G_n : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$  (cf. also [1; p.47]) by

$$G_0(x, y) := 1$$

and, for  $n \in \mathbb{N}$ , by

$$G_n(x, y) := \prod_{k=0}^{n-1} g(f^k(x, y)).$$

In what follows, we accept the convention that the sum over an empty set of indices equals zero.

**Theorem 3.** *Assume (H<sub>2</sub>). Let  $\xi \in I$ ,  $C \in \mathcal{C}_m^+(\xi)$ ,  $g : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R} \setminus \{0\}$  and  $h : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$ . Then every function  $\varphi_0 : A_C^0 \rightarrow \mathbb{R}$  can be uniquely extended to a solution  $\varphi : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$  of equation (1); moreover,*

$$(13) \quad \begin{aligned} & \varphi(x, y) \\ &= G_n(f|_{\Delta})^{-n}(x, y) \left[ \varphi_0(f|_{\Delta})^{-n}(x, y) + \sum_{k=0}^{n-1} \frac{h((f|_{\Delta})^{k-n}(x, y))}{G_{k+1}((f|_{\Delta})^{-n}(x, y))} \right] \end{aligned}$$

if  $(x, y) \in f^n(A_C^0)$  and  $n \in \mathbb{N}_0$  and

$$(14) \quad \varphi(x, y) = \frac{\varphi_0(f^n(x, y))}{G_n(x, y)} - \sum_{k=0}^{n-1} \frac{h(f^k(x, y))}{G_{k+1}(x, y)}$$

whenever  $(x, y) \in (f^n|_{\Delta})^{-1}(A_C^0)$  and  $n \in \mathbb{N}$ .

**Proof.** It follows from Theorem 2 that given  $\varphi_0 : A_C^0 \rightarrow \mathbb{R}$  formulas (13) and (14) define a function  $\varphi : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$ . Putting  $n = 0$  in (13) we see that  $\varphi|_{A_C^0} = \varphi_0$ . A standard computation shows that  $\varphi$  satisfies (1). This and a simple induction prove that any solution  $\varphi : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$  of (1) satisfying

$$(15) \quad \varphi_0(f(x, y)) = g(x, y)\varphi_0(x, y) + h(x, y) \quad \text{for } (x, y) \in C$$

have to be given by (13) and (14). ■

**Corollary 3.** *Assume (H<sub>2</sub>). Let  $\xi \in I$ ,  $C \in \mathcal{C}_m^+(\xi)$  and*

$$A_C := \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C \leq (x, y) \quad \text{and} \quad (x, y) \leq f(C)\}$$

*and let  $g : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R} \setminus \{0\}$  and  $h : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$  be continuous functions. Then every continuous function  $\varphi_0 : A_C \rightarrow \mathbb{R}$  satisfying (15) can be uniquely extended to a solution  $\varphi : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$  of equation (1); moreover, this solution is continuous.*

**Proof.** Let  $\varphi_0 : A_C \rightarrow \mathbb{R}$  be a continuous function satisfying (15). Since  $\varphi_0$  fulfils (1) on  $C$ , by virtue of (15) and Theorem 3 it can be uniquely extended to  $\varphi : \Delta_m^+(\xi) \cap \Delta^0 \rightarrow \mathbb{R}$  given by (13) and (14).

Now we will show that  $\varphi$  is continuous. Fix an  $(x_0, y_0) \in \Delta_m^+(\xi) \cap \Delta^0$  and a sequence  $((x_i, y_i) : i \in \mathbb{N})$  of points of  $\Delta_m^+(\xi) \cap \Delta^0$  convergent to  $(x_0, y_0)$ . First assume that

$$(x_0, y_0) \in f^n \left( \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C < (x, y) \quad \text{and} \quad (x, y) < f(C)\} \right)$$

for some  $n \in \mathbb{N}_0$ . Since  $f|_\Delta$  is a homeomorphism it follows from Lemmas 4 and 6 that the set

$$f^n \left( \{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C < (x, y) \quad \text{and} \quad (x, y) < f(C)\} \right)$$

is open. Therefore almost all  $(x_i, y_i)$  belong to this set and by the continuity of  $\varphi_0$ ,  $g$ ,  $h$  and  $(f|_\Delta)^{-1}$  and on account of (13)

$$(16) \quad \lim_{i \rightarrow \infty} \varphi(x_i, y_i) = \varphi(x_0, y_0).$$

If for some  $n \in \mathbb{N}$  the point  $(x_0, y_0)$  is an element of the set

$$(f^n|_{\Delta})^{-1}\left(\{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C < (x, y) \text{ and } (x, y) < f(C)\}\right)$$

we come to (16) by a similar reasoning.

By virtue of Theorem 2 it remains to consider the case where either  $(x_0, y_0) \in f^n(C)$  for some  $n \in \mathbb{N}_0$ , or  $(x_0, y_0) \in (f^n|_{\Delta})^{-1}(C)$  for some  $n \in \mathbb{N}$ . Assume, for instance, that  $(x_0, y_0) \in C$  (in the remaining cases the proof is similar). Since  $(x_0, y_0)$  is an element of the set

$$(f|_{\Delta})^{-1}\left(\{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C < (x, y) \text{ and } (x, y) < f^2(C)\}\right),$$

which, by Lemmas 4 and 6, is open, almost all  $(x_i, y_i)$  are its elements. Moreover, by the monotonicity of  $f$ ,

$$\begin{aligned} (f|_{\Delta})^{-1}\left(\{(x, y) \in \Delta_m^+(\xi) \cap \Delta^0 : C < (x, y) \text{ and } (x, y) < f^2(C)\}\right) \\ \subset (f|_{\Delta})^{-1}(A_C^0) \cup A_C^0. \end{aligned}$$

We can consider only the case where both  $(f|_{\Delta})^{-1}(A_C^0)$  and  $A_C^0$  include infinitely many  $(x_i, y_i)$ . Let  $(x_{p_i}, y_{p_i})$  be a subsequence of  $(x_i, y_i)$  consisting of all those  $(x_i, y_i)$  which are in  $(f|_{\Delta})^{-1}(A_C^0)$  and  $(x_{q_i}, y_{q_i})$  a subsequence consisting of the remaining pairs. Clearly,  $(x_{q_i}, y_{q_i}) \in A_C^0$  for  $i$ 's large enough. Then, by (14) for  $n = 1$  and by (15), we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \varphi(x_{p_i}, y_{p_i}) &= \lim_{i \rightarrow \infty} \left[ \frac{\varphi_0(f(x_{p_i}, y_{p_i}))}{g(x_{p_i}, y_{p_i})} - \frac{h(x_{p_i}, y_{p_i})}{g(x_{p_i}, y_{p_i})} \right] \\ &= \frac{\varphi_0(f(x_0, y_0))}{g(x_0, y_0)} - \frac{h(x_0, y_0)}{g(x_0, y_0)} = \varphi_0(x_0, y_0) = \varphi(x_0, y_0). \end{aligned}$$

Similarly, using (13) for  $n = 0$ , we have

$$\lim_{i \rightarrow \infty} \varphi(x_{q_i}, y_{q_i}) = \lim_{i \rightarrow \infty} \varphi_0(x_{q_i}, y_{q_i}) = \varphi_0(x_0, y_0) = \varphi(x_0, y_0).$$

This again gives (16) and, consequently, the continuity of  $\varphi$  at  $(x_0, y_0)$ . ■

Repeating the reasoning without essential changes we can obtain analogous results for the set  $\Delta_m^-(\xi) \cap \Delta^0$ .

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