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MULTIVALUED LINEAR OPERATORS AND DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract

In this paper, we study multivalued linear operators (MLO's) and their resolvents in non reflexive Banach spaces, introducing a new condition of a minimal growth at infinity, more general than the Hille-Yosida condition. Then we describe the generalized semigroups induced by MLO's. We present a criterion for an MLO to be a generator

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of a generalized semigroup in an arbitrary Banach space. Finally, we obtain some existence results for differential inclusions with MLO's and various types of multivalued nonlinearities. As a consequence, we give theorems on the existence of local, global and bounded solutions of the Cauchy problem for degenerate differential inclusions.

Keywords: multivalued linear operator, generalized semigroup, minimal growth at infinity, Hille-Yosida condition, degenerate differential inclusion, Cauchy problem, bounded solution.

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1. Introduction

In recent years the method of multivalued linear operators (MLO's) was efficiently applied to the study of degenerate (or Sobolev type) differential equations in Banach spaces (see, e.g. [4, 5, 7, 8, 14] and references therein). In paper [10] this method was used to prove existence results of the Cauchy problem for various types of degenerate differential inclusions in Banach spaces. The solvability of general boundary value problems (including the periodic problem) for degenerate differential inclusions was also considered.

It has to be noted, however, that the method developed in [10] can be applied only to inclusions in reflexive Banach spaces.

In the present paper we want to overcome this limitation. To this aim in Section 2, we study the resolvent operator of an MLO, introducing a new condition of minimal growth at infinity, (see Definition 7), more general than the Hille-Yosida condition. We describe some important properties of MLO's satisfying the above condition, including a necessary and sufficient condition under which the whole space can be represented as $\overline{D(A)} \oplus A\,0$ (see Theorems 12 and 13). In Section 3, we deal with generalized semigroups generated by MLO's. Notice that generalized semigroups were described and studied earlier in a number of works (see, e.g. [4, 5, 7, 8, 11, 12, 14] and others).

In Theorem 16, we give a criterion for an MLO to be a generator of a generalized semigroup in an arbitrary Banach space. Some sufficient conditions are also presented in Corollary 17. In Section 4, we obtain some existence results for differential inclusions with MLO's and various types of nonlinearities. As a consequence, we can give theorems on the existence of local and global solutions of the Cauchy problem for degenerate differential inclusions.

In the conclusions, we consider the existence of bounded solutions on the positive half-line for differential inclusions with MLO's and degenerate differential inclusions (Section 4.3).

2. SOME PROPERTIES OF MULTIVALUED LINEAR OPERATORS

First we present some necessary definitions from the theory of multivalued linear operators. Details can be found in [3] and [5].

Let E be a complex Banach space.

Definition 1. A multivalued map (multimap) $A : E \to 2^E$ is said to be a *multivalued linear operator* (MLO) in E if:

(i) $D(A) = \{x \in E : Ax \neq \emptyset\}$ is a linear subspace of E;

(ii)

$$\begin{cases} Ax + Ay \subset A(x+y), & \forall x, y \in D(A); \\ \lambda\, Ax \subseteq A(\lambda\, x), & \forall \lambda \in \mathbb{C},\ x \in D(A). \end{cases} \tag{1}$$

It is an easy consequence of the definition to note that $Ax + Ay = A(x+y)$ for all $x, y \in D(A)$ and $\lambda\, Ax = A(\lambda\, x)$ for all $x \in D(A)$, $\lambda \neq 0$.

The collection of all MLO's in E will be denoted by $ML(E)$.

Definition 2. The inverse A^{-1} of an MLO is defined as:

(i) $D(A^{-1}) = R(A)$;

(ii) $A^{-1}y = \{x \in D(A) : y \in Ax\}$.

It is easy to verify that $A^{-1} \in ML(E)$.

Definition 3. Let A and B be two MLO's in E. The sum and the product of A, B are defined respectively by the relations:

$$\begin{aligned} &D(A+B) = D(A) \cap D(B), \quad (A+B)x = Ax + Bx \\ &D(AB) = \{x \in D(B) : D(A) \cap Bx \neq \emptyset\}, \quad AB\, x = A(D(A) \cap Bx) \end{aligned} \tag{2}$$

One can observe that $A + B$ and AB are MLO in E and that

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{3}$$

Definition 4. The *resolvent set* $\rho(A)$ of an MLO A is defined as the collection of all $\lambda \in \mathbb{C}$ for which:

(i) $\mathrm{Im}(\lambda I - A) = D((\lambda I - A)^{-1}) = E$;

(ii) $(\lambda I - A)^{-1}$ is a single-valued bounded operator on E.

Denote by $\mathcal{L}(E)$ the collection of all linear bounded operators in E.

Definition 5. The operator-valued function $R(\cdot, A) : \rho(A) \to \mathcal{L}(E)$

$$R(\lambda, A) = (\lambda I - A)^{-1}$$

is called *the resolvent* of an MLO A.

Remark 6. It is easy to verify that, given an $A \in ML(E)$ we have for each $\lambda \in \rho(A)$:

(i) $Ker\, R(\lambda, A) = A0$;

(ii) $\mathrm{Im}\, R(\lambda, A) = D(A)$.

Let us introduce the following notion.

Definition 7. An MLO A *satisfies the* (MGI) *condition* if its resolvent has *a minimal growth at infinity*, that is there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \rho(A)$ such that:

(i) $\lim_{n\to\infty} |\lambda_n| = \infty$;

(ii) $\sup_{n\geq 1} \|\lambda_n R(\lambda_n, A)\|_{\mathcal{L}(E)} < \infty$.

The sequence $\{\lambda_n\}$ will be called *admissible*.

Remark 8. An MLO A satisfies the (MGI) condition if the following Hille-Yosida condition is fulfilled:
(H-Y) there exists $\beta \in \mathbb{R}$ such that $(\beta, +\infty) \subset \rho(A)$ and the operators $R(\lambda, A)$ satisfy the estimates

$$\|R(\lambda, A)^n\|_{\mathcal{L}(E)} \leq \frac{C}{(\lambda - \beta)^n}\,, \quad \lambda > \beta; \quad n = 1, 2, \dots$$

for some constant $C \in \mathbb{R}_+$.

Under the (MGI) condition on $A \in ML(E)$, consider the bounded sequence of operators $\{A_n\}_{n=1}^{\infty} \subset \mathcal{L}(E)$ defined as

$$A_n = I - \lambda_n R(\lambda_n, A), \quad n = 1, 2, \dots \tag{4}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is an admissible sequence. Let $\widetilde{E}$ be the closed subspace defined as

$$\widetilde{E} = \left\{ x \in E : \text{there exists } \lim_{n\to\infty} A_n x \right\}.$$

Lemma 9. *The sequence $\{A_n\}_{n=1}^{\infty}$ has the following properties:*

(i) $\lim_{n\to\infty} \|A_n R(\lambda, A)\|_{\mathcal{L}(E)} = 0$ *for each* $\lambda \in \rho(A)$;

(ii) $(I - A_n)x \in D(A)$ *for each* $x \in E$, $n = 1, 2, \dots$

Proof. (i) We will use a Hilbert type equality: for $\mu_0, \mu_1 \in \rho(A)$

$$R(\mu_1, A) - R(\mu_0, A) = -(\mu_1 - \mu_0) R(\mu_1, A) R(\mu_0, A)$$

(see [5], Theorem 1.8). We have

$$\begin{aligned} \|A_n R(\lambda, A)\|_{\mathcal{L}(E)} &= \|(I - \lambda_n R(\lambda_n, A)) R(\lambda, A)\|_{\mathcal{L}(E)} \\ &= \left\| R(\lambda, A) + \frac{\lambda_n}{\lambda_n - \lambda} (R(\lambda_n, A) - R(\lambda, A)) \right\|_{\mathcal{L}(E)} \\ &= \left\| \frac{1}{\lambda_n - \lambda} (\lambda_n R(\lambda_n, A) - \lambda R(\lambda, A)) \right\| \underset{n\to\infty}{\to} 0 . \end{aligned}$$

(ii) It follows from Remark 6 (ii). ∎

Remark 10. From the Hilbert equality it follows that

$$A_n R(\lambda, A) = R(\lambda, A) A_n$$

for every $\lambda \in \rho(A)$, $n \geq 1$.

Lemma 11. *If $A \in ML(E)$ satisfies the (MGI) condition, then*

$$\widetilde{E} = E_0 \oplus E_1$$

where $E_0 = \overline{D(A)}$ *and* $E_1 = A0$, *and the operator* $P_0 \in \mathcal{L}(\widetilde{E})$ *defined as*

$$P_0 x = \lim_{n\to\infty} (I - A_n)x$$

is the projection on E_0.

Proof. At first, let us mention that if $x \in \widetilde{E}$, then by definition $\lim_{n\to\infty} A_n x$ exists and so the operator P_0 is correctly defined.

Further, since $(I - A_n)x \in D(A)$ (Lemma 9 (ii)) we have that $P_0 x \in \overline{D(A)} = E_0$, i.e., $\operatorname{Im} P_0 \subseteq E_0$.
On the other hand, if $y \in D(A)$ then $y = R(\lambda, A)x$ for some $x \in E$ and any $\lambda \in \rho(A)$ (see Remark 6 (ii)). Since $\lim_{n\to\infty} \|A_n R(\lambda, A)\|_{\mathcal{L}(E)} = 0$ (Lemma 9 (i)), we have that $y \in \widetilde{E}$ and, moreover

$$\lim_{n\to\infty} (I - A_n)y = y. \tag{5}$$

As the subspace $\widetilde{E}$ is closed, for every $y \in \overline{D(A)} = E_0$ we also have $y \in \widetilde{E}$ and relation (5), so we get $E_0 \subset \widetilde{E}$ and $P_{0|E_0} = id_{E_0}$ and hence $\operatorname{Im} P_0 = E_0$.

Furthermore, if $x \in \widetilde{E}$ and $P_0 x = x_0 \in E_0$, then

$$x_1 = x - x_0 = \lim_{n\to\infty} A_n x.$$

For any $\lambda \in \rho(A)$, applying Remark 10 and Lemma 9 (i), we have then

$$\begin{aligned} R(\lambda, A)x_1 &= \lim_{n\to\infty} R(\lambda, A)A_n x \\ &= \lim_{n\to\infty} A_n R(\lambda, A)x = 0, \end{aligned}$$

i.e., $x_1 \in Ker\, R(\lambda, A) = A0 = E_1$.

Now, let us show that $E_1 \subseteq Ker\, P_0 \subset \widetilde{E}$. In fact, if $x_1 \in E_1 = A0 = Ker\, R(\lambda, A)$ for each $\lambda \in \rho(A)$, then

$$\lim_{n\to\infty} A_n x_1 = \lim_{n\to\infty} (I - \lambda_n R(\lambda_n, A))x_1 = x_1,$$

i.e., $x_1 \in \widetilde{E}$ and $P_0 x_1 = 0$.

It remains to show only that $E_0 \cap E_1 = \{0\}$. In fact, if $x \in E_0 \cap E_1$, then $x \in E_1 = A0 = \ker R(\lambda, A)$ for each $\lambda \in \rho(A)$. On the other hand, $x \in E_0$, so

$$x = P_0 x = \lim_{n\to\infty} (I - A_n)x = \lim_{n\to\infty} \lambda_n R(\lambda_n, A)x = 0.$$ ∎

Now we want to know when the subspace $\widetilde{E}$ coincides with the whole E. To clear up this problem, let us recall the notion of a dual MLO (see [3], [5]).

Let E^* be a space dual to E. For $A \in ML(E)$, we will denote by A^* a MLO on E^* defined in the following way:

for $h,\ g \in E^*$, the relation $h \in A^*(g)$ means that $g(y) = h(x)$ for all pairs $(x, y) \in D(A) \times E$ such that $y \in A(x)$. It is easy to verify that

$$A^*0^* = \Big\{h \in E^* : \overline{D(A)} \subset Ker\, h\Big\} = \overline{D(A)}^{\perp};$$

$$\overline{D(A^*)} \subseteq \{g \in E^* : A0 \subset Ker\, g\} = A0^{\perp}.$$

(Here 0^* denotes the zero element of E^*).

Theorem 12. *Under the (MGI) condition on $A \in ML(E)$, for the equality $\widetilde{E} = E$ it is necessary and sufficient that functionals from A^*0^* are separated by elements of $A0$, i.e., for each $h \in A^*0^*$, $h \neq 0^*$ there exists $y \in A0$ such that $h(y) \neq 0$.*

Proof. (1) (Necessity) Let

$$\widetilde{E} = \overline{D(A)} \oplus A0 = E$$

Then $E^* = E_1^* \oplus E_0^*$, where $E_1^* = \overline{D(A)}^{\perp} = A^*0^*$ and $E_0^* = A0^{\perp} \supseteq \overline{D(A^*)}$.

Now, if a functional $h_0 \in A^*0^*$ is non-zero and $h_0 \equiv 0$ on $A0$, then $h_0 \in E_1^* \cap E_0^*$, giving a contradiction.

(2) (Sufficiency) Suppose that $\widetilde{E} \neq E$. Then there exists a non-zero functional $h \in E^*$ with $\widetilde{E} \subset Ker\, h$. Therefore, we have that $\overline{D(A)} \subset Ker\, h$, $A0 \subset Ker\, h$, and so h is a non-zero functional from A^*0^* vanishing on $A0$ and we again have a contradiction. ■

Theorem 13. *Under the (MGI) condition on $A \in ML(E)$ each of the following assumptions implies $\widetilde{E} = E$:*

(i) *the sequence $\{A_n x\}_{n=1}^{\infty}$ is weakly compact for each $x \in E$;*

(ii) *the space E is reflexive;*

(iii) $\dim A0 = \dim A^*0^* < \infty$.

Proof.

(i) For any subsequence $\{A_{n_k}\}$ let us denote by $Erg(E, \{A_{n_k}\}, w)$ the closed subspace of E, consisting of all points $x \in E$ for which the sequence $\{A_{n_k}x\}$ is weakly convergent. It is clear that $\widetilde{E} \subset Erg(E, \{A_{n_k}\}, w)$ for every subsequence $\{A_{n_k}\}$. Moreover, we have the equality

$$\widetilde{E} = Erg(E, \{A_{n_k}\}, w).$$

In fact, if $x \in Erg(E, \{A_{n_k}\}, w)$ then there exists x_0 such that $A_{n_k}x \underset{w}{\rightharpoondown} x_0$. Choosing any $\lambda \in \rho(A)$ and taking into account that the linear operator $R(\lambda, A)$ is weakly continuous, we obtain, applying Remark 10 and Lemma 9 (i), the following:

$$\begin{aligned} R(\lambda, A)x_0 &= R(\lambda, A) \lim_w \{A_{n_k}x\} = \lim_w \{R(\lambda, A)A_{n_k}x\} \\ &= \lim\{A_{n_k}R(\lambda, A)x\} = 0. \end{aligned}$$

So $x_0 \in Ker\, R(\lambda, A) = A0$.
Representing x by

$$x = (I - A_{n_k})x + A_{n_k}x, \tag{6}$$

applying Lemma 9 (ii) and passing to the weak limit in both sides of (6) we obtain that

$$x \in \overline{D(A)} \oplus A0 = \widetilde{E}.$$

(ii) Since the sequence $\{A_n\}$ is bounded, condition (ii) implies (i) (see also [5]).

(iii) Condition (iii) implies that functionals from A^*0^* are separated by elements of $A0$ and we can apply Theorem 12. ■

Let us mention the following property

Theorem 14. *Let $A \in ML(E)$ and $\widetilde{E} = E$. Then the restriction $A_{|E_0.} := A_0$ defined by $A_0x_0 = Ax_0 \cap E_0$ is a closed linear operator, $D(A_0) = D(A)$.*

Proof. It is easy to see that for each $x \in D(A)$, $Ax = y + A0$ with any $y \in Ax$. So, since $E = \widetilde{E} = \overline{D(A)} \oplus A0$, we have $Ax \cap \overline{D(A)} \neq \emptyset$. Moreover, if $y', y'' \in Ax \cap \overline{D(A)}$, then $y' - y'' \in A0$ and hence $y' - y'' \in \overline{D(A)} \cap A0 = 0$. ■

3. Generalized semigroups

Let us recall the following notion (cf. [4, 5, 7, 8, 11, 12, 14]).

Definition 15. A family of bounded linear operators $U : [0, \infty) \to \mathcal{L}(E)$ is said to be *a generalized* (*or degenerate*) C_0*-semigroup* if the following conditions hold:

(i) for each $x \in E$, the function $t \to U(t)x$ is continuous on $[0, \infty)$;

(ii) $U(0) = P$ is a non-zero projection;

(iii) $U(t_0 + t_1) = U(t_1)U(t_0)$, $\forall t_0, t_1 \in [0, \infty)$.

If we denote $E_0 = \operatorname{Im} P$, $E_1 = \operatorname{Im}(I - P) = Ker\, P$, then $E = E_0 \oplus E_1$ and the spaces E_0, E_1 are closed and invariant with respect to $U(t)$, $t \geq 0$. Moreover, the restriction $U_0(t) = U(t)_{|E_0}$ is a C_0 semigroup on E_0 and, hence it admits the estimate

$$\|U_0(t)\|_{\mathcal{L}(E_0)} \leq Ce^{\beta t} \quad \text{for some } C \geq 1, \beta \in \mathbb{R}.$$

Then

$$\|U(t)\|_{\mathcal{L}(E)} = \|U_0(t)P\|_{\mathcal{L}(E)} \leq C\|P\|_{\mathcal{L}(E)}e^{\beta t}.$$

Consider the function $R : C_\beta \to \mathcal{L}(E)$ defined on the open half space $C_\beta = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \beta\}$ by the formula

$$R(\lambda)x = \int_0^\infty U(t)xe^{-\lambda t}dt\,, \quad x \in E. \tag{7}$$

This function satisfies the Hilbert equality and it is the resolvent for the MLO

$$A = \lambda_0 I - R(\lambda_0)^{-1} \tag{8}$$

independently of the choice of $\lambda_0 \in \mathbb{C}_\beta$.

Moreover, it is easy to see that the MLO A given by (8) may be described in the following way:

let $A_0 : D(A) \subset E_0 \to E_0$ be an infinitesimal generator of the C_0-semigroup U_0. Then define A taking $D(A) = D(A_0)$, $A0 = E_1$ and

$$Ax = A_0x + A0, \quad x \in D(A).$$

Further, from (7), (8) it follows that $R(\lambda, A) = R(\lambda)$.

The MLO A is said to be a *generator of a generalized semigroup* $U(t)$, $t \geq 0$.

We can give now the next criterion.

Theorem 16. *The following conditions are necessary and sufficient for the MLO A to be a generator of a generalized semigroup:*

(i) *functionals from A^*0^* are separated by vectors of $A0$;*

(ii) *the Hille-Yosida condition: there exist a constant $C > 0$ and $\beta \in \mathbb{R}$ such that $\mathbb{C}_\beta \subset \rho(A)$ and*

$$\|R(\lambda, A)^n\|_{\mathcal{L}(E)} \leq \frac{C}{(\operatorname{Re}\lambda - \beta)^n}, \quad n = 1, 2, \ldots, \quad \lambda \in \mathbb{C}_\beta\,. \tag{9}$$

Proof. (1) (Necessity) Let $A \in ML(E)$ be a generator of a generalized semigroup $U(t), t \geq 0$. Then, from (7) it follows that there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that $\mathbb{C}_\beta \subset \rho(A)$. The estimates (9) follow from the same formula (7). Then, we know (see Remark 8) that A satisfies the (MGI) condition. Moreover, from the definition of A we see that $\overline{D(A)} = \overline{D(A_0)} = E_0$, $A0 = E_1$ and hence $E = E_0 \oplus E_1 = \overline{D(A)} \oplus A0 = \widetilde{E}$. Applying the necessity part of Theorem 12 we obtain (i).

(2) (Sufficiency) Suppose that conditions (i) and (ii) hold for $A \in ML(E)$. Applying again Remark 8 we conclude that A satisfies the (MGI) condition. Then, from Lemma 11 and Theorem 12 we obtain that $E = E_0 \oplus E_1 = \overline{D(A)} \oplus A0$, and by Theorem 14, the restriction A_0 of A on $\overline{D(A)}$ is a closed linear operator, $D(A_0) = D(A)$. Furthermore, since for $A_0 : D(A_0) \subset E_0 \to E_0$ the Hille-Yosida condition is fulfilled, it generates on E_0 a C_0-semigroup $U_0(t), t \geq 0$. Now, let $P : E \to E_0$ be the projector. Then, $U : [0, \infty) \to \mathcal{L}(E)$ defined as $U(t)x = U_0(t)Px$, $x \in E$ is a generalized semigroup generated by A. ∎

Taking into account Remark 8 and Theorem 12 we get the following

Corollary 17. *If $A \in ML(E)$ satisfies the Hille-Yosida condition* (ii) *of Theorem* 16 *and at least one of conditions of Theorem* 13 *is fulfilled, then A is the generator of a generalized semigroup.*

In the conclusions of this section, we present an example of sufficient conditions under which an MLO A satisfies the Hille-Yosida condition.

Let $J : E \to 2^{E^*}$ be the duality multimap.

Proposition 18 (cf. [10])**.** *Let $A \in ML(E)$ and suppose that for every $x \in D(A)$ there exists $x^* \in J(x)$ such that*

$$\mathrm{Re}\langle y, x^* \rangle \leq \beta \|x\|^2$$

for all $y \in Ax$ where $\beta \in \mathbb{R}$. Let also

$$\mathrm{Im}(\lambda_0 I - A) = E$$

for some $\lambda_0 > \beta$. Then A satisfies (ii) *of Theorem* 16 *with $C = 1$.*

4. DIFFERENTIAL INCLUSIONS WITH MLO'S AND DEGENERATE DIFFERENTIAL INCLUSIONS

At first we recall some notions (see, e.g. [6] for further details).

Let X be a metric space and Y a normed space. Let $P(Y)$ denote the collection of all nonempty subsets of Y. We denote

$$K(Y) = \{S \in P(Y) : S \text{ is compact}\}$$
$$Kv(Y) = \{S \in K(Y) : S \text{ is convex}\}.$$

Definition 19. A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is:

(a) *upper semicontinuous* (u.s.c.) if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of X for every open set $V \subset Y$;

(b) *lower semicontinuous* (l.s.c.) if $\mathcal{F}^{-1}(W)$ is a closed subset of X for every closed set $W \subset Y$.

Definition 20. Let $\mathcal{E}$ be a Fréchet space and $(\mathcal{A}, \geq)$ a partially ordered set. A function $\beta : P(\mathcal{E}) \to \mathcal{A}$ is called a *measure of noncompactness* (MNC) in $\mathcal{E}$ if

$$\beta(\overline{co}\,\Omega) = \beta(\Omega) \quad \text{for every } \Omega \in P(\mathcal{E}).$$

A MNC β is called:

(i) monotone, if Ω_0, $\Omega_1 \in P(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(ii) nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in \mathcal{E}$, $\Omega \in P(\mathcal{E})$;

If $\mathcal{A}$ is a cone, we say that the MNC β is

(iii) regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As an example of an MNC satisfying all the above properties we can consider the *Hausdorff MNC*

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

Definition 21. A multimap $\mathcal{F} : X \subseteq \mathcal{E} \to K(\mathcal{E})$ or a family of multimaps $\mathcal{G} : [0,1] \times X \to K(\mathcal{E})$ is called *condensing* relative to the MNC β (or β-condensing) if for every set $\Omega \subseteq X$ not relatively compact, we have

$$\beta(\mathcal{F}(\Omega)) \ngeq \beta(\Omega) \quad \text{or} \quad \beta(\mathcal{G}([0,1] \times \Omega)) \ngeq \beta(\Omega),$$

respectively.

Let $W \subset \mathcal{E}$ be an open set, $\mathcal{K} \subseteq \mathcal{E}$ a closed convex subset; β a monotone MNC in $\mathcal{E}$ and $\mathcal{F} : \overline{U}_{\mathcal{K}} \to Kv(\mathcal{K})$ a β-condensing u.s.c. multimap such that $x \notin \mathcal{F}(x)$ for all $x \in \partial W_{\mathcal{K}}$, where $\overline{W}_{\mathcal{K}}$ and $\partial W_{\mathcal{K}}$ denote the closure and the boundary of the set $W_{\mathcal{K}} = W \cap \mathcal{K}$ in the relative topology of the space $\mathcal{K}$.

In such a setting, the relative topological degree $\deg_{\mathcal{K}}(i - \mathcal{F}, \overline{W_{\mathcal{K}}})$, satisfying the standard properties, is defined (see [1, 2, 6, 9]).

4.1. Existence theorems for differential inclusions with MLO's

Let E be a real Banach space. We assume that:

(A) $A \in ML(E)$ is a generator of a generalized C_0-semigroup $U(t), t \geq 0$.

We consider the following Cauchy problem

$$\frac{dy}{dt} \in Ay(t) + F(t, y(t)), \quad t \in [0, T] \tag{10}$$

$$y(0) = y_0 \in \overline{D(A)}. \tag{11}$$

We specify conditions under which we study problem (10), (11).

(F0) $F(t,y) \in Kv(E)$ for all $(t,y) \in [0,T] \times E$;

(F1) the multifunction $F(\cdot,x) : [0,T] \to Kv(E)$ has a strongly measurable selection for every $x \in E$;

(F2) the multimap $F(t,\cdot) : E \to Kv(E)$ is u.s.c. for a.a. $t \in [0,T]$;

(F3) for every nonempty bounded set $\Omega \subset E$, there exists a function $\mu_\Omega \in L^1_+[0,T]$ such that

$$\|F(t,\Omega)\| := \sup\{\|z\| : z \in F(t,\Omega)\} \leq \mu_\Omega(t) \text{ for a.a. } t \in [0,T];$$

(F4) there exists a function $k \in L^1_+[0,T]$ such that

$$\chi(F(t,D)) \leq k(t)\chi(D) \text{ for a.a. } t \in [0,T]$$

for every bounded set $D \subset E$.

Definition 22. A function $y : [0,h] \to E$ is *a mild solution to problem* (10), (11) *on interval* $[0,h] \subseteq [0,T]$ if it has the following representation:

$$y(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds$$

for some selection $f \in L^1([0,h];E)$ of the multifunction $t \to F(t,y(t))$.

It follows from conditions (F1) – (F3) that for each h, $0 < h \leq T$, the superposition multioperator

$$\mathcal{P}_F : C([0,h];E) \to L^1([0,h];E),$$

$$\mathcal{P}_F(y) = \left\{f \in L^1([0,h];E) : f(t) \in F(t,y(t)) \text{ a.e. } t \in [0,h]\right\}$$

is well defined, so we can consider the integral multioperator $\Gamma : C([0,h];E) \to C([0,h];E)$ given as

$$\Gamma(y) = \left\{z : z(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds : \ f \in \mathcal{P}_F(y)\right\}. \tag{12}$$

It is clear that fixed points of Γ, $y \in \Gamma(y)$, coincide with mild solutions to (10), (11) on the interval $[0,h]$.

The key point in the study of the multioperator Γ is the fact that *the generalized Cauchy operator* $G : L^1([0,h];E) \to C([0,h];E)$

$$G(f)(t) = \int_0^t U(t-s)f(s)ds$$

satisfies the following conditions:

(G1) there exists $K \geq 0$ such that $\|Gf(t) - Gg(t)\| \leq K \int_0^t \|f(s) - g(s)\| ds$, for all $f, g \in L^1([0,h];E)$, $0 \leq t \leq h$;

(G2) for any compact set $X \subset E$ and sequence $\{f_n\} \subset L^1([0,h];E)$ such that $\{f_n(t)\} \subset X$ for a.a. $t \in [0,T]$, the weak convergence $f_n \rightharpoonup f_0$ implies $Gf_n \to Gf_0$.

These conditions are nothing but conditions (S1) and (S2) from [6], Chapter 4. In fact, as K of (G1) we can obviously take $\sup_{t\in[0,h]} \|U(t)\|$ and the fulfillment of condition (G2) may be verified following the lines of Lemma 4.2.1 of [6].

For a bounded set $\Omega \subset C([0,h];E)$ consider the MNC

$$\nu(\Omega) = \max_{\mathcal{D}\in\Delta(\Omega)} (\gamma(\mathcal{D}), \operatorname{mod}_C(\mathcal{D})) \tag{13}$$

where: $\Delta(\Omega)$ is the collection of all denumerable subsets of Ω; $\gamma(\mathcal{D}) = \sup_{t\in[0,h]} e^{-Lt}\chi(\mathcal{D}(t))$; $L > 0$ is chosen in such a way that

$$\sup_{t\in[0,h]} \left[2K \int_0^t e^{-L(t-s)} k(s) ds \right] < 1$$

(where $k(\cdot)$ is from (F4), the constant K is from (G1)) and

$$\operatorname{mod}_C(\mathcal{D}) = \lim_{\delta\to 0} \sup_{x\in\mathcal{D}} \max_{|t_1-t_2|<\delta} \|x(t_1) - x(t_2)\|$$

is the modulus of equicontinuity.

The range of the MNC ν is the cone $\mathbb{R}^2_+$, the maximum is taken in the sense of the order induced by this cone. It is known ([6]) that the MNC ν is well defined and it is monotone, invariant with respect to the union with compact sets and regular.

From Theorem 5.1.2, Corollary 5.1.2, and Theorem 5.1.3 of [6] we deduce the following assertion describing the main properties of the integral multioperator Γ.

Proposition 23. *The integral multioperator* Γ*, given by* (12)*, is an u.s.c. multioperator with compact convex values and it is* ν*-condensing on bounded sets.*

The above Proposition implies that the topological degree theory for condensing multimaps can be applied to Γ and, as a result, we obtain the following local and global existence theorems which are analogous to Theorems 5.2.1 and 5.2.2 in [6].

Theorem 24. *Under conditions* (A) *and* (F0) – (F4)*, there exists a mild solution to problem* (10), (11) *on some interval* $[0, h]$*,* $0 < h \leq T$*.*

Theorem 25. *Suppose that conditions* (A) *and* (F0) – (F4) *hold, and assume additionally* (F3') *there exists a function* $\alpha \in L^1_+([0,T])$ *such that*

$$\|F(t, x)\| \leq \alpha(t)(1 + \|x\|)$$

for a.a. $t \in [0, T]$ *and all* $x \in E$*.*

Then, the set Σ *of all mild solutions to* (10), (11) *on* $[0, T]$ *is a nonempty compact subset of the space* $C([0, T]; E)$*.*

Remark 26. Analogous existence results can also be obtained if we substitute assumptions (F0) – (F2) with the following condition of almost lower semicontinuity:

(F_L) for a multimap $F : [0, T] \times E \to K(E)$ there exists a sequence of disjoint compact sets $\{I_n\}$, $I_n \subset [0, T]$ such that:

(i) $meas([0, T] \setminus \bigcup_n I_n) = 0$;

(ii) the restriction of F to each open set $I_n \times E$ is lower semicontinuous.

4.2. Degenerate differential inclusions

Let M and L be single-valued linear operators in E satisfying the condition

(ML) $D(L) \subseteq D(M)$ and $\overline{M(D(L))} \subseteq \operatorname{Im} M$.

Consider the following Cauchy problem for a degenerate differential inclusion

$$\frac{dMx(t)}{dt} \in Lx(t) + F(t, Mx(t)), \quad t \in [0,T] \tag{14}$$

$$Mx(0) = y_0 \in \overline{M(D(L))}. \tag{15}$$

With the change $y(t) = Mx(t)$, we can rewrite problem (14), (15) into the form

$$\frac{dy(t)}{dt} \in Ay(t) + F(t, y(t)), \quad t \in [0,T] \tag{16}$$

$$y(0) = y_0 \tag{17}$$

where $A = LM^{-1}$. It is clear that $A \in ML(E)$ if M is non-invertible and that $D(A) = M(D(L))$.

We will suppose that $A = LM^{-1}$ satisfies condition (A).

Remark 27. To present sufficient conditions under which the MLO $A = LM^{-1}$ satisfies the Hille-Yosida condition, let us recall that in a Banach space E a semi-scalar product can be defined as $[u, v] =< u, v^* >$ with $v^* \in J(v)$ (see [14]).

Then we can deduce from Proposition 18 the following assertion:

Proposition 28. *Suppose that*

$$[Lx, Mx] \leq \beta \|Mx\|^2, \quad \forall x \in D(L)$$

for some $\beta \in \mathbb{R}$ *and*

$$\mathrm{Im}(\lambda_0 M - L) = E$$

for some $\lambda_0 > \beta$. *Then the MLO* $A = LM^{-1}$ *satisfies the* (H-Y) *condition of Remark* 8 *with* $C = 1$.

Definition 29. A function $x : [0,h] \to E$ is *a mild solution to problem* (14), (15) *on the interval* $[0,h]$, $0 < h \leq T$ if the function Mx has the form

$$Mx(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds$$

where $f \in L^1([0,h]; E)$ is a selection of the multifunction $t \to F(t, Mx(t))$.

The definition is motivated by the following facts. At first, following [5] Theorem 2.6, it is easy to verify that given a function $f \in L^1([0,h];E)$, every Caratheodory solution to the problem with the MLO A

$$\frac{dy(t)}{dt} \in Ay(t) + f(t)$$

$$y(0) = y_0 \in \overline{D(A)}$$

is necessarily of the form

$$y(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds.$$

Furthermore, the function $t \to U(t)y_0 + \int_0^t U(t-s)f(s)ds$ takes its values in the subspace $\overline{D(A)} = \overline{M(D(L))} \subseteq \mathrm{Im}(M)$ (see condition (ML)). At last, in the non-degenerate case $M = I$, the given definition agrees with the notion of a mild solution for a semilinear differential inclusion (see, e.g. [6]).

From Theorems 24 and 25 we obtain the following existence results.

Theorem 30. *Under conditions* (ML) *and* (A) *for* $A = ML^{-1}$ *and* (F0) – (F4) *for* F*, there exists a mild solution to problem* (14), (15) *on some interval* $[0,h],\ 0 < h \le T$.

Theorem 31. *Under conditions* (ML) *and* (A) *for* $A = ML^{-1}$ *and* (F0), (F1), (F2), (F3') *and* (F4) *for* F*, the set* Σ *of all mild solutions to problem* (14), (15) *on* $[0,T]$ *is nonempty. Moreover, the set*

$$M\Sigma = \{y \in C([0,T];E) : y(t) = Mx(t),\ x \in \Sigma\}$$

is compact in the space $C([0,T];E)$.

4.3. Bounded solutions on the half-line

In this section, we are interested in the existence of bounded mild solutions to the problem

$$\frac{dy}{dt} \in Ay(t) + F(t, y(t)), \quad t \ge 0 \tag{18}$$

$$y(0) = y_0 \in \overline{D(A)} \tag{19}$$

under the following assumptions:

(A_∞) $A \in ML(E)$ is a generator of a generalized C_0-semigroup $\{U(t), t \geq 0\}$ which is uniformly bounded, i.e., there exists a positive constant C such that $\|U(t)\| \leq C$, $t \geq 0$.

Denoting $I = [0, +\infty)$, we will assume that a multimap $F : I \times E \to Kv(E)$ satisfies the conditions similar to those that were considered in the previous section:

($F1_\infty$) for every $x \in E$, the multifunction $F(\cdot, x) : I \to Kv(E)$ has a strongly measurable selection on every compact interval $[a, b] \subset I$;

($F2_\infty$) the multimap $F(t, \cdot) : E \to Kv(E)$ is u.s.c. for a.a. $t \in I$;

($F3_\infty$) there exists a locally integrable function $\alpha \in L^1_{loc}(I)$ such that $\int_0^\infty \alpha(s)ds = r < \infty$ and

$$\|F(t, x)\| \leq \alpha(t)(1 + \|x\|) \text{ for a.a. } t \in I \text{ and } x \in E;$$

($F4_\infty$) there exists a function $k \in L^1_{loc}(I)$ such that

$$\chi(F(t, D)) \leq k(t)\chi(D) \text{ for a.a. } t \in I$$

for every bounded set $D \subset E$.

Denote by $C(I; E)$ the space of all continuous functions $x : I \to E$ with the locally convex topology induced by the topology of uniform convergence on compact subintervals of I. It is known that this topology is completely metrizable by the metric

$$d(x, y) = \sum_{m=0}^{\infty} \frac{2^{-m}\|x - y\|_m}{1 + \|x - y\|_m}$$

where $\|x\|_m := \sup\{\|x(t)\| : t \in I_m = [0, m]\}$, and thus $C(I; E)$ is a Frechet space.

It is easy to see that under conditions ($F1_\infty$) – ($F3_\infty$) the superposition multioperator $\mathcal{P}_F : C(I; E) \to L^1_{loc}(I; E)$,

$$\mathcal{P}_F(x) = \big\{f \in L^1_{loc}(I; E) : f(t) \in F(t, x(t)) \text{ for a.a. } t \in I\big\}$$

is correctly defined, so we can consider the mild solution to the problem (18), (19) as a function $y \in C(I;E)$ satisfying

$$y(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds, \quad f \in \mathcal{P}_F(y).$$

It is also clear that mild solutions to (18), (19) are fixed points of the integral multioperator $\Gamma_\infty : C(I;E) \to C(I;E)$

$$\Gamma_\infty y = \left\{ z : z(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds : f \in \mathcal{P}_F(y) \right\}.$$

Let us denote by $\mathbb{R}_\infty$ the set of sequences $\varkappa = (\varkappa_1, \varkappa_2, \ldots)$ partially ordered by the relation $\varkappa'' \geq \varkappa'$ iff $\varkappa''_i \geq \varkappa'_i$ for all $i = 1, 2, \ldots$ (lexicographic ordering). For a bounded set $\Omega \subset C(I;E)$ define the value $\nu_\infty(\Omega) \in \mathbb{R}_\infty$ as

$$\nu_\infty(\Omega) = \{\nu_1(\Omega_1), \nu_2(\Omega_2), \ldots\}$$

where $\Omega_m = \Omega_{|I_m}$ and ν_m is the MNC on $C(I_m;E)$ defined by (13). It is clear that ν_∞ is a monotone, nonsingular and regular MNC on $C(I;E)$.

From the definition of the topology in $C(I;E)$ and Proposition 23 we can deduce the following

Proposition 32. *The integral multioperator Γ_∞ is an u.s.c. multioperator with compact convex values and it is $\nu_\infty-$condensing on bounded sets.*

We are now in the position to present the main result of this section:

Theorem 33. *Under conditions* (A_∞), $(\mathrm{F1}_\infty)$ – $(\mathrm{F4}_\infty)$ *the set Σ^A_∞ of all mild solutions to problem* (18), (19) *is a nonempty, compact subset of* $C(I;E)$*. Moreover, there exists a constant $N > 0$ such that $\|y\|_m \leq N$, $m = 1, 2, \ldots$ for every $y \in \Sigma^A_\infty$.*

Proof. Consider the convex closed set $\mathcal{K} = \{y \in C(I;E) : y(0) = y_0\}$. Let $\widetilde{y}_0 \in \mathcal{K}$ be defined as $\widetilde{y}_0(t) \equiv y_0$. Consider the following family of multimaps $\Psi : \mathcal{K} \times [0,1] \to Kv(\mathcal{K})$

$$\Psi(y, \lambda) = (1-\lambda)\widetilde{y}_0 + \lambda\Gamma(y).$$

If it easy to see that the family Ψ is u.s.c. and ν_∞-condensing on every bounded set $\Omega \subset \mathcal{K}$. In fact, if we suppose that

$$\nu_\infty(\Psi(\Omega \times [0,1])) \geq \nu_\infty(\Omega)$$

then using nonsingularity property of ν_∞ we obtain that

$$\nu_\infty(\Psi(\Omega \times [0,1])) = \nu_\infty(\overline{co}(\widetilde{y_0} \cup \Gamma(\Omega))) = \nu_\infty(\Gamma(\Omega)),$$

and from Proposition 32 it follows that Ω is relatively compact.

Now we prove that the fixed point set of the family Ψ defined as

$$Fix\Psi = \{y \in \Psi(y, \lambda) \; for \; some \; \lambda \in [0,1]\}$$

is a priori bounded uniformly with respect to each seminorm $\| \cdot \|_m$, $m = 1, 2, \ldots$

Indeed, if $y \in Fix\Psi$, then there exsists $f \in \mathcal{P}_F(y)$ such that for a certain $\lambda \in [0,1]$ and each $t \geq 0$ we have

$$y(t) = (1-\lambda)y_0 + \lambda U(t)y_0 + \lambda \int_0^t U(t-s)f(s)ds.$$

Applying conditions (F3$_\infty$) and (A_∞) we obtain the estimate

$$\begin{aligned} \|y(t)\| &\leq (1+C)\|y_0\| + C\int_0^t \alpha(s)(1+\|y(s)\|)ds \\ &\leq (1+C)\|y_0\| + Cr + C\int_0^t \alpha(s)\|y(s)\|ds. \end{aligned}$$

Applying the Gronwall-Bellmann inequality we obtain that

$$\|y(t)\| \leq Q\exp\{C\int_0^t \alpha(s)ds\},$$

where $Q = (1+C)\|y_0\| + Cr$, and hence

$$\|y\|_m \leq N = Q\exp\{Cr\}, \text{ for } m = 1, 2, \ldots$$

Now we take an open bounded set $W \subset C(I;E)$ containing the set $Fix\Psi$. The ν_∞-condensing family Ψ is fixed point free on the relative boundary $\partial W_{\mathcal{K}}$ and hence it determines the homotopy of the multifield $i-\Gamma$ and the constant field $i-\widetilde{y}_0$. Taking into account that $\widetilde{y}_0 \in W_{\mathcal{K}}$ and using the homotopy and normalization properties of the topological degree, we obtain that $\deg_{\mathcal{K}}(i-\Gamma, \overline{W_{\mathcal{K}}}) = \deg_{\mathcal{K}}(i-\widetilde{y}_0, \overline{W_{\mathcal{K}}}) = 1$ and therefore

$$\emptyset \neq Fix\Gamma \subset W_{\mathcal{K}}.$$

The compactness of $Fix\Gamma$ follows from the fact that Γ is u.s.c. and ν_∞-condensing. ■

As a simple consequence we obtain the following

Theorem 34. *Let M and L be linear operators in E satisfying condition (ML) and let the MLO $A = LM^{-1}$ satisfy (A_∞). Further, let the multimap $F : I \times E \to Kv(E)$ satisfy* (F1$_\infty$) – (F4$_\infty$)*. Then the set Σ_∞ of all mild solutions to the problem*

$$\frac{dMx(t)}{dt} = Lx(t) + F(t, Mx(t)), \quad t \geq 0$$

$$Mx(0) = y_0 \in \overline{M(D(L))}$$

is nonempty. Moreover, the set $M\Sigma_\infty = \{y \in C(I;E) : y(t) = Mx(t), x \in \Sigma_\infty\}$ is compact in $C(I;E)$ and there exists a constant $N > 0$ such that

$$\|Mx\|_m \leq N, \quad m = 1, 2, \ldots$$

for every $x \in \Sigma_\infty$.

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