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OSCILLATION OF DELAY DIFFERENTIAL EQUATION WITH SEVERAL POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract

Some sufficient conditions for oscillation of a first order nonautonomous delay differential equation with several positive and negative coefficients are obtained.

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1. Introduction

In recent years there has been much research onto the oscillation theory of delay differential equations. To a large extent, this is due to the fact that, delay differential equations are widely applied in physics, ecology, biology. They also find many applications in epidemics and infectious diseases. Most of the differential equations models used in vital statistics involve birth and death rates; and thus differential equations with positive and negative coefficients are of crucial importance.

Ladas and Sficas [21] considered a delay differential equation with constant positive and negative coefficients of the form

$$x'(t) + px(t-\tau) - qx(t-\sigma) = 0, \tag{1.1}$$

and proved that the conditions

(h_1) p, q, τ and σ are positive constants,

(h_2) $p < q$, and $\tau \geq \sigma$.

are necessary for the oscillations of all solutions to (1.1).

Arino, Ladas and Sficas [3] improved these results as follows. In addition to the hypotheses (h_1) and (h_2), assume that

(h_3) $q(\tau - \sigma) \leq 1$,

then every non-oscillatory solution to (1.1) tends to zero as $t \to \infty$.

Also Arino, Ladas and Sficas [3] gave new criteria for the oscillation of all solutions to (1.1). Assume that (h_1), (h_2) hold, and $(p-q)\tau > \frac{1}{e}$. Then every solution to (1.1) oscillates.

Equation (1.1) has also been investigated by Gyroi [10]. In [10], the authors proved that if (h_1) and (h_2) hold, $0 \leq q(\tau - \sigma) \leq 1$ and $(p-q)[\tau + q(\tau-\sigma)^2] > \frac{1}{e}$, then (1.1) oscillates. This result is a significant extension of the above mentioned result in [3]. This equation is a particular case of the equation (1.2) below.

Recently, Agwo [1] considered the delay differential equation of several real coefficients and gave a set of necessary and mentioned sufficient conditions for the oscillation of solutions to (1.1) which extended the above results.

In this paper, we consider the first order nonautonomous delay differential equation,

$$x^{'}(t) + \sum_{i=1}^{n} P_i(t)\, x(t-\sigma_i) - \sum_{j=1}^{m} Q_j(t)\, x(t-\tau_j) = 0, \tag{1.2}$$

where

$$P_i,\, Q_j \in C\left([t_0,\infty),\, R^{+}\right) \text{ for } i = 1,\dots,n \text{ and } j = 1,\dots,m,$$

and

$$\sigma_i, \tau_i \in [0,\infty) \text{ for } i = 1,\dots,n \text{ and } j = 1,\dots,m.$$

By a solution to equation (1.2) we mean a function $x \in C([t_0-\rho,\infty),\mathbf{R})$ for some t_0, where $\rho = \max\{\max_{1\leq i\leq n}\ \sigma_i, \max_{1\leq j\leq n}\ \tau_j\}$ satisfies equation (1.2) for all $t \geq t_0$.

As usual, a solution to equation (1.2) is called oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

Qian and Ladas [26] obtained the well-known oscillation criterion

$$(1.3) \qquad \liminf_{t\to\infty} \int_{t-\rho}^{t} [P(s) - Q(s+\tau-\sigma)]\, ds > \frac{1}{e},$$

for (1.2) when $n = m = 1$.

For some contributions to the oscillation theory we refer to the articles by Zhang and Gopalsamy [31], Ladas and Sficas [24], Ladas, Qian and Yan [20], Arino, Gyori and Jawhari [2], Hunt and Yorke [14], Gyori [9], Cheng [4], Kwong [19], Gyori and Ladas [11], Kulenovic, Ladas and Meimardou [15], Kulenovic and Ladas [16, 17, 18], Gopalsamy, Kulenovic and Ladas [8], Ladas and Qian [22, 23], Rodica [28], Wei [20], Hiroshi [12], Hua and Joinshe [13], Xiping Jun and Sui [30], Qiriu [27], Norio [25], Elabbasy, Saker and Saif [5], Elabbasy and Saker [6], and Elabbasy, Hegazi and Saker [7].

Our aim in this paper is to give finite integral conditions for oscillation of all solutions to (1.2) which extend the above results and condition (1.3). In Section 3, we present sufficient conditions for oscillation of (1.2), and give a comparison theorem for the oscillation of (1.2) with the limiting equation with constant coefficients.

In the sequel, when we write a functional inequality we will assume that it holds for all sufficiently large values of t.

2. Lemmas

In this section, we state some preliminary lemmas, taken from [11].

Lemma 2.1. *Assume that* $P_i \in C\left([t_0,\infty), \mathbf{R}^+\right), \tau > 0$ *and*

$$\liminf_{t\to\tau} \int_{t-\tau}^{t} \sum_{i=1}^{n} P_i(s)ds > \frac{1}{e},$$

then the inequality

$$x'(t) + \sum_{i=1}^{n} P_i(t)x(t-\tau) \le 0,$$

has no eventually positive solution, where $\tau = \max_{1\le i\le n}\{\tau_i\}$.

Lemma 2.2. *Let $a \in (-\infty, 0), \tau \in (0, \infty), t_0 \in R$ and suppose that $x \in C([t_0, \infty), \mathbf{R})$ satisfies the inequality*

$$x(t) \leq a + \max_{t-\tau \leq s \leq t} x(s) \qquad for \quad t \geq t_0,$$

then $x(t)$ cannot be a nonnegative function.

Lemma 2.3. *Let p and τ be positive constants, let $z(t)$ be an eventually positive solution to the delay differential inequality $z^{'}(t) + pz(t-\tau) \leq 0$.*

$$z(t-\tau) < \beta z(t)$$

Then for t sufficiently large, where $\beta = \frac{4}{(p\tau)^2}$.

Lemma 2.4. *Let $\upsilon(t)$ be a positive and continuously differentiable function on some interval $[t_0, \infty)$. Assume that there exist positive numbers A and α such that for t sufficiently large*

$$\upsilon^{'}(t) \leq 0 \quad and \quad \upsilon(t-\alpha) < A\,\upsilon(t).$$

Set

$$\Lambda = \Big\{\lambda \geq 0 : \upsilon^{'}(t) + \lambda \upsilon(t) \leq 0 \;\; for\ t\ sufficiently\ large\Big\},$$

then $A > 1$ and $\lambda_0 = \frac{\ln(A)}{\alpha} \notin \Lambda$.

3. Oscillation of (1.2)

In this section we establish sufficient conditions for the oscillation of all solutions to (1.2), and give a comparison theorem for the oscillation with the limiting delay differential equations with constant coefficients.

Theorem 3.1. *Assume that*

(H_1) $P_i, Q_j \in C\left([t_0, \infty), R^+\right), \sigma_i, \tau_j \in [0, \infty)$ *for $i = 1, \ldots, n$ and $j = 1, \ldots, m$;*

(H_2) *There exist a positive number $l \leq m$ and a partition of the set $\{1, \ldots, m\}$ into l disjoint subsets $J_1, J_2, J_3, \ldots, J_l$ such that $j \in J_i$ implies that $\tau_j \leq \sigma_i$;*

(H_3) $P_i(t) \geq \sum_{K\in J_i} Q_k(t+\tau_k-\sigma_i)$ *for* $\geq t_0+\sigma_i-\tau_k$, *and* $i=1,\ldots,l$;
(H_4) $\sum_{i=1}^{l}\sum_{k\in J_i}\int_{t-\sigma_i}^{t-\tau_k} Q_k(s)\,ds \leq 1, \quad t\geq t_0+\sigma_i.$

Let $x(t)$ *be an eventually positive solution to equation* (1.2) *and set*

$$z(t) = x(t) - \sum_{i=1}^{l}\sum_{k\in J_i}\int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k)\,x(s)\,ds, \quad t\geq t_0+\sigma_i-\tau_k. \tag{3.1}$$

Then $z(t)$ *is a nonincreasing and positive function.*

Proof. Assume that $t_1 \geq t_0+\rho$ is such that $x(t)>0$ for $t\geq t_1-\rho$. From (3.1) we have

$$z^{'}(t) = x^{'}(t) - \sum_{i=1}^{l}\sum_{k\in J_i} Q_k(t)\,x(t-\tau_k) + \sum_{i=1}^{l}\sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)x(t-\sigma_i).$$

Hence

$$z^{'}(t) = x^{'}(t) - \sum_{j=1}^{m} Q_j(t)x(t-\tau_j) + \sum_{i=1}^{l}\sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)\,x(t-\sigma_i).$$

From (1.2) we have

$$z^{'}(t) = -\sum_{j=1}^{l} P_i(t)\,x(t-\sigma_j) + \sum_{i=1}^{l}\sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)x(t-\sigma_i) - \sum_{i=p+1}^{n} P_i(t)x(t-\sigma_i).$$

Since

$$\sum_{i=l+1}^{n} P_i(t)\,x(t-\sigma_i) > 0,$$

we have

$$z^{'}(t) \leq -\left[\sum_{i=1}^{l}\Big(P_i(t) - \sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)\Big)x(t-\sigma_i)\right]. \tag{3.2}$$

By using (H_3) we have

$$z^{'}(t) \leq 0 \qquad \text{for} \qquad t\leq t_1+\rho. \tag{3.3}$$

This yields that $z(t)$ is a nonincreasing function. Now we show that $z(t)$ is positive. Otherwise, there exists a $t_2 \geq t_1 + \rho$ and $z^{'}(t) \neq 0$ on $[t_1 + \rho, \infty)$. Then there exists $t_3 \geq t_2$ such that $z(t) \leq z(t_3)$ for $t \geq t_3$.

Thus from (3.1) it follows that for $t \geq t_3$

$$\begin{aligned} x(t) &= z(t) + \sum_{i=1}^{l} \sum_{k \in J_i} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k) x(s)\, ds \\ &\leq z(t_3) + \sum_{i=1}^{l} \sum_{k \in J_i} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k)\, x(s)\, ds. \end{aligned}$$

Hence

$$x(t) \leq z(t_3) + \max_{t-\rho \leq s \leq t} x(s) \left(\sum_{i=1}^{l} \sum_{k \in J_i} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k) ds \right).$$

Therefore, condition (H_4) implies that

$$x(t) \leq z(t_3) + \max_{t-\rho \leq s \leq t} x(s) \qquad \text{for all } t \geq t_3,$$

where $z(t_3) \leq 0$. Thus by Lemma 2.2, we see that $x(t)$ cannot be non-negative $[t_3, \infty)$. This is a contradiction to $x(t) \geq 0$. Then $z(t)$ is a non-increasing and positive function.

Theorem 3.2. *Assume that* $(H_1) - (H_4)$ *hold, and*

$$(H_5) \ \liminf_{t \to \infty} \int_{t-\rho}^{t} \left[\sum_{i=1}^{l} \Big(P_i(s) - \sum_{k \in J_i} Q_k(s+\tau_k - \sigma_i) \Big) \right] ds > \frac{1}{e}.$$

Then every solution to equation (1.2) *oscillates.*

Proof. Assume, for the sake of contradiction, that equation (1.2) has an eventually positive solution $x(t)$. By Theorem 3.1, it follows that the function

$$z(t) = x(t) - \sum_{i=1}^{l} \sum_{k \in J_i} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k) x(s)\, ds,\ t \geq t_0 + \sigma_i - \tau_k,$$

is an eventually positive function. Also by (3.2) we see that eventually,

$$z^{'}(t) + \sum_{i=1}^{l} \left[P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i) \right] x(t - \sigma_i) \leq 0. \tag{3.4}$$

From the fact that $0 < z(t) \leq x(t)$ we have

$$z^{'}(t) + \sum_{i=1}^{p} \left[P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i) \right] z(t - \sigma_i) \leq 0. \tag{3.5}$$

But in view of (H_5) it follows from Lemma 2.1 that the inequality (3.5) cannot have an eventually positive solution. This contradicts the fact that $z(t)$ is eventually positive and completes the proof.

Now we prove that every nonoscillatory solution to (1.2) tends to zero as t tends to infinity.

Theorem 3.3. *Assume that*

(C_1) $\lim_{t\to\infty} P_i(t) = p_i$ *and* $\lim_{t\to\infty} \sup Q_i(t) = q_j$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$;

(C_2) $p_i, q_j, \sigma_i, \tau_j \in (0, \infty)$ *for* $i = 1, \ldots, n,$ *and* $j = 1, \ldots, m$;

(C_3) *There exist a positive number* $l \leq n$ *and a partition of the set* $\{1, \ldots, m\}$ *into* l *disjoint subsets* $J_1, J_2, J_3, \ldots, J_l$ *such that* $j \in J_i$ *implies that* $\tau_j \leq \sigma_i$ *and* $\sum_{k \in J_i} q_k < p_i$;

(C_4) $\sum_{i=1}^{l} \sum_{k \in J_i} q_k(\sigma_i - \tau_k) < 1.$

Then every nonoscillatory solution to (1.2) *tends to zero as* t *tends to infinity.*

Proof. Let $x(t)$ be a nonoscillatory solution to (1.2). We will assume that $x(t)$ is eventually positive (The case where $x(t)$ is eventually negative is similar and will be omitted). Assume that $t_1 \geq t_0 + \rho$ is such that $x(t) > 0$ for $t \geq t_1 - \rho$, $\rho = \max\{\sigma_i, \tau_i\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Set

$$z(t) = x(t) - \sum_{i=1}^{l} \sum_{k \in J_i} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s + \tau_k) x(s)\, ds,\ t \geq t_0 + \sigma_i - \tau_k.$$

From Theorem 3.1 $z(t)$ is nonincreasing and positive. Now we show that $x(t)$ is bounded. Otherwise, there exists a sequence of points $\{t_n\}$ such that

$$\lim_{t\to\infty} t_n = \infty, \lim_{n\to\infty} x(t_n) = \infty \quad \text{and} \quad x(t_n) = \max_{s\le t_n} x(s).$$

From (3.1) and (C_4)

$$z(t_n) = x(t_n) - \sum_{i=1}^{l}\sum_{k\in J_i}\int_{t_n-\sigma_i}^{t_n-\tau_k} Q_k(s+\tau_k)x(s)\,ds$$

$$\ge x(t_n)\left[1 - \sum_{i=1}^{l}\sum_{k\in J_i} q_k(\sigma_i-\tau_k)\right] \to \infty \quad \text{as } n\to\infty,$$

which contradicts (3.2). Then by (3.1) and (3.3) we see that $z(t)$ is also bounded, and $\lim_{t\to\infty} z(t) = k \in R$. By integrating both sides of (3.2) from t_1 to ∞ we obtain

$$\text{(3.6)} \quad k - z(t_1) \le \int_{t_1}^{\infty}\left[\sum_{i=1}^{l}\Big(P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i)\Big)(x(s-\sigma_i))\right] ds.$$

We claim that $\lim_{t\to\infty}\inf x(t) = 0$. Otherwise, there exist a positive constant c, and $t_2 \ge t_1$ such that $x(t) \ge c$ for $t \ge t_2$. Since $x(t) > 0$, thus $x(s-\sigma_i) \ge c^{'}$ for $i = 1, 2, \dots, p$, and some constant $c^{'} > 0$. However, (C_1) implies that

$$\sum_{i=1}^{l}\left(P_i(t) - \sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)\right) \ge \sum_{i=1}^{l}\left(P_i(t) - \sum_{k\in J_i} q_k\right).$$

But by (C_3) we have

$$\sum_{i=1}^{l}\left(P_i(t) - \sum_{k\in J_i} q_k\right) \to \sum_{i=1}^{l}\left(P_i - \sum_{k\in J_i} q_k\right) \text{ as } t\to\infty.$$

Then for t sufficiently large, $\sum_{i=1}^{l}(P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i))$ is bounded below by a positive constant. This contradicts (3.6). Hence $\lim_{t\to\infty}\inf x(t) = 0$. We prove that $\lim_{t\to\infty} x(t) = 0$. Otherwise, let

$$\lim_{t\to\infty}\sup x(t) = \mu > 0,$$

and let $\{t_n\}$ and $\{t_n^{'}\}$ be two sequences such that

$$\lim_{n\to\infty} t_n = \infty,\ \lim_{n\to\infty} t_n^{'} = \infty,\ \lim_{n\to\infty} x(t_n) = 0 \text{ and } \lim_{n\to\infty} x(t_n^{'}) = \mu.$$

Since from (2.1) $z(t_n) \leq x(t_n)$, we have $k \leq 0$. Now choosing $\varepsilon_0 > 0$ sufficiently small, we find

$$\begin{aligned} z(t_n^{'}) &\geq x(t_n^{'}) - \sum_{i=1}^{l} \sum_{k\in J_i} q_k \int_{t_n^{'}-\sigma_i}^{t_n^{'}-\tau_k} x(s)\,ds \\ &\geq x(t_n^{'}) - \sum_{i=1}^{l} \sum_{k\in J_i} q_k(\sigma_i - \tau_k)(\mu + \varepsilon_0). \end{aligned}$$

By taking limits as $n \to \infty$ we obtain

$$k \geq \mu - \sum_{i=1}^{l} \sum_{k\in J_i} q_k(\sigma_i - \tau_k) \geq \mu.$$

As ε_0 is arbitrary we conclude

$$0 \geq k \geq \mu - \sum_{i=1}^{l} \sum_{k\in J_i} q_k(\sigma_i - \tau_k) \geq \mu.$$

This implies that $k = \mu = 0$. Hence $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$ The proof is completed.

Theorem 3.4. *Assume that* $(C_1) - (C_5)$ *hold, and every solution to the delay differential equation*

$$y^{'}(t) + \sum_{i=1}^{l} \left(p_i - \sum_{k\in J_i} q_k \right) y(t - \sigma_i) = 0, \tag{3.7}$$

oscillates. Then every solution to equation (1.2) *also oscillates.*

Proof. Assume for the sake of contradiction that equation (1.2) has a positive solution $x(t)$ such $x(t) > 0$ for $t \geq t_0 - \rho$, $\rho = \max\{\sigma_i, \tau_j\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

As every solution to equation (3.7) oscillates, then the characteristic equation

$$F(\lambda) = \lambda + \sum_{i=1}^{p} \left(P_i - \sum_{k \in J_i} q_k \right) e^{-\lambda \sigma_i} = 0, \tag{3.8}$$

has no real roots. As $F(\infty) = \infty$, it follows $F(\lambda) > 0$ for $\lambda \in R$. In particular,

$$F(0) = \sum_{i=1}^{l} \left(p_i - \sum_{k \in J_i} q_k \right) > 0.$$

Then $F(\infty) = \infty$, and $m = \min_{\lambda \in R} F(\lambda)$ exists and is positive. Thus

$$\lambda + \sum_{i=1}^{l} \left(p_i - \sum_{k \in J_i} q_k \right) e^{-\lambda \sigma_i} \geq m, \quad \lambda \in \mathbf{R},$$

or equivalently

$$\sum_{i=1}^{l} \left(p_i - \sum_{k \in J_i} q_k \right) e^{-\lambda \sigma_i} \geq \lambda + m, \quad \lambda \in \mathbf{R}. \tag{3.9}$$

Since $x(t) > 0$, Theorem 3.1 implies that $z(t)$ is positive, nonincreasing and satisfies

$$z^{'}(t) + \sum_{i=1}^{p} \left(P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i) \right) z(t - \sigma_i) \leq 0.$$

Then

$$z^{'}(t) + \left(P_{i_0}(t) - \sum_{k \in J_{i_0}} Q_k(t + \tau_k - \sigma_{i_0}) \right) z(t - \sigma_{i_0}) \leq 0, \tag{3.10}$$

where the index i_0 is chosen in such a way that

$$P_{i_0}(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_{i_0}) > 0.$$

Clearly, for t sufficiently large from equation (3.10) we have

$$z^{'}(t) + \frac{1}{2} \left(p_{i_0} - \sum_{k \in J_{i_0}} q_k \right) z(t - \sigma_{i_0}) \leq 0.$$

Hence

$$z^{'}(t) + Mz(t - \sigma_{i_0}) \leq 0 \tag{3.11}$$

with $M = \frac{1}{2}(p_{i_0} - \sum_{k \in J_{i0}} q_k)$. By Lemma 2.3 and (3.11) we have $z(t - \sigma_{i_0}) \leq \beta z(t)$ with $\beta = \frac{4}{(M\sigma_{i_0})^2}$.

Set

$$\Lambda = \left\{ \lambda \geq 0 : \, z^{'}(t) + \lambda z(t) \leq 0 \quad \text{for sufficiently large} \right\}.$$

Clearly, Λ is a nonempty subinterval of $\mathbf{R}^+$. Showing that Λ has the following contradictory properties will complete the proof that every solution of equation (1.2) oscillates.

(p_1) Λ is bounded above.

(p_2) $\lambda \in \Lambda \Rightarrow \lambda + \frac{m}{2} \in \Lambda$ where λ is positive and satisfies (3.9). Since

$$z^{'}(t) \leq 0 \text{ and } z(t - \sigma_{i_0}) \leq \beta z(t),$$

then Lemma 2.4 yields $\lambda_0 = \frac{\ln(\beta)}{\sigma_{i_0}} \in \Lambda$. Then Λ is bounded above. In order to establish (p_2), let $\lambda \in \Lambda$, and set

$$\Psi(t) = e^{\lambda t} z(t).$$

Then

$$\Psi^{'}(t) = e^{\lambda t} \left[z^{'}(t) + \lambda z(t) \right] \leq 0,$$

what shows that $\Psi(t)$ is decreasing. Thus $\Psi(t - \sigma_i) \geq \Psi(t)$ for some $i = 1, \ldots, l$. Now, choose $\varepsilon > 0$ such that for t sufficiently large $P_i(t) \geq p_i - \varepsilon$, $Q_j(t) \leq q_j$ and $\sum_{i=1}^{l} e^{\lambda \sigma_i} \varepsilon \leq \frac{m}{2}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then

$$\begin{aligned} & z^{'}(t) + \left(\lambda + \frac{m}{2} \right) z(t) \\ & \leq \sum_{i=1}^{l} \left(P_i(t) + \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i) \right) z(t - \sigma_i) + \left(\lambda + \frac{m}{2} \right) z(t) \\ & \leq \sum_{i=1}^{l} \left(-P_i(t) + \sum_{k \in J_i} q_k \right) z(t - \sigma_i) + \left(\lambda + \frac{m}{2} \right) z(t), \end{aligned}$$

where $Q_k(t) \le q_k$ for $k \in J_i$ and $t \ge t_0 + \sigma_i - \tau_k$. Hence

$$z^{'}(t) + \Big(\lambda + \frac{m}{2}\Big) z(t) \le -\sum_{i=1}^{l}\Bigg((p_i - \varepsilon) - \sum_{k\in J_i} q_k\Bigg) z(t-\sigma_i) + \Big(\lambda + \frac{m}{2}\Big) z(t).$$

As $\Psi(t-\sigma_i) \ge \Psi(t)$, then we have

$$z^{'}(t) + \Big(\lambda + \frac{m}{2}\Big) z(t) \le e^{-\lambda t}\Bigg[-\sum_{i=1}^{l}\Bigg(p_i - \sum_{k\in J_i} q_k\Bigg)e^{\lambda\sigma_i} + \sum_{i=1}^{l}\varepsilon e^{\lambda\sigma_i} + \Big(\lambda + \frac{m}{2}\Big)\Bigg]\Psi(t).$$

Since $\sum_{i=1}^{l}\varepsilon e^{\lambda\sigma_i}\varepsilon \le \frac{m}{2}$, from (3.9) we obtain

$$z^{'}(t) + \Big(\lambda + \frac{m}{2}\Big)\, z(t) \le e^{-\lambda t}\Big(-(\lambda+m) + \frac{m}{2} + \Big(\lambda + \frac{m}{2}\Big)\Big)\Psi(t) = 0.$$

Hence

$$z^{'}(t) + \Big(\lambda + \frac{m}{2}\Big)\, z(t) \le 0.$$

Then $\lambda + \frac{m}{2} \in \Lambda$. Thus (p_2) is proved. Then every solution to equation (1.2) oscillates.

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