

ON THE PICARD PROBLEM FOR HYPERBOLIC DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract

B. Rzepecki in [5] examined the Darboux problem for the hyperbolic equation $z_{xy} = f(x, y, z, z_{xy})$ on the quarter-plane $x \geq 0$, $y \geq 0$ via a fixed point theorem of B.N. Sadovskii [6]. The aim of this paper is to study the Picard problem for the hyperbolic equation $z_{xy} = f(x, y, z, z_x, z_{xy})$ using a method developed by A. Ambrosetti [1], K. Goebel and W. Rzymowski [2] and B. Rzepecki [5].

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1. NOTATIONS AND FORMULATIONS

By $(E, \|\cdot\|)$ we shall denote a real Banach space. The symbol $(R^k, \|\cdot\|)$ is reserved for n -dimensional Euclidean space. We introduce the notion

$$R_+ =]0, \infty[),$$

$$Q = R_+ \times R_+ \subset R^2$$

and

$$\Omega = Q \times E \times E \times E.$$

Let \mathbf{B} be the family of bounded sets of E . Then $\alpha : \mathbf{B} \rightarrow R_+$, defined by

$$\alpha(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter } \leq d\},$$

$B \in \mathbf{B}$, is called Kuratowski's measure of noncompactness.

Let $\sigma, \tau : R_+ \rightarrow E$ be two functions such that $\tau(0) = \sigma(\beta(0))$, where $\beta : R_+ \rightarrow R_+$ is a given function.

We shall consider the following problem

$$\begin{aligned} z_{xy}(x, y) &= f(x, y, z(x, y), z_x(x, y), z_{xy}(x, y)) \\ (P.P) \quad z(x, 0) &= \sigma(x) \\ z(\beta(y), y) &= \tau(y) \end{aligned}$$

where $f : \Omega \rightarrow E$ is a given function.

The above (P.P) problem is usually called the Picard problem for hyperbolic equations.

2. THE MAIN RESULT

The aim of this paper is to prove the following theorem

Theorem 2.1. *Assume that $\sigma, \tau : R_+ \rightarrow E$ are C^1 -mappings such that $\tau(0) = \sigma(\beta(0))$, where $\beta : R_+ \rightarrow R_+$ is a function of class C^1 satisfying the following condition $\beta : [0, M_n] \rightarrow [0, M_n]$, where (M_n) , $n \in N$ is an increasing and unbounded sequence.*

Assume further that $f : \Omega \rightarrow E$ is uniformly continuous on bounded subsets of Ω and

$$(2.1) \quad \|f(x, y, u, v, w)\| \leq G(x, y, \|u\|, \|v\|, \|w\|)$$

for $(x, y, u, v, w) \in \Omega$. Suppose that for each bounded subset P of Q there exist nonnegative constants $k(P)$ and $L(P) < \frac{1}{2}$ such that

$$(2.2) \quad \alpha(f(x, y, U, V, w)) \leq k(P)(\alpha(U) + \alpha(V))$$

and

$$(2.3) \quad \|f(x, y, u, v, w_1) - f(x, y, u, v, w_2)\| \leq L(P) \|w_1 - w_2\|$$

for all $(x, y) \in P$, $u, v, w_1, w_2 \in E$. For any nonempty bounded subsets U, V of E , let α denote Kuratowski's measure of noncompactness in E .

Assume in addition that the function $(x, y, r, s, t) \rightarrow G(x, y, r, s, t)$ is nondecreasing for each $(x, y) \in Q$ (i.e. $0 \leq r_1 \leq r_2$, $0 \leq s_1 \leq s_2$ and $0 \leq t_1 \leq t_2$ implies $G(x, y, r_1, s_1, t_1) \leq G(x, y, r_2, s_2, t_2)$) and the scalar inequality

$$(2.4) \quad G \left(x, y, \left| \int_{\beta(y)}^x \int_0^y g(s, t) ds dt \right|, \int_0^y g(x, t) dt, g(x, y) \right) \leq g(x, y)$$

has a locally bounded solution g_0 on Q .

Under these assumptions, $(P.P)$ has at least one solution on Q .

For the proof we need the following two lemmas.

Lemma 2.1. *Let (M, d) be a metric space and let A_1, A_2 be transformations mapping bounded sets of M into bounded sets of M . Assume that*

$$F : A_1(M) \times A_2(M) \times M \rightarrow M$$

is a mapping such that

$$d(F(A_1x, A_2y, z_1), F(A_1x, A_2y, z_2)) \leq Ld(z_1, z_2)$$

for $(x, y, z_1, z_2) \in M$, $L \geq 0$

and

$$\alpha(F(A_1X \times A_2X \times \{z\})) \leq \psi_1(\alpha(A_1X)) + \psi_2(\alpha(A_2X))$$

for $z \in M$, X being a bounded subset of M , and $\psi_i : R_+ \rightarrow R_+$; $i = 1, 2$.

Then

$$\alpha(F(A_1X \times A_2X \times X)) \leq 2L\alpha(X) + \psi_1(\alpha(A_1X)) + \psi_2(\alpha(A_2X))$$

for any bounded subset X of M .

The proof of this Lemma is similar to that in [4].

Lemma 2.2. *If W is a bounded equicontinuous subset of a Banach space of continuous E -valued functions defined on a compact subset $P = [a_1, a_2] \times [b_1, b_2]$ of Q , then*

$$\alpha \left(\int_{a_1}^{a_2} \int_{b_1}^{b_2} W(s, t) ds dt \right) \leq \int_{a_1}^{a_2} \int_{b_1}^{b_2} \alpha(W(s, t)) ds dt.$$

Lemma 2.2 is an adaptation of the corresponding result of Goebel and Rzymowski [2].

Proof of Theorem 2.1. Without loss of generality, we may assume that $\sigma = 0$ and $\tau = 0$ (see [3]). Problem (P.P) is equivalent to the functional-integral equation

$$(2.5) \quad w(x, y) = f \left(x, y, \int_{\beta(y)}^x \int_0^y w(s, t) ds dt, \int_0^y w(x, t) dt, w(x, y) \right)$$

Denote by $C(Q, E)$ the space of all continuous functions from Q to E ($C(Q, E)$ is a Frechet space whose topology is introduced by seminorms of uniform convergence on compact subsets of Q), and by \mathbf{X} the set of all $w \in C(Q, E)$ with

$$(2.6) \quad \|w(x, y)\| \leq g_0(x, y)$$

$(x, y) \in Q$. Let P be a bounded subset of Q . From the uniform continuity of f on bounded subsets of Ω there follows the existence of a function $\delta_P : (0, \infty) \rightarrow (0, \infty)$ such that

$$(2.7) \quad \left\| \begin{array}{l} f(x', y', \int_{\beta(y')}^{x'} \int_0^{y'} w(s, t) ds dt, \int_0^{y'} w(x', t) dt, w(x', y')) \\ -f(x'', y'', \int_{\beta(y'')}^{x''} \int_0^{y''} w(s, t) ds dt, \int_0^{y''} w(x'', t) dt, w(x'', y'')) \end{array} \right\| < \varepsilon$$

$w \in \mathbf{X}$; (x', y') and $(x'', y'') \in Py$ satisfy the relations $|x' - x''| < \delta_P(\varepsilon)$ and $|y' - y''| < \delta_P(\varepsilon)$.

Consider the set $\mathbf{X}_0 \subset \mathbf{X}$ possessing the following property: for each bounded subset $P \subset Q$, $\varepsilon > 0$ and $|x' - x''| < \delta_P(\varepsilon)$, $|y' - y''| < \delta_P(\varepsilon)$, $(x', y'), (x'', y'') \in P$, the inequality

$$(2.8) \quad \left\| w(x', y') - w(x'', y'') \right\| \leq (1 - L(P))^{-1} \varepsilon$$

holds for every $w \in \mathbf{X}_0$.

The set \mathbf{X}_0 is a closed, convex and almost equicontinuous subset of $C(Q, E)$. To apply the fixed point theorem of B.N. Sadovskii [6] we define the continuous mapping

$$T : C(Q, E) \rightarrow C(Q, E)$$

by the formula

$$(2.9) \quad (Tw)(x, y) = f \left(x, y, \int_{\beta(y)}^x \int_0^y w(s, t) ds dt, \int_0^y w(x, t) dt, w(x, y) \right).$$

Let $w \in \mathbf{X}_0$. Then

$$(2.10) \quad \begin{aligned} & \| (Tw)(x, y) \| \\ & \leq G \left(x, y, \left| \int_{\beta(y)}^x \int_0^y \|w(s, t)\| ds dt \right|, \int_0^y \|w(x, t)\| dt, \|w(x, y)\| \right) \\ & \leq G \left(x, y, \left| \int_{\beta(y)}^x \int_0^y g_0(s, t) ds dt \right|, \int_0^y g_0(x, t) dt, g_0(x, y) \right) \leq g_0(x, y). \end{aligned}$$

Furthermore, for $\varepsilon > 0$ and $(x', y'), (x'', y'') \in P$ such that $|x' - x''| < \delta_P(\varepsilon)$, $|y' - y''| < \delta_P(\varepsilon)$ we have (see (2.3), (2.7) and (2.8))

$$(2.11) \quad \begin{aligned} & \left\| (Tw)(x', y') - (Tw)(x'', y'') \right\| \\ & \leq \left\| \begin{aligned} & f(x', y', \int_{\beta(y')}^x \int_0^{y'} w(s, t) ds dt, \int_0^{y'} w(x', t) dt, w(x', y')) + \\ & - f(x', y', \int_{\beta(y')}^x \int_0^{y'} w(s, t) ds dt, \int_0^{y'} w(x', t) dt, w(x'', y'')) \end{aligned} \right\| \\ & + \left\| \begin{aligned} & f(x', y', \int_{\beta(y')}^x \int_0^{y'} w(s, t) ds dt, \int_0^{y'} w(x', t) dt, w(x'', y'')) + \\ & - f(x'', y'', \int_{\beta(y'')}^x \int_0^{y''} w(s, t) ds dt, \int_0^{y''} w(x'', t) dt, w(x'', y'')) \end{aligned} \right\| \\ & \leq L(P) \left\| w(x', y') - w(x'', y'') \right\| + \varepsilon \leq (1 - L(P))^{-1} \varepsilon. \end{aligned}$$

Thus, the inclusion $T(\mathbf{X}_0) \subset \mathbf{X}_0$ holds.

Let n be a positive integer and let W be a nonempty subset of \mathbf{X}_0 . Put $P_n = [0, M_n] \times [0, M_n]$, $k_n = k(P_n)$ and $L_n = L(P_n)$. Now we shall show the basic inequality (see [5]):

$$(2.12) \quad \begin{aligned} & \sup_{P_n} \exp(-p_n y) \alpha(T(W(x, y))) \\ & \leq (p_n^{-1} k_n (M_n + 1) + 2L_n) \sup_{P_n} \exp(-p_n y) \alpha(W(x, y)) \end{aligned}$$

where $p_n > 0$ ($n = 1, 2, \dots$).

By Lemma 2.2 (see [5]), we obtain for a fixed $(x, y) \in P_n$ the following inequality

$$(2.13) \quad \begin{aligned} & \alpha \left(\int_{\beta(y)}^x \int_0^y W(s, t) ds dt \right) \\ & \leq \left| \int_{\beta(y)}^x \int_0^y \exp(-p_n t) \exp(p_n t) \alpha(W(s, t)) ds dt \right| \\ & \leq \int_0^{M_n} \int_0^y \exp(-p_n t) \exp(p_n t) \alpha(W(s, t)) ds dt \\ & \leq \sup_{P_n} (\exp(-p_n t) \alpha(W(s, t))) \int_0^{M_n} \int_0^y \exp(p_n t) ds dt \\ & \leq p_n^{-1} M_n \exp(p_n y) \sup_{P_n} (\exp(-p_n t) \alpha(W(s, t))). \end{aligned}$$

Analogously, we have

$$(2.14) \quad \begin{aligned} & \alpha \left(\int_0^y W(x, t) dt \right) \leq \int_0^y \alpha(W(x, t)) dt \\ & = \int_0^y \exp(-p_n t) \exp(p_n t) \alpha(W(x, t)) dt \\ & \leq \sup_{P_n} (\exp(-p_n t) \alpha(W(s, t))) \int_0^y \exp(p_n t) dt \\ & \leq p_n^{-1} \exp(p_n y) \sup_{P_n} \exp(-p_n t) \alpha(W(s, t)). \end{aligned}$$

The inequality (2.12) is a simple consequence of (2.13), (2.14) and Lemma 2.1. Let $p_n > (1 - 2L_n)^{-1}k_n(M_n + 1)$ ($n = 1, 2, \dots$). Define

$$\Phi(W) = (\sup_{P_1} \exp(-p_1 y) \alpha(W(x, y)), \sup_{P_2} \exp(-p_2 y) \alpha(W(x, y)), \dots)$$

for any nonempty subset W of \mathbf{X}_0 .

By Ascoli's theorem, the properties of α and inequality (2.12) it follows that all assumptions of B.N. Sadovskii's fixed point theorem are satisfied. Consequently, the mapping T has a fixed point in \mathbf{X}_0 . The proof of the Theorem is complete.

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