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CONTROLLABILITY THEOREM FOR NONLINEAR DYNAMICAL SYSTEMS

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Abstract

Some sufficient conditions for controllability of nonlinear systems described by differential equation $\dot{x} = f(t, x(t), u(t))$ are given.

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1. Introduction

It was proved in the author's paper [2] that for a given multifunction $F : [0,T] \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ satisfying the usual Carathéodory type condition, the boundary value problem

$$(1) \qquad \begin{cases} \dot{x}(t) \in F(t, x(t)) \quad \text{for a.e.} \quad t \in [0,T] \\ x(0) = x_0, \; x(T) = x_1 \end{cases}$$

has for every $x_1, x_0 \in \mathbb{R}^n$ at least one solution if and only if there is a nonempty weakly compact set $\Lambda \subset L^F_{x_1 - x_0} \subset L([0,T], \mathbb{R}^n)$ such that

$$(2) \qquad x_1 - x_0 \in \int_0^T F\left(t, x_0 + \int_0^t v(\tau) d\tau\right) dt$$

for every $v \in \Lambda$, where $L^F_{x_1-x_0}$ denotes the set of all $v \in L([0,T], \mathbb{R}^n)$ such that $|v(t)| \leq \sup_{x \in \mathbb{R}^n} \|F(t,x)\|$ a.e. on $[0,T]$ and $x_1 - x_0 = \int_0^T v(t)dt$. Furthermore, it was proved in [2] that (2) is equivalent to the inequality:

$$< p, x_1 - x_0 > \leq \sum_{i=1}^{N} \mu(E_i) \sup_{t \in E_i} h_{F(t, x_0 + \int_0^t v(\tau)d\tau)dt}(p) \tag{3}$$

for every measurable partition $\{E_1, \ldots, E_N\}$ to $[0,T]$, $v \in \Lambda$ and $p \in S^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$, where h_C denotes the support function of a set $C \subset \mathbb{R}^n$.

We shall use the above equivalences to define some sufficient conditions for the existence of a pair $(x,u) \in AC([0,T], \mathbb{R}^n) \times \mathcal{M}([0,T], \mathbb{R}^m)$, such that

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \ \text{ for a.e. } \ t \in [0,T] \\ x(0) = x_0, x(T) = x_1, \ u(t) \in U(t) \ \text{ for a.e. } \ t \in [0,T], \end{cases} \tag{4}$$

where $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies the Carathéodory conditions, $x_1, x_0 \in \mathbb{R}^n$ and $U : [0,T] \to \mathrm{Conv}(\mathbb{R}^m)$ is a measurable set valued mapping. As usual $AC([0,T], \mathbb{R}^n)$ and $\mathcal{M}([0,T], \mathbb{R}^m)$ denote the spaces of all absolutely continuous and of all measurable, respectively, functions on $[0,T]$. We denote by $\mathrm{Comp}(\mathbb{R}^k)$ and $\mathrm{Conv}(\mathbb{R}^k)$ spaces of all nonempty compact and nonempty compact convex subsets of $\mathbb{R}^k$.

2. Controllability theorem

We begin with the following lemmas.

Lemma 1. *Let $A \in \mathbb{R}^n$ and $G : [0,T] \to \mathrm{Comp}(\mathbb{R}^n)$ be measurable and bounded. Suppose for every measurable partition $\{E_1, \ldots, E_N\}$ to $[0,T]$ and $i = 1, \ldots, N$ there exists $(t_i, x_i) \in \mathrm{Graph}(G|E_i)$ such that*

$$< p, A > \leq \sum_{i=1}^{N} \mu(E_i) < p, x_i > \tag{5}$$

for every $p \in S^n$, where $G|E_i$ denotes the restriction of G to the set E_i. Then for every measurable partition $\{E_1, \ldots, E_N\}$ to $[0,T]$ and $p \in S^n$ one has

$$< p, A > \leq \sum_{i=1}^{N} \mu(E_i) \sup_{t \in E_i} h_{G(t)}(p). \tag{6}$$

Proof. The proof follows immediately from the inequalities

$$< p, A > \leq \sum_{i=1}^{N} \mu(E_i) < p, x_i >$$

$$\leq \sum_{i=1}^{N} \mu(E_i) h_{G(t_i)}(p) \leq \sum_{i=1}^{n} \mu(E_i) \sup_{t \in E_i} h_{G(t)}(p).$$ ■

Lemma 2. *Suppose $F : [0,T] \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ and $x_1, x_0 \in \mathbb{R}^n$ are such that the following conditions are satisfied*

(i) *$F(\cdot, x)$ is measurable for fixed $x \in \mathbb{R}^n$,*
(ii) *$F(t, \cdot)$ is u.s.c. for fixed $t \in [0,T]$,*
(iii) *F is bounded,*
(iv) *$L^F_{x_1 - x_0} \neq \emptyset$.*

If furthermore for every measurable partition $\{E_1, \ldots, E_N\}$ to $[0,T]$, $v \in L^F_{x_1 - x_0}$ and $i = 1, \ldots, N$ there exists $(t_i, x_i) \in \mathrm{Graph}(F \circ v|E_i)$, such that

$$< p, x_1 - x_0 > \leq \sum_{i=1}^{N} \mu(E_i) < p, x_i > \tag{7}$$

for $p \in S^n$, where $(F \circ v)(t) = F(t, x_0 + \int_0^t v(\tau)d\tau)$, then there exists $x \in AC([0,T], \mathbb{R}^n)$ such that conditions (1) are satisfied.

Proof. The proof follows immediately from Theorem 1 given in [2] and Lemma 1. ■

Corollary 1. *Suppose $F : [0,T] \times \mathbb{R}^n \to \mathrm{Conv}(\mathbb{R}^n)$ and $x_1, x_0 \in \mathbb{R}^n$ are such that conditions (i) – (iv) of Lemma 2 are satisfied. If for every measurable partition $\{E_1, \ldots, E_N\}$ to $[0,T], v \in L^F_{x_1 - x_0}$ and $i = 1, \ldots, N$ there exists $t_i \in E_i$ such that*

$$\frac{x_1 - x_0}{T} \in \bigcap_{v \in L^F_{x_1 - x_0}} F\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau\right) \quad \text{for} \quad i = 1, \dots, N$$

then there is $x \in AC([0,T], \mathbb{R}^n)$ *such that conditions* (1) *are satisfied.*

Indeed, for every $i = 1, \dots, N$ and $v \in L^F_{x_1 - x_0}$ there is $x_i \in F(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau)$ such that

$$\mu(E_i) \ < p, \frac{x_1 - x_0}{T} > \leq \mu(E_i) \ < p, x_i > \ \ for \ p \in S^n.$$

Thus (7) is satisfied.

Now we obtain the following controllability theorems.

Theorem 3. *Suppose* $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $x_1, x_0 \in \mathbb{R}^n$ *are such that:*

(i) $f(\cdot, x, u)$ *is measurable for fixed* $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$,

(ii) $f(t, \cdot, \cdot)$ *is continuous for fixed* $t \in [0,T]$,

(iii) $f(t, x, \cdot)$ *is affine for fixed* $(t, x) \in [0,T] \times \mathbb{R}^n$,

(iv) f *is bounded,*

(v) $L^f_{x_1 - x_0} \neq \emptyset$.

Assume furthermore that $U : [0,T] \to \mathrm{Conv}(\mathbb{R}^n)$ *is a measurable and bounded set-valued mapping such that for every measurable partition* $\{E_1, \dots, E_N\}$ *to* $[0,T]$, $v \in L^f_{x_1 - x_0}$, *and* $i = 1, \dots, N$ *there is* $(t_i, u_i) \in \mathrm{Graph}(U|E_i)$ *such that*

$$< p, x_1 - x_0 > \leq \sum_{i=1}^{N} \mu(E_i) < p, f\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau, u_i\right) > \tag{8}$$

for $p \in S^n$. *Then there exists a pair* $(x, u) \in AC([0,T], \mathbb{R}^n) \times \mathcal{M}([0,T], \mathbb{R}^m)$ *such that conditions* (4) *are satisfied.*

Proof. Let us observe ([1], Theorem 1) that there exists a sequence of measurable functions $g_n : [0,T] \to \mathbb{R}^m$ such that $U(t) = \overline{co} \bigcup_{n=1}^{\infty} \{g_n(t)\}$. Then, for a fixed $x \in \mathbb{R}^n$ one has

$$F(t, x) \overset{\triangle}{=} f(t, x, U(t)) = \overline{co} \bigcup_{n=1}^{\infty} \{f(t, x, g_n(t))\}$$

which implies the measurability of $F(\cdot, x)$ for a fixed $x \in \mathbb{R}^n$ and $F(t,x) \in \mathrm{Conv}(\mathbb{R}^n)$ for every $(t,x) \in [0,T] \times \mathbb{R}^n$. Finally ([3], Proposition II.2.5), $F(t,\cdot)$ is continuous for a fixed $t \in [0,T]$. It is also clear that F is bounded and $L^F_{x_1-x_0} \neq \emptyset$. For every measurable partition $\{E_1, \ldots, E_N\}$ to $[0,T], v \in L^F_{x_1-x_0}$, and $i = 1, \ldots, N$ there is $x_i = f(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau, u_i) \in F(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau)$ such that for $p \in S^n$ the inequality (7) is satisfied. Hence, by Lemma 2, there is an $x \in AC([0,T], \mathbb{R}^n)$ such that conditions (1) are satisfied. In particular, we have $\dot{x}(t) \in F(t, x(t))$ for a.e. $[0,T]$. Thus ([3], Theorem II.3.12) there is $u \in \mathcal{M}([0,T], \mathbb{R}^m)$ such that $u(t) \in U(t)$ and $\dot{x}(t) = f(t, x(t), u(t))$ for a.e. $t \in [0,T]$. ■

Immediately from the above theorem we obtain the following.

Theorem 4. *Suppose* $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $x_1, x_0 \in \mathbb{R}^n$ *are such that conditions* (i) – (v) *of Theorem* 3 *are satisfied and let* $U : [0,T] \to \mathrm{Conv}(\mathbb{R}^m)$ *be bounded and measurable. Let* $f(t,x,u) = \alpha(t,x) + \beta(t,x) \cdot u$ *for* $(t,x) \in [0,T] \times \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$, *where* $\beta(t,x)$ *is* $n \times m$*-matrix. If for every measurable partition* $\{E_1, \ldots, E_N\}$ *to* $[0,T], v \in L^f_{x_1-x_0}$ *and* $i = 1, \ldots, N$ *there is* $t_i \in E_i$ *such that.*

$$(9)\quad \frac{x_1 - x_0}{T} - \alpha\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau\right) \in \beta\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau\right) \cdot U(t_i)$$

then there exists $(x,u) \in (AC([0,T], \mathbb{R}^n \times \mathcal{M}([0,T], \mathbb{R}^m)$ *such that conditions* (4) *are satisfied.*

Proof. Let us observe that (9) imples that for every measurable partition $\{E_1, \ldots, E_N\}$, of $[0,T], v \in L^f_{x_1-x_0}$ and $i = 1, \ldots, N$ there exists $(t_i, u_i) \in \mathrm{Graph}(U|E_i)$ such that

$$\sum_{i=1}^{N} \mu(E_i) < p, \frac{x_1 - x_0}{T} - \alpha\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau\right)$$

$$-\beta\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau\right) \cdot u_i > \leq 0,$$

for $p \in S^n$ which in particular implies the inequality (8) for every $p \in S^n$. Thus the result follows immediately from Theorem 3. ■

Theorem 5. *Suppose* $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $x_1, x_0 \in \mathbb{R}^n$ *satisfy conditions* (i) – (v) *of Theorem* 3 *and let* $U : [0,T] \to \mathrm{Conv}(\mathbb{R}^m)$ *be bounded* *and measurable and such that* $0 \in U(t)$ *for* $t \in [0,T]$. *Let* $f(x,x,u) = \alpha + \beta(t,x) \cdot u$ *for* $(t,x) \in [0,T] \times \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$, *where* $\alpha \in \mathbb{R}^n$ *and* $\beta(t,x)$ *is* $n \times m$*-matrix. If* $(x_1 - x_0)/T = \alpha$, *then there exists* $(x,u) \in (AC([0,T],\mathbb{R}^n \times \mathcal{M}([0,T],\mathbb{R}^m)$ *such that conditions* (4) *are satisfied.*

Proof. Let us observe that for every measurable partition $\{E_1,\ldots,E_N\}$ to $[0,T]$, $v \in L^f_{x_1-x_0}$, $t_i \in E_i$ for $i = 1,2,\ldots,N$ one has

$$0 \in \beta\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau\right) \cdot U(t_i).$$

This in particular implies that conditions (9) of Theorem 4 are satisfied. Thus the result follows immediately from Theorem 4. ∎

Finally, we get the following approximation theorems.

Theorem 6. *Suppose* $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $x_1, x_0 \in \mathbb{R}^n$ *satisfy conditions* (i), (ii), (iv) *and* (v) *of Theorem* 3 *and let* $f(t,x,\cdot)$ *be differentiable on* $O_h = \{u \in \mathbb{R}^m : \|u\| < h\}$ *for* $h > 0$. *If* $U : [0,T] \to \mathrm{Conv}(\mathbb{R}^m)$ *is bounded and measurable and for every measurable partition* $\{E_1,\ldots,E_N\}$ *to* $[0,T]$, $v \in L^f_{x_1-x_0}$ *and* $i = 1,\ldots,N$ *there is* $t_i \in E_i$ *such that*

$$\frac{x_1 - x_0}{T} - f\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau, 0\right) \in \beta\left(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau\right) \cdot U(t_i) \tag{10}$$

where $\beta(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau)$ *is the Jacobi matrix of* $f(t_i, x_0 + \int_0^{t_i} v(\tau)d\tau, \cdot)$ *at* $u = 0$, *then for every* $\varepsilon > 0$ *there is* $\delta_\varepsilon > 0$ *such that for every pair* $(x_\varepsilon, u_\varepsilon) \in AC([0,T] \times \mathbb{R}^n) \times \mathcal{M}([0,T],\mathbb{R}^m)$ *satisfying conditions* $\dot{x}_\varepsilon(t) = f(t,x_\varepsilon(t),0) + \beta(t,x_\varepsilon(t)) \cdot u_\varepsilon(t)$, $x_\varepsilon(0) = x_0$, $x_\varepsilon(T) = x_1$, $u_\varepsilon(t) \in U(t)$ *for a.e.* $t \in [0,T]$ *and such that* $\|u_\varepsilon(t)\| \le \delta_\varepsilon$ *for a.e.* $t \in [0,T]$ *one has* $\|\dot{x_\varepsilon}(t) - f(t,x_\varepsilon(t),u_\varepsilon(t))\| \le \varepsilon$ *for a.e.* $t \in [0,T]$.

Proof. For every $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that

$$\|f(t,x,u) - [f(t,x,0) + \beta(t,x) \cdot u]\| \le \varepsilon$$

for every $u \in (-h,h)$ such that $\|u\| < \delta_\varepsilon$.

By Theorem 4 there exists $(x,u) \in AC([0,T],\mathbb{R}^n) \times \mathcal{M}([0,T],\mathbb{R}^m)$ such that

$$\begin{cases} \dot{x}(t) = f(t,x(t),0) + \beta(t,x(t)) \cdot u(t) & \text{for a.e. } t \in [0,T] \\ x(0) = x_0, \ x(T) = x_1, \ u(t) \in U(t) & \text{for a.e. } t \in [0,T]. \end{cases}$$

If there is a pair $(x_\varepsilon, u_\varepsilon)$ such that $\|u_\varepsilon(t)\| \le \delta_\varepsilon$, then $\|f(t,x_\varepsilon(t),u_\varepsilon(t)) - \dot{x}_\varepsilon(t)\| \le \varepsilon$ for a.e. $t \in [0,T]$. ∎

Theorem 7. *Suppose* $F : [0,T] \times \mathbb{R}^n \to \mathrm{Comp}(\mathbb{R}^n)$ *and* $x_1, x_2 \in \mathbb{R}^n$ *are such that conditions* (i), (iii), (iv) *of Lemma* 2 *are satisfied. If* $\{F(t,\cdot)\}_{t\in[0,T]}$ *is uniformly equicontinuous and for every measurable partition* $\{E_1,\dots,E_N\}$ *to* $[0,T]$, $v \in L^F_{x_1-x_0}$ *and* $i = 1,\dots,N$ *there exists* $t_i \in E_i$ *such that*

$$\frac{x_1 - x_0}{T} \in \bigcap_{v \in L^F_{x_1-x_0}} F\left(t, x_0 + \int_0^{t_i} v(\tau)d\tau\right) \quad \text{for} \quad i = 1,\dots,N$$

then for every $\varepsilon > 0$ *there is* $x_\varepsilon \in AC([0,T],\mathbb{R}^n)$ *such that* $\mathrm{dist}(\dot{x}_\varepsilon(t), F(t,x_\varepsilon(t))) \le \varepsilon$ *for a.e.* $t \in [0,T]$ *and* $x_\varepsilon(0) = x_0$, $x_\varepsilon(T) = x_1$.

Proof. Let $G(v) = cl_w S(F)(v) \cap L^F_{x_1-x_0}$, for $v \in L^F_{x_1-x_0}$, where cl_w denotes the weak closure in $L([0,T],\mathbb{R}^n)$ and $S(F)(v) = \{z \in L([0,T],\mathbb{R}^n) : z(t) \in F(t, x_0 + \int_0^t v(\tau)d\tau)$ a.e. on $[0,T]\}$. Since $S(F)(v)$ is decomposable, then ([3], Theorem III.2.3) $cl_w S(F)(v) = S(coF)(v)$. Thus, by Theorem 1 given in [2] there is $x \in AC([0,T],\mathbb{R}^n)$ such that $\dot{x} \in cl_w S(F)(x) \cap L^F_{x_1-x_0}$, $x(0) = x_0$ and $x(T) = x_1$. Let $(v_k)_{k=1}^\infty$ be a sequence of $S(F)(x) \cap L^F_{x_1-x_0}$ such that $v_k \rightharpoonup \dot{x}$ as $k \to \infty$ and let $x_k(t) = x_0 + \int_0^t v_k(\tau)d\tau$ for $t \in [0,T]$. We get $x_k \to x$ as $k \to \infty$ in the norm topology of $C([0,T]),\mathbb{R}^n)$. Furthermore, we have $\dot{x}_k(t) \in F(t,x(t))$ for $k = 1,2,\dots$ and a.e. $t \in [0,T]$. Hence, in particular, we obtain

$$\begin{aligned} &\mathrm{dist}(\dot{x}_k(t),\, F(t,x_k(t)) \\ &\le \mathrm{dist}\,(\dot{x}_k(t),\, F(t,x(t))) \\ &+ \sup\nolimits_{0\le t\le T} h(F(t,x(t)),\, F(t,x_k(t)) \end{aligned}$$

for $k = 1,2,\dots$ and a.e. $t \in [0,T]$. Therefore $\lim_{k\to\infty} \mathrm{dist}(\dot{x}_k(t),\, F(t,x_k(t))$ $= 0$ uniformly for a.e. $t \in [0,T]$, which ends the proof. ∎

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