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OSCILLATION OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract

Oscillation criteria, extended Kamenev and Philos-type oscillation theorems for the nonlinear second order neutral delay differential equation with and without the forced term are given. These results extend and improve the well known results of Grammatikopoulos et. al., Graef et. al., Tanaka for the nonlinear neutral case and the recent results of Dzurina and Mihalikova for the neutral linear case. Some examples are considered to illustrate our main results.

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1. Introduction

In this paper, we are concerned with the oscillation of all solutions of the second order nonlinear neutral delay differential equations

$$[y(t)+p(t)y(t-\tau))]'' + q(t)f(y(t-\sigma)) = 0, \quad t \in [t_0, \infty), \tag{1.1}$$

$$(1.2) \qquad [y(t)+p(t)y(t-\tau))]^{''}+q(t)f(y(t-\sigma))=F(t), \;\; t\in[t_0,\infty).$$

where

(H$_1$) $p,\, q\in C([t_0,\infty),\mathbf{R}^+)$, $\tau,\sigma\geq 0$ and $0\leq p(t)<1$;

(H$_2$) $f\in C(\mathbf{R},\mathbf{R})$, $uf(u)\geq 0$ for $u\neq 0$, $f(uv)\geq f(u)f(v)$ for $uv>0$, and $f(u)\geq \beta u$, $\beta>0$,

(H$_3$) There exists $\theta\in C^2([t_0,\infty),\mathbf{R})$ such that $\theta(t)$ is oscillatory, periodic of period τ and $\theta^{''}(t)=F(t)$.

Let $\rho=\max\{\sigma,\tau\}$ and let $t_1\geqslant t_0$. By a solution of equation (1.1) (or (1.2)) on $[t_1,\infty)$ we mean a function $y\in C([t_1-\rho,\infty),\mathbf{R})$, such that $y(t)+p(t)y(t-\tau)$ is twice continuously differentiable on $[t_1,\infty)$ and such that (1.1) is satisfied for $t\geqslant t_1$. Our attention is restricted to those solutions of (1.1) that satisfy $\sup\{|y(t)|: t\geq T\}>0$. We make a standing hypothesis that (1.1) does possess such solutions. For further questions concerning the existence and uniqueness of solutions of neutral delay differential equations see Hale [15]. A solution y(t) of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise the solution is called non-oscillatory. Equation (1.1) is said to be oscillatory if its all solutions are oscillatory.

In recent years the literature on the oscillation theory of neutral delay differential equations has been growing very fast. This is due to the fact that neutral delay differential equations are a new field with interesting applications in real world life problems. In fact, neutral delay differential equations appear in modelling of networks containing lossless transmission lines (as in high-speed computers where lossless transmission lines are used to interconnect switching circuits), second order neutral delay differential equations appear in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, the theory of automatic control and in neuromechanical systems in which inertia plays an important role (see Hale [15], Popove [23] and Boe and Chang [4] and reference cited therein).

There has been considerable research into the oscillation and asymptotic behavior of solutions of second order equations of neutral type with deviating argument (see for example [5, 6, 8–13, 21, 24, 29]). For more results on neutral delay differential equations and other various functional differential equations we refer to the monographs [1–3, 7, 14, 19].

For the oscillation of (1.1) Grammatikopoulos et al. [11] extended the results of Waltman [26] and Travis [25] for the oscillation of a second order differential equation in the linear case, i.e., when $f(u) = u$, to (1.1) and proved that if $0 \le p(t) < 1$ and

$$\int_{t_0}^{\infty} q(s)[1 - p(s-\sigma)]ds = \infty. \tag{1.3}$$

then every solution of (1.1) oscillates. But one can see that the condition (1.3) cannot be applied to the equation

$$[y(t) + p(t)y(t-\tau)]'' + \frac{\mu}{t^2} y(t-\sigma) = 0, \quad t \ge t_0. \tag{1.4}$$

where $\mu > 0$ and $0 \le p(t) < 1$. However, if $p(t) = 0$, then (1.4) reduces to the well known Euler equation and every solution of this equation oscillates if $\mu > \frac{1}{4}$. Recently Dzurina and Mihalikova [6] considered (1.1) when $p(t) = p$ is a constant and $f(x) = x$ and gave the following oscillation criteria: If

$$\int_{t_0}^{\infty} \left[q(s)(s-\sigma)\frac{1-p^{n+1}}{1-p} - \frac{1}{4(s-\sigma)} \right] ds = \infty \tag{1.5}$$

then every solution of (1.1) oscillates. It is clear that the results of Dzurina and Mihalikova [6] can be applied to (1.4) when p is a constant. But this result applies to the linear case with constant p and cannot be applied to a more general case (1.1). In fact, we will see below that the result of Dzurina and Mihalikova will be considered as a special case of our results.

In [8] Graef et al. extend the condition (1.3) to the nonlinear equation (1.1), and proved that every solution of (1.1) oscillates if

$$\int_{t_0}^{\infty} q(s) f((1 - p(s-\sigma)c)ds = \infty, \quad c > 0. \tag{1.6}$$

In [24] Tanaka extended the condition (1.6) to (1.2) and proved that every solution of (1.2) oscillates if

$$\int_{t_0}^{\infty} q(s) f((1 - p(s-\sigma)c + \Theta(s-\sigma))ds = \infty, \quad c > 0 \tag{1.7}$$

where $\Theta(t) = \theta(t) - p(t)\theta(t-\tau)$.

Note that the results of Graef et. al., and Tanaka cannot be applied to (1.4) in the linear case.

The before mentioned results have motivated the present research and the principle reasons are the following: The results of Grammatikopoulos et al and Dzurina and Mihalikova considered $f(u) = u$, the linear case without a forced term. The results of Tanaka also cannot be applied to the linear case and it may be somewhat restrictive for applications, so it is useful to prove results in the nonlinear case with a forced term.

The purpose of this paper is to give some new oscillation criteria of (1.1) and extend our results to (1.2). We present some new sufficient conditions which guarantee oscillation of all proper solutions of the nonlinear delay differential equation (1.1) and (1.2). Also we give some Kamenev-type and Philos-type theorems for oscillation due to Kamenev and Philos Methods [17, 22], and discuss a number of carefully chosen examples which clarify the relevance of our results. Our results extend and improve the results of Grammatikopoulos et al. [11], Dzurina and Mihalikova [6], Graef et. al. [8] and the results of Tanaka [24]. Moreover our results, immediately improve the results of Waltman [26] and Wintner [27], Travis [25] and Leighton [20], Kamenev [17], Philos [22] and Yan [28] for second order differential equations.

In the sequel, when we write a functional inequality, we will assume that it holds for all sufficiently large values of t.

2. Main results

In this section we will establish some new oscillation criteria for the oscillation of (1.1), and extend these results to (1.2), and also presented some extended Kamenev-type and Philos-type theorems for oscillations.

Theorem 2.1. *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_2)$ *hold, and there exists a function* $\rho \in C^1[[t_0, \infty), \mathbf{R}^+]$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(s)Q(s) - \frac{\rho'(s)}{4\rho(s)} \right) ds = \infty. \tag{2.1}$$

where $Q(t) = \beta q(t) f((1 - p(t - \sigma)))$*. Then every solution of* (1.1) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we assume that $y(t) \neq 0$ for $t \geqslant t_0$. Further, we suppose that $y(t) > 0$, $y(t-\tau) > 0$ and $y(t-\sigma) > 0$ for $t \geqslant t_1 \geq t_0$, since the substitution $u = -y$ transforms (1.1) into an equation of the same form subject to the assumption of the Theorem. Let

$$z(t) = y(t) + p(t)y(t-\tau). \tag{2.2}$$

By (H_1) we see that $z(t) \geq y(t) > 0$ for $t \geqslant t_1$, and from (1.1) it follows that

$$z^{''}(t) = -q(t)f(y(t-\sigma)) < 0, \ \text{ for } t \geqslant t_1. \tag{2.3}$$

Therefore $z^{'}(t)$ is a decreasing function. Now as $z(t) > 0$ and $z^{''}(t) < 0$ for $t \geq t_1$, then by Kiguradze Lemma [18] we have immediately

$$z^{'}(t) > 0, \quad \text{for } t \geqslant t_1. \tag{2.4}$$

Now using (2.4) in (2.2) we have

$$\begin{aligned} y(t) &= z(t) - p(t)y(t-\tau) = z(t) - p(t)[z(t-\tau) - p(t-\tau)y(t-2\tau)] \\ &\geq z(t) - p(t)z(t-\tau) > (1-p(t))z(t). \end{aligned}$$

Thus there exists a $t_2 \geq t_1$ such that

$$y(t-\sigma) \geq (1-p(t-\sigma))z(t-\sigma) \qquad \text{for } \ t \geq t_2. \tag{2.5}$$

Then by using (H_2) we have

$$f(y(t-\sigma)) \geq f(1-p(t-\sigma))f(z(t-\sigma)) \qquad \text{for } \ t \geq t_2. \tag{2.6}$$

By substituting (2.6) in (2.3), we obtain

$$z^{''}(t) + Q(t)z(t-\sigma) \leq 0, \ \ t \geq t_2, \tag{2.7}$$

where $Q(t) = \beta q(t) f(1 - p(t - \sigma))$. Define

$$w(t) = \frac{\rho(t) z^{'}(t)}{z(t - \sigma)},$$

then $w(t) > 0$. Since $z^{'}(t - \sigma) > z^{'}(t)$, then from (2.7) we have

$$w^{'}(t) - \frac{\rho^{'}(t)}{\rho(t)} w(t) + \rho(t) Q(t) + \frac{1}{\rho(t)} w^2(t) \leq 0. \tag{2.8}$$

Hence

$$\begin{aligned} w^{'}(t) &\leq -\rho(t) Q(t) + \frac{\rho^{'}(t)}{\rho(t)} w(t) - \frac{1}{\rho(t)} w^2(t) \\ &= -\rho(t) Q(t) - \left[\frac{1}{\sqrt{\rho(t)}} w(t) - \frac{1}{2} \sqrt{\frac{\rho^{'}(t)}{\rho(t)}} \right]^2 + \frac{\rho^{'}(t)}{4\rho(t)}. \end{aligned}$$

Thus

$$w^{'}(t) < - \left[\rho(t) Q(t) - \frac{1}{4} \frac{\rho^{'}(t)}{\rho(t)} \right].$$

Integrating the last inequality from t_2 to t, we get

$$w(t) \leq w(t_2) - \int_{t_2}^{t} \left[\rho(s) Q(s) - \frac{\rho^{'}(s)}{4\rho(s)} \right] ds.$$

Taking $t \to \infty$, we deduce by (2.1) that $w(t) \to -\infty$, a contradiction. Then every solution of (1.1) oscillates. ■

Now we extend Theorem 2.1 to (1.2) with a forced term.

Theorem 2.2. *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold,* $\lim_{t \to \infty} \Theta(t) = 0$ *and there exists a function* $\rho \in C^1[[t_0, \infty), \mathbf{R}^+]$ *such that*

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left(\rho(s) Q_1(s) - \frac{\rho^{'}(s)}{4\rho(s)} \right) ds = \infty, \tag{2.9}$$

where $Q_1(t) = \beta q(t) f(\lambda(1 - p(t - \sigma)))$. *Then every solution of* (1.2) *oscillates.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.2). Without loss of generality, we assume that $y(t) \neq 0$ for $t \geqslant t_0$. Further, we suppose that $y(t) > 0$, $y(t-\tau) > 0$ and $y(t-\sigma) > 0$ for $t \geqslant t_1 \geq t_0$, since the substitution $u = -y$ transforms (1.2) into an equation of the same form subject to the assumption of the Theorem. Let

$$z(t) = y(t) + p(t)y(t-\tau) - \theta(t). \tag{2.10}$$

By (H_1) and as in [24] we see that $z(t) \geq y(t) > 0$ for $t \geqslant t_1$, and from (1.2) it follows that

$$z^{''}(t) = -q(t)f(y(t-\sigma)) < 0, \text{ for } t \geqslant t_1. \tag{2.11}$$

Therefore $z^{'}(t)$ is a decreasing function. Now as $z(t) > 0$ and $z^{''}(t) < 0$ for $t \geq t_1$, then by Kiguradze Lemma [18] we have immediately again

$$z^{'}(t) > 0, \quad for\ t \geqslant t_1. \tag{2.12}$$

Now using (2.12) in (2.10) and using the fact that $\theta(t) = \theta(t-\tau)$, we have

$$\begin{aligned} y(t) &= z(t) - p(t)y(t-\tau) + \theta(t) \\ &\geq z(t) - p(t)[z(t-\tau) + \theta(t-\tau)] + \theta(t) \\ &\geq (1-p(t))z(t) - p(t)\theta(t-\tau) + \theta(t) = (1-p(t))z(t) + \Theta(t). \end{aligned}$$

On the other hand, since $\lim_{t\to\infty} \Theta(t) = 0$, we can find $t_2 \geq t_1$ such that

$$y(t-\sigma) \geq \lambda(1-p(t-\sigma))z(t-\sigma) \quad \text{for } t \geq t_2, \tag{2.13}$$

where $\lambda \in (0,1)$. Then by using (H_2) we have

$$f(y(t-\sigma)) \geq f(\lambda(1-p(t-\sigma)))f(z(t-\sigma)) \quad \text{for } t \geq t_2. \tag{2.14}$$

Substituting (2.14) in (2.11), we obtain

$$z^{''}(t) + Q_1(t)z(t-\sigma) \leq 0, \ t \geq t_2. \tag{2.15}$$

where $Q_1(t) = \beta q(t) f(\lambda(1 - p(t - \sigma)))$. Defining again $w(t)$ as in Theorem 2.1 we obtain

$$w^{'}(t) - \frac{\rho^{'}(t)}{\rho(t)} w(t) + \rho(t) Q_1(t) + \frac{1}{\rho(t)} w^2(t) \leq 0. \tag{2.16}$$

The remainder of the proof is now similar to that of Theorem 2.1 and will be omitted. ∎

Corollary 2.1. *Assume that* (H_1) *and* (H_2) *hold,* $f(u) = u$ *and* $\rho(t) = t$, *such that*

$$\lim_{t \to \infty} \sup \int_{t_0}^{t} \left(sq(s)(1 - p(s - \sigma) - \frac{1}{4s} \right) ds = \infty.$$

Then every solution of (1.1) *oscillates.*

Corollary 2.2. *Assume that* (H_1) *and* (H_2) *hold,* $p(t) = 0$, $\rho(t) = t$, *and* $f(u) = u$ *such that*

$$\lim_{t \to \infty} \sup \int_{t_0}^{t} \left(sq(s) - \frac{1}{4s} \right) ds = \infty.$$

Then every solution of the delay differential equation

$$y^{''}(t) + q(t) y(\sigma(t)) = 0, \quad t \geq t_0.$$

oscillates.

Remark 2.1. Note that the results of Dzurina and Mihalikova depend on a positive integer $n > 0$, and in Corollary 2.1 we do not require any additional constants. Then Corollary 2.1 extends and improves the results of Dzurina and Mihalikova [6].

To illustrate our main results, we consider the following two examples

Example 2.1. Consider the Euler Equation

$$y^{''}(t) + \frac{\nu}{t^2} y(t) = 0, \quad t \geq 1. \tag{E}$$

where $\nu > 0$ is a constant. Here $p(t) = 0$, $\sigma = 0$, and $q(t) = \frac{\nu}{t^2}$. Note that:

(Wintner) $\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(x)dxds = \lim_{t\to\infty} \frac{1}{t}\int_{1}^{t}\int_{1}^{s} \frac{\nu}{x^2}dxds =$
$\lim_{t\to\infty} \frac{\nu}{t}\int_{1}^{t}(-\frac{1}{s}+1)ds = \lim_{t\to\infty} \frac{\nu}{t}(-\ln(t)+t-1) = \nu < \infty$.

(Leighton) $\int_{t_0}^{\infty} q(s)ds = \int_{1}^{\infty} \frac{\nu}{s^2}ds = \nu[-\frac{1}{s}]_1^{\infty} = \nu < \infty$.

(Kamenev) $\lim_{t\to\infty}\sup \frac{1}{t^n}\int_{t_0}^{t}(t-s)^n q(s)ds = \lim_{t\to\infty}\sup \frac{1}{t^n}\int_{1}^{t}(t-s)^n \frac{\nu}{s^2}ds <$
$\lim_{t\to\infty}\sup \int_{1}^{t} \frac{\nu}{s^2}ds = \nu < \infty$, for some $n > 1$.

(Hartman) A(T)$\leq \lim_{t\to\infty}\inf \frac{1}{t^n}\int_{T}^{t}(t-s)^n q(s)ds \leq$
$\lim_{t\to\infty}\sup \frac{1}{t^n}\int_{T}^{t}(t-s)^n q(s)ds \leq \frac{\nu}{T}$ for every $T \geq 1$.

(Yan) $\int_{1}^{\infty} A^2(t)dt \leq \int_{1}^{\infty} \frac{\nu^2}{s^2}ds = \nu^2 < \infty$.

(Philos) $\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)q(s) - \frac{1}{4}h^2(t,s)]ds \leq$
$\lim_{t\to\infty}\sup \int_{1}^{t} \frac{\nu}{s^2}ds = \nu < \infty$.

That is, none of the above mentioned oscillation criteria holds. Thus, the above mentioned oscillation criteria of Wintner, Leighton, Hartman, Kamenev, Yan and Philos cannot be applied to the Euler equation (E). But

$$\begin{aligned}\lim_{t\to\infty}\sup \int_{t_0}^{t}\left(sq(s) - \frac{1}{4s}\right)ds &= \lim_{t\to\infty}\sup \int_{2\pi}^{t}\left(s\frac{\nu}{s^2} - \frac{1}{4s}\right)ds \\ &= \lim_{t\to\infty}\sup \int_{2\pi}^{t}\left(\frac{4\nu-1}{4s}\right)ds = \infty, \quad \text{if } \nu > \frac{1}{4}.\end{aligned}$$

Hence, by Corollary 2.2, every solution of (E) oscillates if $\nu > \frac{1}{4}$.

Then Corollary 2.2 improves the results of Wintner, Leighton, Hartman, Yan, Kamenev, Philos.

Example 2.2. Consider the following neutral delay differential equation

$$\left[y(t) + \frac{1}{t+\pi}y(t-2\pi)\right]'' + \frac{\lambda}{t^2}y(t-\pi) = 0, \quad t \geq 2\pi. \tag{2.17}$$

where $\lambda > 0$ is a constant. Here, $f(u) = u$, $q(t) = \frac{\lambda}{t^2}$, $p(t) = \frac{1}{t+\pi}$ and $p(t-\pi) = \frac{1}{t}$. Then

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left(sq(s)(1 - p(s-\sigma)) - \frac{1}{4s} \right) ds$$

$$= \lim_{t\to\infty} \sup \int_{2\pi}^{t} \left(s\frac{\lambda}{s^2}(1 - \frac{1}{s}) - \frac{1}{4s} \right) ds$$

$$= \lim_{t\to\infty} \sup \int_{2\pi}^{t} \left(\frac{4\lambda - 1}{4s} - \frac{\lambda}{s^2} \right) ds = \infty, \quad \text{if } \lambda > \frac{1}{4}.$$

Hence, by Corollary 2.1, every solution of (2.17) oscillates. Note that the results of Grammatikopoulos et al. [11] and Dzurina and Mihalikova [6] and Tanaka [24] cannot be applied to (2.17). Then Corollary 2.1 improves the results of [11], [6] and [24].

Below we obtain some Kamenev-type oscillation results for the cases (1.1) and (1.2).

Theorem 2.3. *Assume that* (H_1), (H_2), *hold and there exists a functsion* $\rho \in C^1[[t_0, \infty), \mathbf{R}^+]$ *such that*

(2.18)
$$\lim_{t\to\infty} \sup \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n \left(\beta\rho(s)q(s)f((1 - p(s-\sigma))) - \frac{\rho'(s)}{4\rho(s)} \right) ds = \infty.$$

Then every solution of (1.1) *oscillates.*

Proof. Assume to the contrary that (1.1) has a nonoscillatory solution. We may assume that without loss of generality $y(t) > 0$ and $y(t-\tau) > 0$ and $y(t-\sigma) > 0$ for $t \geqslant t_1 \geq t_0$ since the substitution $u = -y$ transforms (1.1) into an equation of the same form subject to the assumption of the Theorem. Defining again $w(t)$ as in Theorem 2.1 and going through as in the proof of Theorem 2.1, we find that $w(t)$ is greater than 0 and it satisfies the inequality (2.8) which can be rewritten in the form

$$\begin{aligned} w^{'}(t) &\leq -\rho(t)Q(t) + \frac{\rho^{'}(t)}{\rho(t)}w(t) - \frac{1}{\rho(t)}w^2(t) \\ &= -\rho(t)Q(t) - \left[\frac{1}{\sqrt{\rho(t)}}w(t) - \frac{1}{2}\sqrt{\frac{\rho^{'}(t)}{\rho(t)}}\,\right]^2 + \frac{\rho^{'}(t)}{4\rho(t)}. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{t_0}^{t}(t-s)^n w^{'}(s)ds + \int_{t_0}^{t}(t-s)^n\rho(s)Q_2(s)ds \\ &\leq -\int_{t_0}^{t}(t-s)^n\left[\frac{1}{\sqrt{\rho(s)}}w(s) + \frac{1}{2}\sqrt{\frac{\rho^{'}(s)}{\rho(s)}}\right]^2 ds, \end{aligned} \tag{2.19}$$

where $Q_2(s) = (\rho(s)Q(s) - \frac{\rho^{'}(s)}{4\rho(s)})$. Since

$$\int_{t_0}^{t}(t-s)^n w^{'}(s)ds = n\int_{t_0}^{t}(t-s)^{n-1}w(s)ds - w(t_0)(t-t_0)^n,$$

we get

$$\begin{aligned} &\frac{1}{t^n}\int_{t_0}^{t}(t-s)^{n-1}Q_2(s)ds \leq w(t_0)\left(\frac{t-t_0}{t}\right)^n - \frac{n}{t^n}\int_{t_0}^{t}(t-s)^{n-1}w(s)ds \\ &-\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left[\frac{1}{\sqrt{\rho(s)}}w(s) + \frac{1}{2}\sqrt{\frac{\rho^{'}(s)}{\rho(s)}}\,\right]^2 ds \leq w(t_0)\left(\frac{t-t_0}{t}\right)^n, \end{aligned}$$

where $w(t) > 0$. Then

$$\lim_{t\to\infty}\sup\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n Q_2(s)ds \to w(t_0) \equiv \text{finite number},$$

which contradicts the condition (2.18). Therefore every solution of (1.1) oscillates. ∎

Theorem 2.4. *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold,* $\lim_{t\to\infty}\Theta(t)=0$ *and there exists a function* $\rho\in C^1[[t_0,\infty),\mathbf{R}^+]$ *such that*

$$\lim_{t\to\infty}\sup\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(Q_3(s)-\frac{\rho'(s)}{4\rho(s)}\right)ds=\infty, \tag{2.18}$$

where $Q_3(s)=\beta\rho(s)q(s)f(\lambda(1-p(s-\sigma)))$. *Then every solution of* (1.2) *oscillates.*

Proof. Assume to the contrary that (1.2) has a nonoscillatory solution. We may assume that without loss of generality $y(t)>0$ and $y(t-\tau)>0$ and $y(t-\sigma)>0$ for $t\geqslant t_1\geq t_0$, since the substitution $u=-y$ transforms (1.2) into an equation of the same form subject to the assumption of the Theorem. Defining again w(t) as in Theorem 2.1 and going through as in the proof of Theorem 2.2, we find that $w(t)$ is greater than 0 and it satisfies the inequality (2.16). The proof is similar to that of Theorem 2.3 and will be omitted. ■

If $\rho(t)=t$ in Theorems 2.3 and 2.4, we have immediately the following Corollaries for oscillation of (1.1) and (1.2) respectively.

Corollary 2.3. *Assume that* (H_1), (H_2) *hold. If*

$$\lim_{t\to\infty}\sup\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(\beta sq(s)f(1-p(s-\sigma))-\frac{1}{4s}\right)ds=\infty. \tag{2.20}$$

Then every solution of (1.1) *oscillates.*

Corollary 2.4. *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold, and* $\lim_{t\to\infty}\Theta(t)=0$. *If* (2.18)

$$\lim_{t\to\infty}\sup\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left(\beta sq(s)f(\lambda(1-p(s-\sigma)))-\frac{\rho'(s)}{4\rho(s)}\right)ds=\infty.$$

Then every solution of (1.2) *oscillates.*

In the following theorems we will extend the Philos Theorems for oscillations to (1.1) and (1.2). Following Philos [22], we introduce a class of functions $\Re$. Let

$$\mathbf{D}_0 = \{(t,s) : t > s \geq t_0\} \quad \text{and} \quad \mathbf{D} = \{(t,s) : t \geq s \geq t_0\}.$$

The function $H \in C(\mathbf{D}, \mathbf{R})$ is said to belong to the class $\Re$ if

(I) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on $\mathbf{D}_0$;

(II) H has a continuous and nonpositive partial derivative on $\mathbf{D}_0$ with respect to the second variable such that

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \text{ for all } (t,s) \in \mathbf{D}_0.$$

Theorem 2.5. *Assume that* (H_1) *and* (H_2) *hold. Let* H *belong to the class* $\Re$, *and*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big(H(t,s)Q_4(s) - \frac{s}{4}Q^2(t,s)\Big)\, ds = \infty, \tag{2.21}$$

where $Q_4(t) = \beta t q(t)(1 - p(t-\sigma))$ *and* $Q(t,s) = h(t,s) - \frac{\sqrt{H(t,s)}}{s}$. *Then every solution of* (1.1) *oscillates.*

Proof. Assume to the contrary that (1.1) has a nonoscillatory solution. We may assume that without loss of generality $y(t) > 0$, $y(t-\tau) > 0$ and $y(t-\sigma) > 0$ for $t \geqslant t_1 \geq t_0$, since a similar argument holds also for the case when $y(t) < 0$. Defining $z(t)$ as in (2.2) and going through the proof as in Theorem 2.1, we obtain (2.7). Let us define the function $w(t)$ as follows

$$w(t) = \frac{t z'(t)}{z(t-\sigma)}. \tag{2.22}$$

Differentiating (2.22) and using (2.7), we obtain

$$w'(t) \leq -Q_4(t) + \frac{1}{t} w(t) - \frac{w^2(t)}{t}. \tag{2.23}$$

where $Q_4(t) = \beta t q(t) f(1-p(t-\sigma))$. Hence, by (2.23) for all $t > T \geq t_2$, we have

$$\begin{aligned}
&\int_T^t H(t,s)Q_4(s)ds \\
&\leq \int_T^t H(t,s)\frac{w(s)}{s}ds - \int_T^t H(t,s)w'(s)ds - \int_T^t H(t,s)\frac{w^2(s)}{s}ds \\
&= -H(t,s)w(s)|_T^t - \int_T^t \left[-\frac{\partial H(t,s)}{\partial s}w(s) - H(t,s)\frac{w(s)}{s} + H(t,s)\frac{w^2(s)}{s}\right] ds \\
&= H(t,T)w(T) - \int_T^t \left[\sqrt{\frac{H(t,s)}{s}}w(s) + \frac{1}{2}\sqrt{s}Q(t,s)\right]^2 ds + \int_T^t \frac{sQ^2(t,s)}{4}ds.
\end{aligned}$$

where $Q(t,s) = h(t,s) - \frac{\sqrt{H(t,s)}}{s}$. Thereby, for all $t > T \geq t_2$, we conclude that

$$\begin{aligned}
&\int_T^t \left[H(t,s)Q_4(s) - \frac{s}{4}Q^2(t,s)\right] ds \\
&\leq H(t,T)w(T) - \int_T^t \left[\sqrt{\frac{H(t,s)}{s}}w(s) + \frac{1}{2}\sqrt{s}Q(t,s)\right]^2 ds.
\end{aligned} \tag{2.24}$$

By virtue of (2.24) and (II) for all $t > T \geq t_2$, we obtain

$$\int_{t_2}^t \left[H(t,s)Q_4(s) - \frac{sQ^2(t,s)}{4}\right] ds \leq H(t,t_2)\left|w(t_2)\right| \leq H(t,t_0)w(t_2). \tag{2.25}$$

Then by (2.25) and (II), we have

$$\frac{1}{H(t,t_0)}\int_{t_0}^t \left[H(t,s)Q_4(s) - \frac{sQ^2(t,s)}{4k}\right] ds \leq \int_{t_0}^{t_2} Q_4(s)ds + \left|w(t_2)\right|. \tag{2.26}$$

Inequality (2.26) yields

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)Q_4(s) - \frac{sQ^2(t,s)}{4}\right] ds$$
$$\leq \int_{t_0}^{t_2} Q_4(s)ds + |w(t_2)| < \infty,$$

and the latter inequality contradicts the assumption (2.21). Hence, every solution of (1.1) oscillates. ■

Corollary 2.5. *Assume that the assumptions of Theorem* 2.5 *hold with* (2.21) *replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)Q_4(s)ds = \infty, \tag{2.27}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} sQ^2(t,s)ds < \infty.$$

Then every solution of (1.1) *oscillates.*

For the oscillation of (1.2) we have the following oscillation results immediately.

Theorem 2.6. *Assume that* $(H_1) - (H_3)$ *hold,* $\lim_{t\to\infty} \Theta(t) = 0$. *Let* H *belong to the class* $\Re$, *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)Q_5(s) - \frac{sQ^2(t,s)}{4}\right] ds = \infty, \tag{2.28}$$

where $Q_5(s) = \beta s q(s) f(\lambda(1 - p(s-\sigma)))$. *Then every solution of* (1.2) *oscillates.*

Corollary 2.6. *Assume that* $(H_1) - (H_3)$ *hold,* $\lim_{t\to\infty} \Theta(t) = 0$. *Let* H *belong to the class* $\Re$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)Q_5(s)ds = \infty, \tag{2.29}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} sQ^2(t,s)ds < \infty.$$

Then every solution of (1.2) *oscillates.*

Remark 2.2. With the appropriate choice of functions H and h, it is possible to derive from Theorems 2.5 and 2.6 a number of oscillation criteria for (1.1) and (1.2). Defining, for example, for some integer $n > 1$, the function $H(t,s)$ by

$$H(t,s) = (t-s)^n, \quad (t,s) \in \mathbf{D}. \tag{2.30}$$

we can easily check that $H \in \Re$. Furthermore, the function

$$h(t,s) = n(t-s)^{(n-2)/2}, \quad (t,s) \in \mathbf{D} \tag{2.31}$$

is continuous and satisfies condition (II). Therefore, as a consequence of Theorem 2.5, we obtain the following oscillation criteria.

Corollary 2.7. *Let the assumption of* (H_1) *and* (H_2) *hold, and*

$$\begin{aligned}&\lim_{t\to\infty} \sup \frac{1}{t^n} \int_{t_0}^{t} \Bigg[(t-s)^n \beta s q(s) f(1-p(s-\sigma)) \\ &-\frac{s}{4}(t-s)^{n-2}\left(n-\left(\frac{t-s}{s}\right)\right)^2 \Bigg] ds = \infty.\end{aligned} \tag{2.32}$$

Then every solution of (1.1) *oscillates.*

Corollary 2.8. *Assume that* (H_1) – (H_3) *hold,* $\lim_{t\to\infty} \Theta(t) = 0$*, and*

$$\begin{aligned}&\lim_{t\to\infty} \sup \frac{1}{t^n} \int_{t_0}^{t} \Bigg[(t-s)^n \beta s q(s) f(\lambda(1-p(s-\sigma))) \\ &-\frac{s}{4}(t-s)^{n-2}\left(n-\left(\frac{t-s}{s}\right)\right)^2 \Bigg] ds = \infty.\end{aligned}$$

Then every solution of (1.2) *oscillates.*

The following two oscillation criteria apply to the case when it is not possible to verify easily conditions (2.21) and (2.28).

Theorem 2.7. *Assume that* (H_1) *and* (H_2) *hold. Let* H *belong to the class* $\Re$, *and assume that*

$$0 < \inf_{s\geq t_0}\left[\lim_{t\to\infty}\inf\frac{H(t,s)}{H(t,t_0)}\right]\leq\infty. \tag{2.33}$$

Let $\phi\in C[[t_0,\infty),\mathbf{R}]$ *such that for* $t>t_0$, $T\geq t_0$

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t} sQ^2(t,s)ds<\infty, \tag{2.34}$$

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\frac{\phi_+^2(s)}{s}ds=\infty, \tag{2.35}$$

and

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big(H(t,s)Q_4(s)-\frac{s}{4}Q^2(t,s)\Big)\,ds\geq\phi(T), \tag{2.36}$$

where Q(t,s) as in Theorem 2.5 *and* $\phi_+=\max\{\phi(t),0\}$. *Then every solution of* (1.1) *oscillates.*

Proof. As above, in Theorem 2.5, we assume that (1.1) has a nonoscillatory solution. We may assume that without loss of generality $y(t)>0$, $y(t-\tau)>0$ and $y(t-\sigma)>0$ for $t\geqslant t_1\geq t_0$, since a similar argument holds also for the case when $y(t)<0$ defining $w(t)$ by (2.22), and in the same way as in Theorem 2.5, we obtain the inequality (2.24). By (2.24) we have for $t>T\geq T_0=t_2$

$$\begin{aligned}&\frac{1}{H(t,T)}\int_T^t\Big[H(t,s)Q_4(s)-\frac{s}{4}Q^2(t,s)\Big]\,ds\\&\leq w(T)-\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{H(t,s)}{s}}w(s)+\frac{1}{2}\sqrt{s}Q(t,s)\right]^2ds\end{aligned}$$

By (2.36), we have for $T \geq T_0$

(2.37)
$$w(T) \geq \phi(T) + \lim_{t\to\infty} \inf \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{H(t,s)}{s}} w(s) + \frac{1}{2}\sqrt{s} Q(t,s) \right]^2 ds.$$

It follows from (2.37) that for $T \geq T_0$

$$w(T) \geq \phi(T), \tag{2.38}$$

and

$$\lim_{t\to\infty} \inf \frac{1}{H(t,T_0)} \int_{T_0}^t \left[\sqrt{\frac{H(t,s)}{s}} w(s) + \frac{1}{2}\sqrt{s} Q(t,s) \right]^2 ds$$
$$\leq w(T_0) - \phi(T_0) = M < \infty.$$

Therefore, for $t \geq T_0$, we have

(2.39)
$$\begin{aligned} \infty &> \lim_{t\to\infty} \inf \frac{1}{H(t,T_0)} \int_{T_0}^t \left[\sqrt{\frac{H(t,s)}{s}} w(s) + \frac{1}{2}\sqrt{s} Q(t,s) \right]^2 ds \\ &\geq \lim_{t\to\infty} \inf \frac{1}{H(t,T_0)} \int_{T_0}^t \left[\frac{H(t,s)}{s} w^2(s) + \sqrt{H(t,s)} Q(t,s) w(s) \right] ds. \end{aligned}$$

Define the functions $\alpha(t)$ and $\beta(t)$ as follows

$$\begin{aligned} \alpha(t) &= \frac{1}{H(t,T_0)} \int_{T_0}^t \frac{H(t,s)}{s} w^2(s) ds, \quad \beta(t) \\ &= \frac{1}{H(t,T_0)} \int_{T_0}^t \sqrt{H(t,s)} Q(t,s) w(s) ds. \end{aligned}$$

Then (2.39) may be written as

$$\lim_{t\to\infty} [\alpha(t) + \beta(t)] < \infty. \tag{2.40}$$

Now we claim that

$$\int_{T_0}^{\infty} \frac{w^2(s)}{s} ds < \infty. \tag{2.41}$$

Suppose to the contrary that

$$\int_{T_0}^{\infty} \frac{w^2(s)}{s} ds = \infty. \tag{2.42}$$

By (2.33), there is a positive constant ζ satisfying

$$\inf_{s \geq t_o} \left[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)} \right] > \zeta > 0. \tag{2.43}$$

Let μ be any arbitrary positive number, then it follows from (2.42) that there exists a $T_1 \geq T_0$ such that

$$\int_{T_0}^{t} \frac{w^2(s)}{s} ds \geq \frac{\mu}{\zeta} \text{ for all } t \geq T_1.$$

Therefore, for $t \geq T_1$, we obtain

$$\begin{aligned}
\alpha(t) &= \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) d\left[\int_{T_0}^{s} \frac{w^2(u)}{u} du\right] \\
&= \frac{1}{H(t,T_0)} \int_{T_0}^{t} -\frac{\partial H(t,s)}{\partial s} \left[\int_{T_0}^{s} \frac{w^2(u)}{u} du\right] ds \\
&\geq \frac{1}{H(t,T_0)} \int_{T_1}^{t} -\frac{\partial H(t,s)}{\partial s} \left[\int_{T_1}^{s} \frac{w^2(u)}{u} du\right] ds \\
&\geq \frac{\mu}{\zeta} \frac{1}{H(t,T_0)} \int_{T_1}^{t} -\frac{\partial H(t,s)}{\partial s} ds = \frac{\mu}{\zeta} \frac{H(t,T_1)}{H(t,T_0)},
\end{aligned}$$

for all $t \geq T_1$. By (2.43), there is a $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,T_0)} \geq \zeta \quad \text{for all } t \geq T_2.$$

which implies that $\alpha(t) \geq \mu$ for all $t \geq T_2$. Since μ is arbitrary,

$$\lim_{t\to\infty} \alpha(t) = \infty. \tag{2.44}$$

Next, consider a sequence $\{t_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} t_n = \infty$ satisfying

$$\lim_{n\to\infty} [\alpha(t_n) + \beta(t_n)] = \lim_{t\to\infty} [\alpha(t) + \beta(t)].$$

In view of (2.40), there exists a constant M such that

$$\alpha(t_n) + \beta(t_n) \leq M, \quad n = 1, 2, \ldots. \tag{2.45}$$

It follows from (2.44) that

$$\lim_{n\to\infty} \alpha(t_n) = \infty. \tag{2.46}$$

This and (2.45) give

$$\lim_{n\to\infty} \beta(t_n) = -\infty. \tag{2.47}$$

Then, by (2.45) and (2.47)

$$1 + \frac{\beta(t_n)}{\alpha(t_n)} \leq \frac{M}{\alpha(t_n)} < \frac{1}{2} \quad \text{for } n \text{ large enough.}$$

Thus

$$\frac{\beta(t_n)}{\alpha(t_n)} \leq -\frac{1}{2} \quad \text{for all large } n.$$

This implies that

$$\lim_{n\to\infty} \frac{\beta(t_n)}{\alpha(t_n)} \beta(t_n) = \infty. \tag{2.48}$$

On the other hand, by Schwarz's inequality, we have

$$\begin{aligned}
\beta^2(t_n) &= \left[\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\sqrt{H(t_n,s)}Q(t_n,s)w(s)ds\right]^2\\
&\le \left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n} sQ^2(t_n,s)ds\right\}\left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n}\frac{H(t_n,s)}{s}w^2(s)ds\right\}\\
&\le \alpha(t_n)\left\{\frac{1}{H(t_n,T_0)}\int_{T_0}^{t_n} sQ^2(t_n,s)ds\right\},
\end{aligned}$$

for any positive integer n. But (2.43) guarantee that

$$\lim_{t\to\infty}\inf\frac{H(t,T_0)}{H(t,t_0)} > \zeta.$$

This means that there exists a $T_3 > T_0$ such that

$$\frac{H(t,T_0)}{H(t,t_0)} > \zeta \quad \text{for every } t \ge T_3.$$

Then

$$\frac{H(t_n,T_0)}{H(t_n,t_0)} > \zeta \quad \text{for } n \text{ large enough}$$

and therefore

$$\frac{\beta^2(t_n)}{\alpha(t_n)} \le \frac{1}{\zeta H(t_n,t_0)}\int_{T_0}^{t_n} sQ^2(t_n,s)ds.$$

It follows from (2.48) that

$$\lim_{n\to\infty}\frac{1}{H(t_n,t_0)}\int_{T_0}^{t_n} sQ^2(t_n,s)ds = \infty.$$

This gives

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{T_0}^{t} sQ^2(t,s)ds = \infty, \tag{2.49}$$

which contradicts (2.34). Thus, (2.41) holds. Then by (2.38) we have

$$\int_{T_0}^{\infty} \frac{\phi_+^2(s)}{s} ds \le \int_{T_0}^{\infty} \frac{w^2(s)}{s} ds < \infty$$

which contradicts (2.35). Then every solution of (1.1) oscillates. ■

For the oscillation of (1.2) we have the following oscillation results immediately and the proof is similar to that of Theorem 2.7 and the details are left to the reader.

Theorem 2.8. *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold,* $\lim_{t\to\infty} \Theta(t) = 0$. *Let* H *belongs to the class* $\Re$, *such that*

$$0 < \inf_{s \ge t_0} \left[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)} \right] \le \infty.$$

Let $\phi \in C[[t_{0,}\infty), \mathbf{R}]$ *such that for* $t \ge t_0$, $T \ge t_0$

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} sQ^2(t,s) ds < \infty,$$

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \frac{\phi_+^2(s)}{s} ds = \infty,$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s) Q_5(s) - \frac{s}{4} Q^2(t,s) \right) ds \ge \phi(T)$$

where Q(t,s) as in Theorem 2.5 *and* $\phi_+ = \max\{\phi(t), 0\}$. *Then every solution of* (1.2) *oscillates.*

Theorem 2.9. *Assume that* (H_1) *and* (H_2) *hold. Let* H *belong to the class* $\Re$, *and assume that* (2.33) *holds. Suppose there exists a function* $\phi \in C[[t_{0,}\infty), \mathbf{R}]$ *such that for* $t \ge t_0$, $T \ge t_0$ (2.35) *holds, and*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) Q_4(s) ds < \infty, \tag{2.50}$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big(H(t,s)Q_4(s) - \frac{s}{4}Q^2(t,s) \Big)\, ds \geq \phi(T). \tag{2.51}$$

where $Q(t,s)$ *as in Theorem* 2.5 *and* $\phi_+ = \max\{\phi(t),0\}$. *Then every solution of* (1.1) *oscillates.*

Proof. As above, in Theorem 2.7, we assume that (1.1) has a nonoscillatory solution. We may assume that without loss of generality $y(t) > 0$, $y(t-\tau) > 0$ and $y(t-\sigma) > 0$ for $t \geqslant t_1 \geq t_0$, since a similar argument holds also for the case when $y(t) < 0$. defining $w(t)$ by (2.22), and in the same way as in Theorem 2.7, we obtain the inequality (2.24). By (2.24) we have for $t > T \geq T_0 = t_2$

$$\begin{aligned} &\lim_{t\to\infty} \inf \frac{1}{H(t,T)} \int_{T}^{t} \Big[H(t,s)Q_4(s) - \frac{s}{4}Q^2(t,s) \Big]\, ds \\ &\leq\ w(T) - \lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_{T}^{t} \left[\sqrt{\frac{H(t,s)}{s}} w(s) + \frac{1}{2}\sqrt{s}Q(t,s) \right]^2 ds \end{aligned}$$

It follows from (2.51) that $T \geq T_0$

$$w(T) \geq \phi(T) + \lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_{T}^{t} \left[\sqrt{\frac{H(t,s)}{s}} w(s) + \frac{1}{2}\sqrt{s}Q(t,s) \right]^2 ds$$

Hence, (2.38) holds for all $T \geq T_0$, and

$$\begin{aligned} &\lim_{t\to\infty} \sup \frac{1}{H(t,T_0)} \int_{T_0}^{t} \left[\sqrt{\frac{H(t,s)}{s}} w(s) + \frac{1}{2}\sqrt{s}Q(t,s) \right]^2 ds \\ &\qquad \leq w(T_0) - \phi(T_0) = M < \infty. \end{aligned}$$

This implies that

$$\lim_{t\to\infty} \sup[\alpha(t)+\beta(t)]$$

$$\le \lim_{t\to\infty} \sup \frac{1}{H(t,T_0)} \int_{T_0}^{t} \left[\sqrt{\frac{H(t,s)}{s}} w(s) + \frac{1}{2}\sqrt{s} Q(t,s)\right]^2 ds, \tag{2.52}$$

where $\alpha(t)$ and $\beta(t)$ are defined as in the proof of Theorem 2.7. By (2.51)

$$\begin{aligned}\phi(T_0) \;&\le\; \lim_{t\to\infty} \inf \frac{1}{H(t,T_0)} \int_{T_0}^{t} \left[H(t,s)Q_4(s) - s\frac{Q^2(t,s)}{4}\right] ds \\ &\le\; \lim_{t\to\infty} \inf \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)Q_4(s) \\ &\quad -\frac{1}{4} \lim_{t\to\infty} \inf \frac{1}{H(t,T_0)} \int_{T_0}^{t} sQ^2(t,s)ds,\end{aligned}$$

this and (2.50) imply that

$$\lim_{t\to\infty} \inf \frac{1}{H(t,T_0)} \int_{T_0}^{t} sQ^2(t,s)ds < \infty.$$

Then, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} t_n = \infty$ satisfying

$$\lim_{n\to\infty} \frac{1}{H(t_n,T_0)} \int_{T_0}^{t_n} sQ^2(t_n,s)ds = \lim_{t\to\infty} \inf \frac{1}{H(t,T_0)} \int_{T_0}^{t} sQ^2(t,s)ds < \infty. \tag{2.53}$$

Now suppose that (2.42) holds. Using the procedure of the proof of Theorem 2.7, we conclude that (2.44) is satisfied. It follows from (2.52) that there exists a constant M such that (2.45) is satisfied. Thus as in the proof of Theorem 2.7, we see that (2.49) holds, which contradicts (2.53). This contradiction prove that (2.42) fails. Since the remainder of the proof is similar to that of Theorem 2.7, we omit the details. ■

Theorem 2.10. *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold,* $\lim_{t\to\infty}\Theta(t) = 0$. *Let* H *belong to the class* $\Re$, *and assume that* (2.33) *holds. Suppose there exists a function* $\phi \in C[[t_{0,}\infty), \mathbf{R}]$ *such that for* $t \geq t_0$, $T \geq t_0$ (2.35) *holds and*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) Q_5(s) ds < \infty,$$

and

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big(H(t,s) Q_5(s) - \frac{s}{4} Q^2(t,s) \Big)\, ds \geq \phi(T).$$

where $Q(t,s)$ *as in Theorem* 2.5, Q_5 *as in Theorem* 2.6 *and* $\phi_+ = \max\{\phi(t), 0\}$. *Then every solution of* (1.2) *oscillates.*

Proof. The proof is left to the reader. ■

Remark 2.3. We point out that we can deduce corollaries similar to Corollary 2.7 from Theorems 2.5 and 2.6 as well. Of course, we are not limited only to choice of functions H and h defined, by (2.30), (2.31) respectively which has become standard and goes back to the well known Kamenev-type conditions. Changing these functions it is possible to derive from Theorems 2.7, 2.8, 2.9 and 2.10 other Corollaries. In fact, another possibility is to choose the functions H and h as follows:

$$H(t,s) = \left(\ln \frac{t}{s}\right)^n, \; h(t,s) = \frac{n}{s}\left(\ln \frac{t}{s}\right)^{n/2-1}, \;\; t \geq s \geq t_0, \tag{2.54}$$

One may also choose the more general forms for the functions H and h:

$$H(t,s) = \left(\int_s^t \frac{du}{\theta(u)}\right)^n, \; h(t,s) = \frac{n}{\theta(s)} \left(\int_s^t \frac{du}{\theta(u)}\right)^{n/2-1} \quad t \geq s \geq t_0, \tag{2.55}$$

where $n > 1$ is an integer, and $\theta : [t_0, \infty) \to \mathbf{R}^+$ is a continuous function satisfying the condition

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{du}{\theta(u)} = \infty.$$

and

$$H(t,s) = (e^t - e^s)^n, h(t,s) = ne^s\left(e^t - e^s\right)^{(n-2)/2} \quad t \geq s \geq t_0, \tag{2.56}$$

It is a simple matter to check that in all these cases assumptions (I) and (II) are verified.

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