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OPTIMAL CONTROL OF IMPULSIVE STOCHASTIC EVOLUTION INCLUSIONS

N.U. Ahmed

SITE
and Department of Mathematics
University of Ottawa
Ottawa, Ontario, Canada

Abstract

In this paper, we consider a class of infinite dimensional stochastic impulsive evolution inclusions driven by vector measures. We use stochastic vector measures as controls adapted to an increasing family of complete sigma algebras and prove the existence of optimal controls.

Keywords: impulsive perturbations, C_0-semigroups, stochastic systems, differential inclusions, vector measures, optimal controls.

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1. Introduction

In this paper, we consider a class of controlled stochastic impulsive systems where the principal operator is the generator of a C_0-semigroup which is impulsively perturbed multiplicatively. A nonlinear drift (additive) term driven by a vector measure also represents impulsive behavior of the system. The third (drift) term represents control, again generated by a stochastic vector measure u which may contain both continuous and impulsive forces. The diffusion term is given by a multivalued map. Symbolically, the system is governed by the differential inclusion

$$(1)\quad dx(t) - Ax(t)d\beta(t) - F(t,x)d\mu(t) - G(t,x)du \in C(t,x)dW, x(0) = \xi,$$

where W is a cylindrical Brownian motion defined on some probability space $(\Omega, \mathcal{F}, P)$ and taking values in a Hilbert space U. We have recently shown

[2] that under certain assumptions on the pair $(A, \beta(\cdot))$, the nonlinear maps F, G and the vector measures μ, u and the multivalued operator C, the stochastic inclusion has a nonempty set of solutions. In a recent paper [1], we have studied the following system of evolution inclusions in general Banach spaces.

$$dx - Axd\beta \in F(t,x)dt,\ t \geq 0, x(0) = \xi \tag{2}$$

$$dx - Axd\beta - B(t,x)dt \in C(t,x)d\mu,\ t \geq 0, x(0) = \xi, \tag{3}$$

where F and C are multivalued maps. There, in the context of general Banach spaces, we proved the existence and regularity properties of solutions for such systems under mild assumptions on the operators and multivalued maps. We shall freely use the basic results of the papers [1] and [2], in particular the results related to the transition operator corresponding to the generator $(A, \beta(\cdot))$ and the existence result for equation (1). Here we assume that the operator A is the infinitesimal generator of a C_0-semigroup in the Hilbert space H; and F, G are suitable nonlinear operators, C a multivalued map, and β is generally a nonnegative nondecreasing (except for A generating groups) scalar valued function of bounded variation on bounded intervals of $R_0 \equiv [0, \infty)$ and μ and the control u are suitable vector measures on the sigma algebra of Borel subsets $\mathcal{B}_0$ of R_0. These models are much more general and cover all classical models of impulsive systems as widely used in the literature [4, 9, 12, 14, 17]. In fact they also cover the models used to develop control theory in recent years like [3–4, 6, 8]. The admissible controls considered in [7, 8] are deterministic vector measures and so may also be impulsive. Here in this paper, the controls are stochastic vector measures as described later in details.

Recently we have considered stochastic evolution inclusions of the form (1) in [2], where we proved the existence of solutions and studied some topological properties of the solution set. Here we are interested in the question of existence of optimal controls presented later in the paper.

Examples of impulsive systems can be found in many engineering applications such as optical communication, pulsed radars, spacecraft antennas etc., see [1].

The rest of the paper is organized as follows. In Section 2, basic notations are introduced. In Section 3, we present some results from [1] on the basic evolution operator associated with the pair $(A, \beta(\cdot))$ and its properties. This is used to construct solutions of non homogeneous Cauchy problems like

$$(4) \qquad dx(t) = Ax(t)d\beta(t) + f(t), t \geq 0, x(0) = \xi.$$

In Section 4 we consider the questions of existence, uniqueness, and regularity properties of solutions of stochastic evolution equations associated with the evolution inclusions. In Section 5, stochastic differential inclusions are considered. For convenience of the reader, here we present some recent results of the author [2] on the question of existence of a nonempty set of solutions. The main contribution of this paper is presented in Section 6. Before we study the control problems and prove the existence of optimal controls we present some simple examples of potential admissible controls from the class of adapted stochastic vector measures.

2. Some notations and terminologies

For any metrizable topological space $\mathcal{Z}$, $2^{\mathcal{Z}} \setminus \emptyset$ will denote the class of all nonempty subsets of $\mathcal{Z}$, and $c(\mathcal{Z})(cb(\mathcal{Z}), cc(\mathcal{Z}), cbc(\mathcal{Z}), ck(\mathcal{Z}))$, denotes the class of nonempty closed (closed bounded, closed convex, closed bounded convex, compact convex) subsets of $\mathcal{Z}$.

Let $(\Omega, \mathcal{B})$ be an arbitrary measurable space and $\mathcal{Z}$ a Polish space. A multifunction $G : \Omega \longrightarrow 2^{\mathcal{Z}} \setminus \emptyset$ is said to be measurable (weakly measurable) if for every closed (open) set $C \subset \mathcal{Z}$ the set

$$G^{-1}(C) \equiv \{\omega \in \Omega : G(\omega) \cap C \neq \emptyset\} \in \mathcal{B}.$$

Let d be any metric induced by the topology of the Polish space $\mathcal{Z}$. It is known that the measurability of the multifunction G is equivalent to the measurability of the function $\omega \rightarrow d(x, G(\omega))$ for every $x \in \mathcal{Z}$. Even more, it is also equivalent to the graph measurability of G in the sense that

$$\{(x, \omega) \in \mathcal{Z} \times \Omega : x \in G(\omega)\} \in \mathcal{B}(\mathcal{Z}) \times \mathcal{B}$$

where $\mathcal{B}(\mathcal{Z})$ denotes the sigma algebra of Borel sets of $\mathcal{Z}$. Let X, Y be any two topological spaces and $G : X \longrightarrow c(Y)$ be a multifunction. G is said to be upper semicontinuous (USC) if for each set $C \in c(Y)$

$$G^{-1}(C) \equiv \{x \in X : G(x) \cap C \neq \emptyset\} \in c(X).$$

If Y is a metric space with a metric d, we can introduce a metric d_H on $c(Y)$, called the Hausdorff metric, as follows:

$$d_H(K, L) \equiv \max\{\sup\{d(k, L), k \in K\}, \sup\{d(K, \ell), \ell \in L\}\}$$

where $d(x, K) \equiv \inf\{d(x, y), y \in K\}$ is the distance of x from the set K. If Y is a complete metric space, then $(c(Y), d_H)$ is also a complete metric space.

Let E be a Banach space and let $\mathcal{M}_c(J, E)$ denote the space of bounded countably additive vector measures on the sigma algebra $\mathcal{B}$ of subsets of the set $J \subset R_0 \equiv [0, \infty)$ with values in the Banach space E, furnished with the strong total variation norm. That is, for each $\mu \in \mathcal{M}_c(J, E)$, we write

$$|\mu|_v \equiv |\mu|(J) \equiv \sup_{\pi} \Big\{ \sum_{\sigma \in \pi} \| \mu(\sigma) \|_E \Big\}$$

where the supremum is taken over all partitions π of the interval J into a finite number of disjoint members of $\mathcal{B}$. With respect to this topology, $\mathcal{M}_c(J, E)$ is a Banach space. For any $\Gamma \in \mathcal{B}$ define the variation of μ on Γ by

$$V(\mu)(\Gamma) \equiv V(\mu, \Gamma) \equiv |\mu|(\Gamma).$$

Since μ is countably additive and bounded, this defines a countably additive bounded positive measure on $\mathcal{B}$. In the case $E = R$, the real line, we have the space of real valued signed measures. We denote this by simply $\mathcal{M}_c(J)$ in place of $\mathcal{M}_c(J, R)$. Clearly, for $\nu \in \mathcal{M}_c(J)$, $V(\nu)$ is also a countably additive bounded positive measure. For uniformity of notation we use λ to denote the Lebesgue measure. For any Banach space X, we let X^* denote the dual. Strong convergence of a sequence $\{\xi_n\} \in X$ to an element $\xi \in X$ is denoted by $\xi_n \xrightarrow{s} \xi$ and its weak convergence by $\xi_n \xrightarrow{w} \xi$. For any pair of Banach spaces X, Y, $\mathcal{L}(X, Y)$ will denote the space of bounded linear operators from X to Y.

For any real Banach space X, and any arbitrary set J, the space of all bounded X valued functions defined on J and denoted by $B_b(J, X)$ and furnished with the sup norm topology,

$$\| z \|_0 \equiv \sup\{\| z(t) \|_X, t \in J\},$$

is a Banach space. If X is separable and J is furnished with the sigma algebra $\mathcal{B}_J$ of Borel subsets of the set J, then the family $B(J,X)$ of bounded Borel measurable functions, furnished with the same sup norm topology, is a closed subspace of $B_b(J,X)$ and hence a Banach space also. In fact, $B(J,X)$ is given by the uniform limits, in the topology of $B_b(J,X)$, of characteristic functions of sets from $\mathcal{B}_J$ with values from any countable set dense in X. We use $PWC(J,X)$ to denote the class of all bounded piece wise continuous functions with values in X and furnished with the same topology. Clearly, this is a dense linear subspace of $B(J,X)$.

3. Basic evolution operator

We start with the Cauchy problem

$$dx(t) = Ax(t)d\beta(t), t \geq 0, x(0) = \xi. \tag{5}$$

Let D denote the collection of an ordered sequence of discrete points from R_0 given by

$$D \equiv \{0 = t_0 < t_1 < t_2, \cdots t_n < t_{n+1}, \cdots n \in N_0\}$$

and let S denote the step function

$$S(t) = \begin{cases} 1 & \text{if } t \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Without loss of generality we may assume that A is the infinitesimal generator of a C_0-semigroup of contraction $T(t), t \geq 0$, in a Banach space X and that the function β is given by

$$\beta(t) \equiv t + \sum_{k \geq 0} \alpha_k S(t - t_k), t \geq 0, t_k \in D, \tag{6}$$

where generally $\alpha_k \in R \cup \{+/-)\infty\}$, with $\alpha_0 = 0$. Define the intervals $\sigma_k \equiv [t_k, t_{k+1}); k \in N_0$ and note that

$$R_0 = \bigcup_{k \geq 0} \sigma_k.$$

In a recent paper [1], it was shown that, under some reasonable assumptions, the pair $(A, \beta(\cdot))$ generates an evolution operator,

$$U_\beta(t,s), \quad 0 \leq s \leq t < \infty,$$

in X. This is reproduced below. For arbitrary $t \in R_0$, define the following integer valued function

$$i(t) \equiv k, \text{ for } t \in \sigma_k, k \in N_0.$$

Using this notation one can express the evolution operator corresponding to the pair (A, β) [Ref. 1] as

$$U_\beta(t,r) \equiv \sum_{\ell=0}^{i(t)} \Big(\prod_{k=\ell+1}^{i(t)} (I - \alpha_k A)^{-1} \Big) \chi_{\sigma_\ell}(r) T(t-r), \tag{7}$$

for any $t \in R_0$ and $0 \le r < t$, where χ_σ denotes the indicator function of the set σ.

From the expression (7), it is clear that for $r = 0$ all the terms except the one with $\ell = 0$ vanish and hence

$$U_\beta(t,0) = \Big(\prod_{k=1}^{i(t)} (I - \alpha_k A)^{-1} \Big) T(t). \tag{8}$$

The following result is fundamental and can be found in [1].

Lemma 3.1. *Consider the system* (5) *and suppose A is the infinitesimal generator of a C_0-semigroup of contractions $T(t), t \ge 0$ in the Banach space X and the function β is given by the expression* (6) *where the coefficients $\{\alpha_k\}$ are nonnegative with $\alpha_0 = 0$. Then, there exists a unique evolution operator U_β satisfying the following properties:*

(P1): *$t \to U_\beta(t,r), t > r$ is continuous from the right in the strong operator topology in X; that is, $s - \lim_{t \downarrow r} U_\beta(t,r)\xi = \xi, \xi \in X$.*

(P2): *$s - \lim_{t \uparrow \tau > r} U_\beta(t,r)\xi$ exists $\forall\ \xi \in X$* and $\tau > r$.

(P3): $\| U_\beta(t,s)\xi \| \le \| \xi \|,\ \forall\ \xi \in X,$ and $0 \le s \le t < \infty$.

(P4): *$r \to g_t(r) \equiv U_\beta(t,r)\xi, 0 \le r \le t$, is piecewise continuous having simple discontinuities at $r \in \{t_k, t_k < t, k \ge 1\}$.*

(P5): $U_\beta(t,s)U_\beta(s,r) = U_\beta(t,r)\ \forall\ 0 \le r < s < t < \infty$.

Proof. See [1].

A similar result holds for groups, see [1, 2].

4. Stochastic evolution equations

Let $(\Omega, \mathcal{F}, \mathcal{F}_t \uparrow, t \geq 0, P)$ denote a complete filtered probability space with $\mathcal{F}_t, t \geq 0$, denoting an increasing family of complete sub sigma algebras of the sigma algebra $\mathcal{F}$. For any $\mathcal{F}$-measurable random variable z, we use the standard notation Ez to denote the integral of z with respect to the probability measure P, that is,

$$Ez = \int_\Omega z(\omega) P(d\omega).$$

Let U, H be any pair of separable Hilbert spaces and let $\mathcal{L}_{HS}(U, H)$ denote the space of Hilbert-Schmidt operators from U to H furnished with the scalar product and the associated norm

$$< B, C > \equiv Tr(BC^*), \text{ and } \parallel B \parallel_{HS} = \sqrt{Tr(BB^*)}$$

respectively. It is easy to show that $Y \equiv \mathcal{L}_{HS}(U, H)$ is a separable Hilbert space. Assuming that Y is furnished with its topological Borel field $\mathcal{B}(Y)$, we have a measurable space $(Y, \mathcal{B}(Y))$. We consider random variables $\{\sigma\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and taking values from $Y = \mathcal{L}_{HS}(U, H)$ In fact, we are more interested in stochastic processes taking values from the separable Hilbert space Y.

All the random processes considered in this paper are assumed to be adapted to the filtration $\mathcal{F}_t, t \geq 0$. Let $J \equiv [0, a]$ denote a finite interval and $\mathcal{P}$ the σ-algebra of progressively measurable subsets of the set $J \times \Omega$. Let $L_2(\mathcal{P}, Y) \equiv L_2(\mathcal{P}, \mathcal{L}_{HS}(U, H))$ denote the class of progressively measurable random processes taking values from the space of Hilbert-Schmidt operators $Y \equiv \mathcal{L}_{HS}(U, H)$ with square integrable Hilbert-Schmidt norms. For convenience of presentation and to emphasize time we denote this by

$$M_{2,2}(J, \mathcal{L}_{HS}(U, H)) \equiv L_2(\mathcal{P}, \mathcal{L}_{HS}(U, H))$$

or briefly as

$$M_{2,2}(J, Y) \equiv L_2(\mathcal{P}, Y).$$

Its topology is induced by the scalar product

$$< K, L > \equiv E \int_J Tr(K(t) L^*(t)) dt,$$

where L^* denotes the adjoint of the operator L. Clearly, the norm is given by

$$\| K \| \equiv \Big(\int_J E\{\| K(t) \|_{HS}^2\}dt\Big)^{1/2}.$$

With respect to this norm topology, it is again a Hilbert space. We shall also use the notation $M_{\infty,2}(J,H)$ for the space $L_\infty(J,L_2(\Omega,H))$ of all progressively measurable random processes with values in H having essentially bounded second moments. This is furnished with the norm topology

$$\| x \| \equiv esssup\Big\{\sqrt{E \| x(t) \|_H^2}, t \in J\Big\}.$$

With respect to this topology $M_{\infty,2}(J,H)$ is a Banach space. For initial states we choose the Hilbert space $L_2(\mathcal{F}_0,H)$ and denote this by $M_2(H)$. Note that this consists of all H-valued $\mathcal{F}_0$-measurable random variables having finite second moments. Since $\mathcal{F}_0$ is complete, this is a closed subspace of the Hilbert space $L_2(\mathcal{F},H)$ and hence a Hilbert space. For the study of the differential inclusion (1), we need some basic results. Before we can consider the Differential Inclusion (1), we must consider the stochastic differential equation,

$$dx(t) = Ax(t)d\beta(t) + F(t,x)d\mu(t) + L(t)dW, x(0) = \xi, \tag{9}$$

where L is a suitable operator valued random process to be defined shortly. Let H,U be any pair of separable Hilbert spaces as introduced above and E an arbitrary Hilbert space. The pair $(A,\beta(\cdot))$ is as described in Section 3 and μ is any countably additive E valued vector measure of bounded total variation. The process $W \equiv \{W(t), t \geq 0\}$ with $P(W(0) = 0) = 1$, is a cylindrical Brownian motion with values in U. We quote the following result from [2] without proof.

Theorem 4.1. *Suppose A and β satisfy the assumptions of Lemma 3.1 with X replaced by H. Let $F : J \times H \longrightarrow \mathcal{L}(E,H)$ be measurable in t on J and Lipschitz in x on E and that there exists a $K \in L_2^+(J,|\mu|)$, so that the following growth and Lipschitz conditions hold*

$$\| F(t,x) \|_{\mathcal{L}(E,H)} \leq K(t)(1 + |x|_H) \tag{10}$$

$$\| F(t,x) - F(t,y) \|_{\mathcal{L}(E,H)} \leq K(t)\ |x - y|_H. \tag{11}$$

Then, for every $\xi \in M_2(H)$*, and every* $L \in M_{2,2}(J, \mathcal{L}_{HS}(U,H))$*, independent of* $\{\xi\}$*, equation* (9) *has a unique mild solution* $x \in M_{\infty,2}(J,H)$*.*

Proof. See [2].

Remark. It is clear that the solution to equation (9) is certainly not continuous. However, if all the jumps of β are zero and the vector measure μ is absolutely continuous with respect to the Lebesgue measure, that is, E has RNP with respect to Lebesgue measure and that μ is λ continuous, then the solution to (9) has continuous trajectories almost surely. Since $L \in M_{2,2}(J, \mathcal{L}(U,H))$, this follows from the classical results of Da Prato and Zabczyk [9, Theorem 7.4].

Remark. If β is as given and the vector measure μ has RND (Radon-Nikodym derivative) with respect to the Lebesgue measure, then the solution $x \in PWC(J,H)$ almost surely.

Corollary 4.2. *Under the assumptions of Theorem* 4.1*, for fixed* $\xi \in M_2(H)$*, the map* $L \longrightarrow x$ *is Lipschitz continuous from* $M_{2,2}(J, \mathcal{L}_{HS}(U,H))$ *to* $M_{\infty,2}(J,H)$ *satisfying*

$$\| x_1 - x_2 \|^2_{M_{\infty,2}(J,H)} \leq \tilde{c}\Big\{E \int_J \| L_1(s) - L_2(s) \|^2_Y \, ds\Big\}, \tag{12}$$

for some constant $\tilde{c}$ *finite.*

Remark. It would be interesting to consider $t \rightarrow \beta(t)$ to be a nonnegative nondecreasing random process of bounded variation on any finite interval and prove the existence of a family of measurable evolution operators, $U_\beta(t,s), 0 \leq s \leq t < \infty$, giving the transition operator for the problem (5).

5. Stochastic differential inclusions

Often, deterministic systems governed by parabolic (or hyperbolic) variational inequalities, systems with uncertain parameters, systems with discontinuous vector fields, and control systems can be modeled as differential inclusions. The same remark applies to stochastic systems as well. Stochastic differential inclusions of the classical type, like

$$dx \in Axdt + F(t,x)dt + C(t,x)dW,$$

were studied by the author in [6] where the existence of solutions in an appropriate weak sense was established under different situations. For example, cases like F multivalued and C single valued, F single valued and C multivalued and both multivalued were considered under the assumptions that the multivalued maps are weakly inward and α-condensing where α denotes the Kuratowski's measure of non compactness. Also nonlinear systems with A monotone hemicontinuous with respect to the so called Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ were covered. Here we consider the differential inclusion given by

$$dx \in Axd\beta + F(t,x)d\mu + C(t,x)dW \tag{13}$$

and its controlled version given by inclusion (1). Clearly, this model is significantly different from the classical ones and in fact generalizes them.

Recall that we have used Y to denote the Hilbert space $\mathcal{L}_{HS}(U,H)$ with the scalar product as defined at the beginning of Section 4. Since we have assumed that both U and H are separable Hilbert spaces, it is clear that $Y \equiv \mathcal{L}_{HS}(U,H)$ is also a separable Hilbert space and therefore, a complete separable metric space with the metric induced by the Hilbert-Schmidt norm. According to our earlier notation, $L_2(\mathcal{P},Y) = M_{2,2}(J,Y)$. Let $cb(Y)$ denote the class of nonempty closed bounded subsets of Y and d_H denote the Hausdorff metric on it. It is easy to verify that $cb(Y)$, furnished with this metric, is a complete separable metric space and hence a Polish space.

The multivalued diffusion C is a map

$$C : J \times H \longrightarrow cb(Y). \tag{14}$$

We need the notion of solution for stochastic differential inclusions as introduced in [6]. By a solution, of course, we always mean a mild solution.

Definition 5.1. An element $x \in M_{\infty,2}(J,H)$ is a (mild) solution of the evolution inclusion (13), if there exists an $L \in M_{2,2}(J,Y)$ such that x is a (mild) solution of the evolution equation

$$dx = Axd\beta + F(t,x)d\mu + L(t)dW,\ t \geq 0, x(0) = \xi, \tag{15}$$

and

$$L(t) \in C(t, x(t)) \text{ a.e. } t \in J,\ P - \text{a.s.} \tag{16}$$

For a fixed $\xi \in M_2(H)$, let $\mathcal{S}$ denote the solution map $L \longrightarrow x(L)(\cdot) \equiv \mathcal{S}(L)(\cdot)$ corresponding to the system (15). It follows from Theorem 4.1, that equation (15) has a unique mild solution for each given $L \in M_{2,2}(J, Y)$. Thus it is evident from the definition, that if the pair $\{L, x\}$ satisfies the relations (15) and (16), and $\mathcal{S}$ is the solution map as introduced above, L must satisfy the following inclusion relation

$$L(t) \in C(t, \mathcal{S}(L)(t)),\ \text{a.e. } t \in J,\ P - \text{a.s.}$$

In other words, the question of existence of a solution of the stochastic evolution inclusion (13) is equivalent to the question of existence of a fixed point of the multivalued map $\hat{C}$ in the Hilbert space $L_2(\mathcal{P}, Y) \equiv M_{2,2}(J, H)$, where

$$\hat{C}(L) \equiv \{\Gamma \in L_2(\mathcal{P}, Y) : \Gamma(t) \in C(t, \mathcal{S}(L)(t)) \text{ a.e.}, P - \text{a.s}\}. \tag{17}$$

Theorem 5.2. *Suppose* 0 *is not an atom of* μ *and the pair* (A, β) *satisfies the assumptions of Lemma* 3.1 *and* F *satisfies the assumptions of Theorem* 4.1*, and the multifunction* C *satisfies the following assumptions:*

(C1): $C : J \times H \longrightarrow cb(Y)$, *measurable in* t *on* J *for each fixed* $x \in H$*, and, for almost all* $t \in J$*, it is (USC) upper semi continuous on* H,

(C2): *there exists an* $\ell_0 \in L_2^+(J)$ *such that*

$$\inf\{\| L \|_Y, L \in C(t, e)\} \leq \ell_0(t)(1 + |e|_H), t \in J,$$

(C3): *there exists an* $\ell \in L_2^+(J)$ *such that*

$$d_H(C(t, x), C(t, y)) \leq \ell(t)|x - y|_H,\ \forall\ x, y \in H, t \in J.$$

Then for each $x(0) = \xi \in M_2(H)$*, the evolution inclusion* (13) *has at least one solution* $x \in M_{\infty,2}(J, H)$.

Proof. See [2].

Solution Set

In general, differential inclusions possess many solutions. Hence it is natural to consider the solution set. Consider the system (13) with a given $\xi \in M_2(H)$. Let $Fix(\hat{C})$ denote the set of fixed points of the multifunction $\hat{C}$ mapping $L_2(\mathcal{P}, Y)$ to $2^{L_2(\mathcal{P},Y)}$. That is,

$$Fix(\hat{C}) \equiv \{L \in L_2(\mathcal{P}, Y) : L \in \hat{C}(L)\}. \tag{18}$$

It is clear from Theorem 5.2 that $Fix(\hat{C}) \neq \emptyset$. Let X_ξ denote the set of solutions of the evolution inclusion (13) corresponding to the initial state $\xi \in M_2(H)$. Define the linear operator $\mathcal{K}$ mapping $M_{2,2}(J, Y)$ to $M_{\infty,2}(J, H)$ by

$$\mathcal{K}(L)(t) \equiv \int_0^t U_\beta(t, s)L(s)dW(s),\ t \in J.$$

Clearly, this is a bounded linear operator from $M_{2,2}(J, Y)$ to $M_{\infty,2}(J, H)$ and for each $L \in M_{2,2}(J, Y)$ we have

$$E|\mathcal{K}(L)(t)|_H^2 = \int_0^t E \parallel U_\beta(t, s)L(s) \parallel_Y^2 ds,\ t \in J.$$

It follows from this that $\mathcal{K}$ is an isometric map from $M_{2,2}(J, Y)$ to $M_{\infty,2}(J, H)$. Using the properties of the transition operator one can also verify that the operator $\mathcal{K}$ is also injective.

The following result has important applications in the study of optimal controls. From now on we assume that $x(0) = \xi \in M_2(H)$ is fixed.

Corollary 5.3. *Suppose the assumptions of Theorem* 5.2 *hold and that the linear operator* $\mathcal{K}$ *maps every closed subset of* $M_{2,2}(J, Y)$ *into a closed subset of* $M_{\infty,2}(J, H)$. *Then, the solution set* $\mathcal{X}$ *of the evolution inclusion* (13), *corresponding to a fixed initial state* $x(0) = \xi \in M_2(H)$, *is a nonempty sequentially closed subset of* $M_{\infty,2}(J, H)$.

Proof. See [2].

Remark. The assumption that the linear operator $\mathcal{K}$ maps closed subsets of $M_{2,2}(J, Y)$ into closed subsets of $M_{\infty,2}(J, H)$ holds if the range of the operator $\mathcal{K}$ is closed.

Another assumption under which the Corollary is valid is as follows. The multifunction

$$\Gamma_t(s) \equiv \{U_\beta(t, s)L(s), L \in Fix(\hat{C})\},\ s \leq t,\ t \in J$$

is $\mathcal{F}_t$ measurable with values from the class of nonempty closed subsets of $M_{2,2}(J_t, Y)$ where $J_t \equiv [0,t], t \in J$. In this case one invokes the theory of measurable selections.

Remark. Even though the solution set $\mathcal{X}$ is closed, as stated in Corollary 5.3, it may not be bounded. For boundedness we need an additional condition on the multifunction C. This is given in the following theorem.

Theorem 5.4. *Consider the system* (13) *and suppose the assumptions of Theorem* 5.2 *hold. Further suppose that C satisfies the following growth condition: there exists a $\zeta \in L_2^+(J)$ such that*

$$\sup\{\| L \|_Y : L \in C(t,x)\} \leq \zeta(t)(1 + |x|_H) \ \forall \ x \in H.$$

Then the solution set $\mathcal{X}$ of the evolution inclusion (13) *is a closed bounded subset of $M_{\infty,2}(J,H)$.*

Proof. See [2].

6. Optimal impulsive control

The results mentioned above can be easily extended to include the controlled system,

$$dx \in Axd\beta + F(t,x)d\mu + G(t,x)du + C(t,x)dW,\ x(0) = \xi, \tag{19}$$

where u belongs to a suitable class of vector measures representing controls. The operators $\{A, \beta, F, C\}$ are as in the previous sections. Let V be another Hilbert space, or, in general, a reflexive Banach space. The operator G is a single valued map mapping $J \times H$ to $\mathcal{L}(V, H)$ satisfying similar properties with respect to the vector measure $u \in \mathcal{M}_c(J, V)$ as those of F with respect to the vector measure μ.

We are interested in control problems for this system. As in classical stochastic control problems, it is natural to consider admissible controls to be only those which are non anticipative with respect to the filtration $\mathcal{F}_t, t \geq 0$, or simply $\mathcal{F}_t$ adapted. In general, there controls can be chosen from the class of progressively measurable stochastic processes. Since our controls are V-valued measures defined on $\mathcal{B}_J$ it is necessary to clarify what is meant by non anticipating. We assume that the Banach space V is furnished with its topological Borel field $\mathcal{B}_V$ so that $(V, \mathcal{B}_V)$ is a measurable space. The following definition was introduced in [2].

Definition 6.1. A random vector valued measure u defined by the mapping

$$u : \mathcal{B}_J \times \Omega \longrightarrow V$$

is said to be $\mathcal{F}_t$-progressively measurable if, for every $t \in J$ and every set $\sigma \in \mathcal{B}_{[0,t)}$, the V-valued random variable $\omega \longrightarrow u(\sigma, \omega) = u(\sigma)(\omega)$ is $\mathcal{F}_t$ measurable. That is,

$$\{\omega \in \Omega : u(\sigma)(\omega) \in \Gamma\} \in \mathcal{F}_t$$

for every $\Gamma \in \mathcal{B}_V$.

We denote this class of vector measures by $\mathcal{M}_0$. For $1 \leq p \leq \infty$, let $L_p(\Omega, \mathcal{M}_c(J, V))$ denote the Banach space of random vector measures with the norm topology given by

$$\| u \|_p \equiv (E|u|_v^p)^{1/p}$$

where $|u|_v$ denotes the total variation norm. The variation norm is given by

$$|u|_v \equiv |u|(J) \equiv \sup_{\pi} \Big\{ \sum_{\sigma \in \pi} \| u(\sigma) \|_V \Big\} < \infty,$$

where the supremum is taken over all partitions π of the interval J into a finite number of disjoint members of $\mathcal{B}_J$. With respect to this norm topology, $L_p(\Omega, \mathcal{M}_c(J, V))$ is a Banach space.

Let V be a reflexive Banach space. For $1 \leq p, q < \infty$ satisfying $(1/p) + (1/q) = 1$, using the theory of "lifting" [17, Theorem 7, p. 94], one can verify that the dual of $L_q(\Omega, C(J, V^*))$ is precisely $L_p^w(\Omega, \mathcal{M}_c(J, V))$, where the superscript w is used to denote the class of weakly measurable functions on Ω with values in $\mathcal{M}_c(J, V)$. The theory of lifting is required here since the spaces $C(J, V^*)$ and $\mathcal{M}_c(J, V)$ do not have the Radon-Nikodym property (RNP). In particular, we are interested in the space $L_\infty^w(\Omega, \mathcal{M}_c(J, V))$ which is the dual of $L_1(\Omega, C(J, V^*))$. It is interesting to note that any continuous linear functional ℓ on $L_1(\Omega, C(J, V^*))$ has the representation

$$\ell(g) = E \int_J < g(t, \omega), u(\omega)(dt) >_{V^*, V},$$

for some $u \in L^w_\infty(\Omega, \mathcal{M}_c(J,V))$ uniquely determined by ℓ and conversely every $u \in L^w_\infty(\Omega, \mathcal{M}_c(J,V))$ induces a continuous (equivalently bounded) linear functional given by

$$\ell_u(g) = E \int_J < g(t,\omega), u(\omega)(dt) >_{V^*,V} .$$

In other words, $(L_1(\Omega, C(J,V^*)))^*$ is isometrically isomorphic to $L^w_\infty(\Omega, \mathcal{M}_c(J,V))$.

Some Simple Examples of Admissible Controls

Here we present some choices of admissible controls.

(AC1): Let ν be a countably additive bounded positive measure on J. For admissible controls, one may choose the family

$$\mathcal{U}_{ad} \equiv \{u \in \mathcal{M}_0 \cap L^w_\infty(\Omega, \mathcal{M}_c(J,V)) : |u|(\sigma) \leq \nu(\sigma) \ P - \text{a.s.}, \forall \ \sigma \in \mathcal{B}_J\} .$$

(AC2): Define the set

$$\Gamma \equiv \{\mu_k, 1 \leq k \leq d, \mu_k \in \mathcal{M}_c(J,V)\}$$

where d is any finite positive integer. Consider the family of measurable random processes

$$M \equiv \{\alpha \in L_\infty(J \times \Omega, R^d) : \alpha_k(t,\omega) \in [-1,+1],$$
$$\alpha_k(t,\omega) - \mathcal{F}_t - \text{adapted}, 1 \leq k \leq d\}.$$

Then we define the set of admissible controls as

$$\mathcal{U}_{ad} \equiv \Big\{u : \text{ for every } \sigma \in \mathcal{B}_J, u(\omega)(\sigma) = \int_\sigma \sum_{i=1}^{d} \alpha_i(t,\omega)\mu_i(dt), \alpha \in M\Big\}.$$

Clearly, the measure defined by $\nu(\sigma) \equiv \sum_{i=1}^d |\mu_i|(\sigma)$ is a countably additive bounded positive measure and the elements of $\mathcal{U}_{ad}$ are dominated by ν in the sense that

$$V(u,\sigma) \leq \nu(\sigma) \ \forall \ u \in \mathcal{U}_{ad}, \sigma \in \mathcal{B}_J.$$

(AC3): Consider the infinite set $\{\mu_i, i \in N\}$ where each $\mu_i \in \mathcal{M}_c(J,V)$ and that

$$\sup_{i \in N} \| \mu_i \| < \infty.$$

Consider the infinite family of $\mathcal{F}_t$ adapted essentially bounded random processes

$$M \equiv \Big\{ m_i, i \in N : m_i(t,\omega) \in \{0,1\},\ \forall (t,\omega) \in J \times \Omega, \\ m_i(t) \text{ is } \mathcal{F}_t \text{ measurable},\ \textstyle\sum_{i=1}^{\infty} m_i = 1 \Big\}.$$

Choose for the admissible controls the set given by

$$\mathcal{U}_{ad} \equiv \Big\{ u : u(\omega)(\sigma) \equiv \int_\sigma \sum_{i=1}^{\infty} m_i(t,\omega)\mu_i(dt), \sigma \in \mathcal{B}_J \{m_i\} \in M \Big\}.$$

(AC4): Let $\Gamma \subset \mathcal{M}_c(J,V)$ be a weakly compact set satisfying the celebrated Bartle-Dunford-Schwartz theorem [16, Theorem IV.5, p. 105]. Let $m(t,\omega)$ be an essentially bounded measurable function, that is $m \in L_\infty(J \times \Omega)$, and that $m(t)$ is $\mathcal{F}_t$ adapted. Define the set of admissible controls as

$$\mathcal{U}_{ad} \equiv \Big\{ u : \forall \sigma \in \mathcal{B}_J, \omega \in \Omega, u(\omega)(\sigma) = \int_\sigma m(t,\omega)\mu(dt),\ \mu \in \Gamma \Big\}.$$

(AC5): The class (AC4) can be generalized to the following class. Define

$$M_b \equiv \Big\{ m \in L_\infty(J \times \Omega) : m(t) \ \text{ is } \mathcal{F}_t \ \text{ adapted and } \ |m(t,\omega)| \le b \Big\},$$

and the set of admissible controls as

$$\mathcal{U}_{ad} \equiv \Big\{ u : \forall \sigma \in \mathcal{B}_J, \omega \in \Omega, u(\omega)(\sigma) = \int_\sigma m(t,\omega)\mu(dt),\ m \in M, \mu \in \Gamma \Big\}.$$

The class of controls defined above are constructed from deterministic countably additive bounded vector measures $\mathcal{M}_c(J,V)$ multiplied by $\mathcal{F}_t$ adapted random processes. In contrast, we can also construct admissible controls from $L_1(J,V)$. Let γ be a countably additive bounded positive measure on J not necessarily absolutely continuous with respect to the Lebesgue measure.

To emphasize the measure we let $L_1(\gamma, V)$ denote the Lebesgue-Bochner space of γ-measurable V-valued functions on J which are integrable with respect to the measure γ.

(AC6): Take any weakly compact set $D \subset L_1(\gamma, V)$. For characterization of weakly compact subsets of $L_1(\gamma, V)$ see the Dunford theorem [16, Theorem IV.1, p. 101]. Let M denote the class of $\mathcal{F}_t$ adapted essentially bounded real random processes satisfying $|m(t,\omega)| \leq 1$. For admissible controls one may then choose the set

$$\mathcal{U}_{ad} \equiv \Big\{u : \text{for } \omega \in \Omega, \sigma \in \mathcal{B}_J, u(\omega)(\sigma) \equiv \int_\sigma m(t,\omega)f(t)\gamma(dt), m \in M, f \in D\Big\}.$$

(AC7): Take any weakly compact set $\Gamma \subset \mathcal{M}_c(J, V)$. Since V is reflexive, both V and V^* satisfy Radon Nikodym property and therefore by the Bartle-Dunford-Schwartz theorem [16, Theorem IV.5, p. 105], there exists a countably additive bounded positive measure π such that

$$\lim_{\pi(\sigma)\to 0} \mu(\sigma) = 0 \text{ uniformly with respect to } \mu \in \Gamma.$$

Consider the Lebesgue-Bochner space $L_1(\pi, V)$ and define the linear operator

$$(L_\pi f)(\sigma) \equiv \int_\sigma f(t)\pi(dt), f \in L_1(\pi, V).$$

Clearly, L_π is a bounded linear map from $L_1(\pi, V)$ to $\mathcal{M}_c(J, V)$ and that $Range(L_\pi) \supset \Gamma$. Thus $L_\pi^{-1}(\Gamma)$ is a weakly compact subset of $L_1(\pi, V)$. For M as in (AC6), define the admissible controls as

$$\begin{aligned}\mathcal{U}_{ad} \equiv \Big\{u :\ & \text{for } \omega \in \Omega, \sigma \in \mathcal{B}_J, \\ & u(\omega)(\sigma) = \int_\sigma m(t,\omega)f(t)\pi(dt), m \in M, f \in L_\pi^{-1}(\Gamma)\Big\}.\end{aligned}$$

The following result is proved exactly in the same way as Theorem 5.2. See [2].

Theorem 6.2. *Consider the system* (19) *with the admissible controls $\mathcal{U}_{ad}$ being weakly compact as described above and suppose all the assumptions of Theorem* 5.4 *hold and that there exists a constant $c > 0$ such that*

$$
\begin{aligned}
&\| G(t,x) \|_{\mathcal{L}(V,H)} \leq c\ (1+|x|_H) \\
&\| G(t,x) - G(t,y) \|_{\mathcal{L}(V,H)} \leq c\ |x-y|_H.
\end{aligned}
\tag{20}
$$

Then, for every $x(0)=\xi \in M_2(H)$ and $u \in \mathcal{U}_{ad}$, the system has a nonempty set of solutions $\mathcal{X}(u)$ and that it is a closed bounded subset of the Banach space $M_{\infty,2}(J,H)$.

Remark. We note that not all the admissible controls introduced above are weakly compact subsets of $L^w_\infty(\Omega, \mathcal{M}_c(J,V)) \cap \mathcal{M}_0$. Clearly, the class (AC4) is weakly compact. For weak compactness of the sets (AC5) – (AC7), it suffices to choose the family of real random processes M weakly compact in the following sense.

Lemma 6.3. *Let M denote the family of $\mathcal{F}_t$-adapted real random processes defined on J satisfying the following conditions:*

(1): *the elements of M are stochastically right or left continuous,*

(2): $\lim_{r\to\infty} \sup_{\xi\in M} \sup_{t\in J} P\{|\xi(t)| > r\} = 0,$

(3): $\lim_{h\downarrow 0} \sup_{\xi\in M} \sup_{|t-s|<h} P\{|\xi(t)-\xi(s)| > \varepsilon\} = 0\ \forall \varepsilon > 0.$

Then, corresponding to any sequence $\{\xi_n\}$ from M, there exist a subsequence $\{\xi_{n_k}\}$, defined in general on another probability space, and a stochastically continuous process $\xi_o \in M$ such that

$$
\xi_{n_k}(t) \longrightarrow \xi_o(t)\ \forall\ t \in J\ \text{in probability.} \tag{21}
$$

Further, if M is essentially bounded, or more generally, if there exists an integrable $\mathcal{F}_t$-adapted stochastically continuous process ζ so that

$$
|\xi_{n_k}(t,\omega)| \leq \zeta(t,\omega), (t,\omega) \in J \times \Omega,
$$

then the sequence also converges in the mean.

The result of Lemma 6.3 is essentially due to Skorohod [15, p. 9]. In view of this result, if the family M chosen for the classes (AC5) – (AC7) is sequentially compact in the sense of Lemma 6.3, we can conclude that the admissible controls given by (AC5) – (AC7) are sequentially compact. This is presented in the following lemma.

Lemma 6.4. *Consider any of the families* (AC5) – (AC7) *denoted by* $\mathcal{U}_{ad}$. *Suppose* $\Gamma \subset \mathcal{M}_c(J, V)$ *is weakly compact and* M *is an essentially bounded subset of* $L_\infty(\Omega \times J)$ *satisfying the assumptions of Lemma* 6.3. *Further suppose that* (21) *holds uniformly on* J. *Then* $\mathcal{U}_{ad}$ *is weakly compact.*

Proof. Let $\{u_n\}$ be any sequence from $\mathcal{U}_{ad}$. Then by definition there exist a sequence $\xi_n \in M$ and a sequence $\mu_n \in \Gamma$ such that

$$u_n(\omega)(\sigma) = \int_\sigma \xi_n(t, \omega)\mu_n(dt) \text{ for every } \sigma \in \mathcal{B}_J, \omega \in \Omega.$$

Suppressing ω we write this as

$$u_n(\sigma) \equiv \int_\sigma \xi_n(t)\mu_n(dt).$$

Clearly, this is a V-valued random variable. Take any $g \in L_1(\Omega, C(J, V^*))$ which is $\mathcal{F}_t$-adapted. Then

$$\begin{aligned} \ell_n(g) &\equiv E\int_J < g(t), u_n(dt) >_{V^*,V} \\ &= E\int_J < g(t)\xi_n(t), \mu_n(dt) > \\ &= \int_J < E(g(t)\xi_n(t)), \mu_n(dt) >_{V^*,V} \end{aligned}$$

with the last identity following from Fubini's theorem. By virtue of Lemma 6.3, there exist a subsequence of the sequence $\{\xi_n\}$, relabeled as the original sequence, and an element $\xi_o \in M$ such that

$$E(g\xi_n)(t) \longrightarrow E(g\xi_o)(t)$$

for each $t \in J$. In fact, it follows from the additional assumption that this convergence holds uniformly in t on J. Similarly, Γ being weakly compact, there exist a subsequence, again relabeled as the original sequence, and an element $\mu_o \in \Gamma$ such that

$$\mu_n \xrightarrow{w} \mu_o \text{ in } \mathcal{M}_c(J, V).$$

Using these facts one can easily verify that

$$\ell_n(g) \longrightarrow \ell_o(g)$$

for every $\mathcal{F}_t$-adapted $g \in L_1(\Omega, C(J, V^*))$ where

$$\ell_o(g) \equiv E \int_J < g(t)\xi_o(t), \mu_o(dt) > .$$

We denote the weak limit of u_n by u_o given by

$$u_o(\sigma) \equiv \int_\sigma \xi_o(t)\mu_o(dt).$$

This completes the proof. ■

For suitable f, φ and Ψ, the natural cost functional for a control problem may be given by,

$$J_0(u) \equiv \sup\Big\{\lambda(x,u) = E\Big(\int_J \ell(t,x(t))dt + \varphi(x(T)) + \Psi(u)\Big), x \in \mathcal{X}(u)\Big\}, \tag{22}$$

which is an appropriate measure of the maximum risk or cost. The problem here is to find a control $u_o \in \mathcal{U}_{ad}$ that minimizes the maximum risk, that is,

$$J_0(u_o) \leq J_0(u) \ \forall \ u \in \mathcal{U}_{ad}.$$

In general, the function ℓ is measurable in t on J and continuous in x on H satisfying

$$h_0(t) \leq \ell(t,x) \leq h(t)\left[1 + |x|_H^2\right], \tag{23}$$

for some $h_0 \in L_1(J)$ and $h \in L_1^+(J)$, and φ is required to satisfy

$$\alpha_0 \leq \varphi(x) \leq \alpha_1 + \alpha_2|x|_H^2, \alpha_0, \alpha_1 \in R, \quad \alpha_2 \geq 0. \tag{24}$$

The function Ψ is a nonnegative real valued weakly lower semi continuous functional defined on $\mathcal{M}_c(J,V)$ signifying a measure of cost of control. For example

$$\Psi(\mu) \leq \beta_1 + \beta_2|\mu|^\gamma, \ \gamma \geq 1, \tag{25}$$

where $\beta_1, \beta_2 \geq 0$ and $|\mu|$ denotes the total variation norm of the measure $\mu \in \mathcal{M}_c(J,V)$ as defined earlier. In view of the choice of the admissible controls $\mathcal{U}_{ad}$ and the assumption $(24-25)$, the control problem (19), (22) is well defined.

The questions of existence of optimal controls and also necessary conditions of optimality are of fundamental importance in control theory. For deterministic problems involving impulsive systems some results have been proved recently in [7]. Here we want to prove the existence of optimal controls for the stochastic problem. At this time we can only prove an existence result for systems of the form (19) with control appearing linearly,

$$dx \in Axd\beta + F(t,x)d\mu + G(t)du + C(t,x)dW, x(0) = \xi, \tag{26}$$

where the control operator G is a suitable linear operator valued function with values in $\mathcal{L}(V,H)$. We consider the system (26) with the cost functional given by (22).

The following is the main result of the paper.

Theorem 6.5. *Consider the optimal control problem* (22) *and* (26) *with the admissible controls $\mathcal{U}_{ad}$ being weakly compact. Suppose $\{A, \beta, F, \mu, C\}$ satisfy the assumptions of Theorem* 6.2 *and the functions ℓ φ are continuous in x on H and satisfy the growth conditions* $(23)-(24)$ *and Ψ is a nonnegative lower semi continuous functional on $\mathcal{M}_c(J,V)$ bounded on bounded sets satisfying* (25)*. Further suppose the semi group $T(t), t > 0$, corresponding to the generator A, is compact and that the control operator $G : J \longrightarrow \mathcal{L}(V,H)$ is bounded uniformly measurable and compact for each $t \in J$. Then there exists an optimal control for the problem* (22) *and* (26).

Proof. Since $\mathcal{U}_{ad}$ is a weakly compact subset of $L_\infty^w(\Omega, \mathcal{M}_c(J,V)) \cap \mathcal{M}_0$ it is clearly bounded and thus it follows from the assumption on Ψ that $E\{\Psi(|u|_v)\} < \infty$. Using this fact and the assumptions $(23-25)$ and the fact that $\mathcal{X}(u)$ is a bounded subset of $M_{\infty,2}(J,H)$ (Theorem 6.2), we conclude that

$$J_0(u) \equiv \sup\{\lambda(x,u), x \in \mathcal{X}(u)\} \equiv M_u < \infty.$$

Since no concavity assumption is imposed on the functions ℓ and φ, and $\mathcal{X}(u)$ is not necessarily compact, the supremum may not be attained in $\mathcal{X}(u)$.

However, for any $\varepsilon > 0$, we can find an element $x^u \in \mathcal{X}(u)$ such that

$$M_u \geq \lambda(x^u, u) \geq M_u - \varepsilon \ \ \forall \ \ u \in \mathcal{U}_{ad}. \tag{27}$$

It is also clear form $(23-25)$ and the fact that Ψ is nonnegative, that

$$-\infty < m \equiv \inf\{J_o(u),\, u \in \mathcal{U}_{ad}\}. \tag{28}$$

Let $\{u_n\} \in \mathcal{U}_{ad}$ be a minimizing sequence, that is,

$$\lim_{n\to\infty} J_o(u_n) = m.$$

By virtue of (27) we can find an $x_n \in \mathcal{X}(u_n)$ so that

$$J_o(u_n) \equiv M_{u_n} \geq \lambda(x_n, u_n) \geq M_{u_n} - \varepsilon \equiv J_o(u_n) - \varepsilon. \tag{29}$$

Since $x_n \in \mathcal{X}(u_n)$ there exists an $L_n \in M_{2,2}(J, Y)$ such that x_n is the solution of the stochastic differential equation,

$$dx_n = Ax_n d\beta + F(t, x_n)d\mu + G(t)du_n + L_n(t)dW, x(0) = \xi, \tag{30}$$

and that the pair $\{x_n, L_n\}$ is related by the inclusion relation,

$$L_n(t) \in C(t, x_n(t)) \text{ a.e., } P - \text{a.s.} \tag{31}$$

In other words, x_n satisfies the integral equation

$$\begin{aligned} x_n(t) = U_\beta(t,0)\xi \ &+ \int_0^t U_\beta(t,s)F(s,x_n(s))d\mu(s) + \int_0^t U_\beta(t,s)G(s)du_n(s) \\ &+ \int_0^t U_\beta(t,s)L_n(s)dW(s), t \in J \end{aligned} \tag{32}$$

for some $L_n \in Fix(\hat{C}^{u_n})$ where, for any $L \in M_{2,2}(J, Y)$,

$$\hat{C}^u(L) = \{\Gamma \in M_{2,2}(J, Y) : \Gamma(t) \in C(t, \mathcal{S}^u(L)(t)) \text{ a.e. and } P - \text{a.s.}\} \tag{33}$$

with $\mathcal{S}^u(L)$ denoting the mild solution of the stochastic differential equation (SDE),

$$dx = Axd\beta + F(t,x)d\mu + G(t)du + LdW, x(0) = \xi, \tag{34}$$

corresponding to the control $u \in \mathcal{U}_{ad}$ and $L \in M_{2,2}(J,Y)$. Since the set of admissible controls $\mathcal{U}_{ad}$ is bounded and, by Theorem 5.4, each solution set $\mathcal{X}(u)$ is a bounded subset of $M_{\infty,2}(J,H)$, the set

$$\mathcal{X}(\mathcal{U}_{ad}) \equiv \cup\{\mathcal{X}(u), u \in \mathcal{U}_{ad}\}$$

is also a bounded subset of $M_{\infty,2}(J,H)$. This can be easily proved by using the growth assumption for the multifunction C, as stated in Theorem 5.4, and the boundedness of the set $\mathcal{U}_{ad}$. Hence there exists a finite positive number $b > 0$ such that

$$\sup\Big\{E|x(t)|_H^2, t \in J, x \in \mathcal{X}(\mathcal{U}_{ad})\Big\} \leq b,$$

where the constant b depends on the set of parameters $\{K, \zeta, |\mu|_v, |\nu|_v, E|\xi|_H^2\}$. Thus it follows from the growth assumption of the multifunction C, as mentioned above, that the sequence $\{L_n\}$ is contained in a bounded subset of $M_{2,2}(J,Y)$. Since $\mathcal{U}_{ad}$ is weakly compact and $M_{2,2}(J,Y)$ is a Hilbert space (hence a bounded set is relatively weakly compact) there exist a subsequence of the sequence $\{u_n, L_n\}$, relabeled as the original sequence, and a pair $\{u^o, L_o\}$ such that

$$u_n \xrightarrow{w} u^o \text{ in } \mathcal{U}_{ad} \tag{35}$$

$$L_n \xrightarrow{w} L_o \text{ in } M_{2,2}(J,Y). \tag{36}$$

Let $x_o \in M_{\infty,2}(J,H)$ denote the solution of the of the integral equation,

$$x(t) = U_\beta(t,0)\xi + \int_0^t U_\beta(t,s)F(s,x(s))d\mu(s) + \int_0^t U_\beta(t,s)G(s)du^o(s)$$

$$+ \int_0^t U_\beta(t,s)L_o(s)dW(s),\ t \in J, \tag{37}$$

corresponding to the pair $\{u^o, L_o\}$. We proceed with the proof assuming for the moment that $x_n(t) \longrightarrow x_o(t)$ in H pointwise in t, $P-$a.s. and then complete the proof by demonstrating the correctness of this hypothesis. We prove that u_o is the optimal control. Since $\{u_n\}$ is a minimizing sequence it follows from (28) and (29) that

$$\begin{aligned} m &= \lim_{n\to\infty} J_o(u_n) = \liminf_{n\to\infty} J_o(u_n) \\ &\geq \liminf_{n\to\infty} \lambda(x_n, u_n) \geq \liminf_{n\to\infty} J_o(u_n) - \varepsilon. \end{aligned} \tag{38}$$

Since ℓ and φ are continuous in x on H and they satisfy the growth conditions $(23 - 24)$, using Fatou's Lemma one can verify that the functional $x \longrightarrow \lambda(x, u)$ is lower semi continuous and bounded on $M_{\infty,2}(J, H)$ for each fixed $u \in \mathcal{U}_{ad}$. Recalling that Ψ is a lower semi continuous real valued functional on $\mathcal{M}_c(J, V)$, using Fatou's Lemma once again we have

$$E\left\{\liminf_{n\to\infty} \Psi(u_n)\right\} \leq \liminf_{n\to\infty} E\left\{\Psi(u_n)\right\}.$$

Clearly, this means that the functional $u \longrightarrow \lambda(x, u)$ is weakly lower semi continuous on $\mathcal{U}_{ad}$ for each $x \in M_{\infty,2}(J, H)$. Thus

$$\liminf_{n\to\infty} \lambda(x_n, u_n) \geq \lambda(x_o, u^o). \tag{39}$$

In view of (29) it follows from (38) and (39) that

$$m \geq \lambda(x_o, u^o) \geq m - \varepsilon. \tag{40}$$

Since $x_o \in \mathcal{X}(u^o)$, it follows from this inequality that

$$J_o(u^o) \geq m \geq \lambda(x_o, u_o) \geq m - \varepsilon$$

and that

$$J_o(u) \geq m, \forall u \in \mathcal{U}_{ad}.$$

Clearly, if $J_o(u^o) = m$, the control u^o is optimal. So it suffices to prove that $J_o(u^o) = m$. We prove this by actually showing that we can construct another sequence of minimizing controls whose weak limit is an admissible control at which J_o equals m. If u^o is not optimal, there exists a control $u^1 \in \mathcal{U}_{ad}$ such that

$$J_o(u^o) > J_o(u^1) \geq m.$$

But then we can find another minimizing sequence, say, $\{u_n^1\} \subset \mathcal{U}_{ad}$ converging weakly to u^1 and a corresponding sequence pair $\{L_n^1, x_n^1\}$ such that

$$L_n^1(t) \in C(t, x_n^1(t)), \text{a.e.}, \ P - \text{a.s.},$$

$$L_n^1 \xrightarrow{w} L^1 \text{ in } M_{2,2}(J, H)$$

$$x_n^1(t) \longrightarrow x^1(t), \text{ a.e., } P - \text{a.s. in } H,$$

and

$$J_o(u_n^1) \longrightarrow m.$$

Clearly, again if u^1 is not optimal (that is, $J_o(u^1) > m$), there exist a better control u^2 and, by the same argument, an infinite sequence of such better controls such that

$$J_o(u^o) > J_o(u^1) > J_o(u^2) > J_o(u^3) > \cdots > J_o(u^k) \cdots \geq m, \forall, k \in N.$$

Since $\{J_o(u^k)\}$ is a monotone decreasing sequence, bounded from below by m, it must converge to m. Letting $u^\infty \in \mathcal{U}_{ad}$ denote the weak limit of the above sequence of controls and noting that $u \longrightarrow \mathcal{X}(u)$ is continuous from $\mathcal{U}_{ad}$ to $cb(M_{\infty,2}(J, H))$ with respect to the relative weak topology on $\mathcal{U}_{ad}$ and the Hausdorff metric topology on $cb(M_{\infty,2}(J, H))$, it follows from weak lower semi continuity of $u \longrightarrow \lambda(x, u)$ that

$$\lim_{k \to \infty} J_o(u^k) = \underline{\lim}_{k \to \infty} J_o(u^k) \geq J_o(u^\infty).$$

Hence $J_o(u^\infty) = m$ and u^∞ is the optimal control. Thus, at the very outset one may assume that the minimizing sequence $\{u_n\}$ chosen has a limit u^o for which $J_o(u^o) = m$. Thus we have justified that

$$J_o(u) \geq J_o(u^o) = m \quad \forall u \in \mathcal{U}_{ad}$$

proving that u^o is optimal. Now it remains to justify that $x_n(t) \longrightarrow x_o(t)$ in H for all $t \in J$, $P -$ a.s. Subtracting (37) from (32), with the solution of the former denoted by x_o and that of the later denoted by x_n, and using the notations,

$$e_n(t) \equiv |x_n(t) - x_o(t)|_H, \quad \varphi_n(t) \equiv |z_n(t) - z_o(t)|_H, t \in J \tag{41}$$

$$\alpha(t) \equiv \int_0^t K(s) d|\mu|(s), t \in J, \tag{42}$$

$$z_n(t) \;\equiv\; \int_0^t U_\beta(t,s)G(s)du_n(s) + \int_0^t U_\beta(t,s)L_n(s)dW(s), \tag{43}$$

$$z_o(t) \;\equiv\; \int_0^t U_\beta(t,s)G(s)du_o(s) + \int_0^t U_\beta(t,s)L_o(s)dW(s), \tag{44}$$

we deduce the following inequality

$$e_n(t) \leq \varphi_n(t) + \int_0^t e_n(s)d\alpha(s). \tag{45}$$

Since μ is a countably additive vector measure of bounded total variation, and $K \in L_2^+(J,|\mu|) \subset L_1^+(J,|\mu|)$, α is a positive nondecreasing function of bounded variation.

Hence, by virtue of a generalized Gronwall type inequality, it follows from this that

$$e_n(t) \leq \varphi_n(t) + exp(\alpha(t)) \int_0^t \varphi_n(s)d\alpha(s). \tag{46}$$

Clearly, if $\varphi_n(t) \longrightarrow 0$ as $n \to \infty$ for all $t \in J$, $P-$a.s, then $e_n(t) \longrightarrow 0$ for all $t \in J$, $P-$ a.s. Thus it suffices to prove that $\varphi_n(t) \longrightarrow 0$ for each $t \in J$, $P-$ a.s.

We prove this as follows. Define

$$\Pi_n(t) \equiv \eta_n(t) - \eta_o(t) = \int_0^t U_\beta(t,s)(L_n(s) - L_o(s))dW(s) \tag{47}$$

$$\Lambda_n(t) \equiv \xi_n(t) - \xi_o(t) \equiv \int_0^t U_\beta(t,s)G(s)d(u_n(s) - u_o(s)), \tag{48}$$

and note that $z_n(t) - z_o(t) = \Pi_n(t) + \Lambda_n(t)$. To complete the proof it suffices to show that for each $t \in J$, $\Pi_n(t) \longrightarrow 0$ and $\Lambda_n(t) \longrightarrow 0$ in H strongly $P-$ a.s.

Considering Π_n, it is clear that

$$\begin{aligned} E|\Pi_n(t)|_H^2 \;&=\; E\int_0^t \parallel U_\beta(t,s)(L_n(s) - L_o(s)) \parallel_Y^2 ds \\ &=\; E\int_0^{t-\delta} \parallel U_\beta(t,s)(L_n(s) - L_o(s)) \parallel_Y^2 ds \\ &\quad + E\int_{t-\delta}^t \parallel U_\beta(t,s)(L_n(s) - L_o(s)) \parallel_Y^2 ds \end{aligned} \tag{49}$$

for every $\delta > 0$ so that $t > \delta$.

By our hypothesis the semigroup $T(t), t > 0$, is compact and the resolvents are bounded operators. Therefore, it follows from the expression (7) that the evolution operator $U_\beta(t,s), 0 \leq s < t \leq T$, is also a family of compact operators. Since L_n converges weakly to L_o in $M_{2,2}(J,Y)$, it follows from the compactness of the evolution operator that the first term of the above expression converges to zero as $n \to \infty$. For the second term, recall that both $\{L_n\}$ and L_o are dominated in the sense that

$$E \parallel L_n(t) \parallel_Y^2 \leq 2\zeta^2(t)(1 + E|x_n(t)|_H^2) \leq 2(1+b)\zeta^2(t), t \in J,$$

$$E \parallel L_o(t) \parallel_Y^2 \leq 2\zeta^2(t)(1 + E|x_o(t)|_H^2) \leq 2(1+b)\zeta^2(t), t \in J,$$

where b is the apriori bound already mentioned. Since the transition operator U_β is non expansive , this implies that

$$\begin{aligned} & E \int_{t-\delta}^{t} \parallel U_\beta(t,s)(L_n(s) - L_o(s) \parallel_Y^2 ds \\ & \leq \int_{t-\delta}^{t} 4(1+b)\zeta^2(s) ds \longrightarrow 0, \text{ as } \delta \to 0. \end{aligned} \tag{50}$$

Thus, it follows from (49) that $E|\Pi_n(t)|_H^2 \longrightarrow 0$ for each $t \in J$, and hence, along a subsequence if necessary,

$$\Pi_n(t) \xrightarrow{s} 0 \text{ in } H, \ P - \text{a.s.} \tag{51}$$

Now consider Λ_n. Clearly

$$|\Lambda_n(t)|_H = \sup\{(\Lambda_n(t), h), h \in B_1(H)\} \tag{52}$$

$$= \sup\left\{\int_0^t < \Phi_{t,h}(s), u_n(ds) - u^o(ds) >, h \in B_1(H)\right\} \tag{53}$$

where

$$\Phi_{t,h}(s) \equiv G^*(s)U_\beta^*(t,s)h, 0 \leq s \leq t \leq T.$$

We show that $\Lambda_n(t) \xrightarrow{s} 0$, $P -$ a.s. To show this, it suffices to verify that

$$E\{|\Lambda_n(t)|_H^2\} \longrightarrow 0.$$

Suppose the contrary. Then there exists a $\delta > 0$ and a sequence $h_n \in B_1(H)$ such that

$$0 < \delta \leq E\{(\Lambda_n(t), h_n)\} \ \ \forall \ \ n \in N.$$

Since $B_1(H)$ is weakly compact there exist a subsequence of the sequence $\{h_n\}$, relabeled as the original sequence, and an element $h_o \in B_1(H)$ such that $h_n \xrightarrow{w} h_o$. Then

$$\begin{aligned} 0 < \delta &\leq E\{(\Lambda_n(t), h_n)\} \\ &= E\{(\Lambda_n(t), h_n - h_o)\} + E\{(\Lambda_n(t), h_o)\} \ \ \forall \ \ n \in N. \end{aligned} \tag{54}$$

Since $u_n \xrightarrow{w} u^o$, $\Lambda_n(t) \xrightarrow{w} 0$. Hence we can choose $n_\delta \in N$ such that

$$|E\{(\Lambda_n(t), h_o)\}| \leq (\delta/4) \tag{55}$$

for all $n \geq n_\delta$. By our assumption G is a compact operator valued function and consequently

$$\| \Phi_{t,h_n - h_o}(s) \|_{V^*} = \| G^*(s) U^*_\beta(t,s)(h_n - h_o) \|_{V^*} \longrightarrow 0$$

for every $s \in [0,t]$. Using this result and the assumption that the set of admissible controls is a weakly compact subset of $L^w_\infty(\Omega, M_c(J,V)) \cap \mathcal{M}_o$, it is easy to verify that $E\{(\Lambda_n(t), h_n - h_o)\} \longrightarrow 0$ for every fixed $t \in J$. Thus there exists an $m_\delta \in N$ such that for $n \geq m_\delta$ we have

$$|E\{(\Lambda_n(t), h_n - h_o)\}| \leq (\delta/4). \tag{56}$$

Hence, for $n \geq \max\{n_\delta, m_\delta\}$, it follows from (55) and (56) that (54) is contradicted thereby proving that $x_n(t) \xrightarrow{s} x_o(t)$ point wise in $t \in J$, $P-$ a.s. Since the multifunction $C(t,x)$ is closed convex valued in Y, we can show that $L_o(t) \in C(t, x_o(t))$ a.e. $P-$ a.s. Thus $x_o \in \mathcal{X}(u^o)$ and so is a solution of the evolution inclusion corresponding to the control u^o. This completes the proof. ■

Remark. Under some additional assumptions it should be possible to extend the result of Theorem 6.5 to the case of nonlinear operator G as in equation (19).

Remark. It would be interesting to relax the compactness assumptions of the semigroup operators $\{T(t), t > 0\}$ and the control operator $\{G(t), t \in J\}$.

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