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ON THE EXISTENCE OF VIABLE SOLUTIONS FOR A CLASS OF SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract

We prove the existence of viable solutions to the Cauchy problem $x'' \in F(x, x'), x(0) = x_0, x'(0) = y_0$, where F is a set-valued map defined on a locally compact set $M \subset R^{2n}$, contained in the Fréchet subdifferential of a ϕ-convex function of order two.

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1. Introduction

In the present paper, we consider the second order differential inclusion of the form

$$x'' \in F(x, x'), \quad x(0) = x_0, x'(0) = y_0, \tag{1.1}$$

where $F(.,.) : M \subset R^n \times R^n \to \mathcal{P}(R^n)$ is a given set-valued map and $x_0, y_0 \in R^n$.

Existence of viable solutions to problem (1.1) has been studied by many authors, mainly in the case when the multifunction is convex valued ([2], [6], [8], [10] etc.).

Recently in [9], the situation when the multifunction is not convex valued is considered. More exactly, in [9] the existence of viable solutions to the problem (1.1) is proved when $F(.,.)$ is an upper semicontinuous, compact valued multifunction contained in the subdifferential of a proper convex function.

The aim of this paper is to extend the result of [9] to the case when the multifunction F is contained in the Fréchet subdifferential of a ϕ-convex function of order two. Since the class of proper convex functions is strictly contained in the class of ϕ-convex functions of order two, our result generalizes the one in [9].

On the other hand, our result may be considered as an extension of our previous existence result ([5]) obtained for first order differential inclusion of the form

$$x' \in F(x), \quad x(0) = x_0 \tag{1.2}$$

to the more general problem (1.1). The proof of our main result follows the general ideas of [5] and [9].

The paper is organized as follows: in Section 2, we recall some preliminary facts that we need in the sequel and in Section 3, we prove our main result.

2. Preliminaries

We denote by $\mathcal{P}(R^n)$ the set of all subsets of R^n and by R_+ the set of all positive real numbers. For $\epsilon > 0$ we put $B(x,\epsilon) = \{y \in R^n; \|y-x\| < \epsilon\}$. With B we denote the unit ball in R^n. By $cl(A)$ we denote the closure of the set $A \subset R^n$, by $co(A)$ we denote the convex hull of A and we put $\|A\| = \sup\{\|a\|; a \in A\}$.

Let $\Omega \subset R^n$ be an open set and let $V : \Omega \to R \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in R^n; V(x) < +\infty\}$.

Definition 2.1. The multifunction $\partial_F V : \Omega \to \mathcal{P}(R^n)$, defined as:

$$\partial_F V(x) = \{\alpha \in R^n, \liminf_{y\to x} \frac{V(y) - V(x) - <\alpha, y-x>}{\|y-x\|} \geq 0\} \text{ if } V(x) < +\infty$$

and $\partial_F V(x) = \emptyset$ if $V(x) = +\infty$ is called the *Fréchet subdifferential* of V.

We also put $D(\partial_F V) = \{x \in R^n; \partial_F V(x) \neq \emptyset\}$.

According to [4] the values of ∂_F are closed and convex.

Definition 2.2. Let $V : \Omega \to R \cup \{+\infty\}$ be a lower semicontinuous function. We say that V is a *ϕ-convex of order* 2 if there exists a continuous map

$\phi_V : (D(V))^2 \times R^2 \to R_+$ such that for every $x, y \in D(\partial_F V)$ and every $\alpha \in \partial_F V(x)$ we have

$$(2.1)\quad V(y) \geq V(x) + < \alpha, x - y > - \phi_V(x, y, V(x), V(y))(1 + \|\alpha\|^2)\|x - y\|^2.$$

In [4] there are several examples and properties of such maps.

For $M \subset R^n$ and $x \in M$ we recall that the contingent cone to M at x is defined by

$$K_M(x) = \{v \in R^n; \quad \liminf_{h \to 0+} \frac{1}{h} d(x + hv, M) = 0\}$$

and the second-order contingent set to M at $(x, y) \in M \times R^n$ is defined by:

$$K_M^2(x, y) = \{v \in R^n; \quad \liminf_{h \to 0+} \frac{d(x + hy + \frac{h^2}{2}, M)}{h^2/2} = 0\}.$$

Remark 2.3 ([2], [6], [10]). If $F(.,.)$ is upper semicontinuous, compact convex valued and $x(.) : [0, T] \to R^n$ is a solution to the Cauchy problem (1.1) such that $x(t) \in M$, $\forall t \in I$, then

$$(x(t), x'(t)) \in graph(K_M(.)), \quad \forall t \in [0, T].$$

In what follows, for a multifunction $F : M \subset R^n \times R^n \to \mathcal{P}(R^n)$ and for any $(x_0, y_0) \in M$ we consider the problem (1.1) under the following assumptions.

Hypothesis 2.4.

(i) $M = M_1 \times M_2 \subset R^n \times R^n$ is a locally compact set such that

$$graph(K_{M_1}(.)) \subset M.$$

(ii) F is upper semicontinuous (i.e., $\forall z \in M, \forall \epsilon > 0$ there exists $\delta > 0$ such that $\|z - z'\| < \delta$ implies $F(z') \subset F(z) + \epsilon B$) with compact values and such that

$$F(x, y) \cap K_{M_1}^2(x, y) \neq \emptyset, \quad \forall (x, y) \in M.$$

(iii) There exists a proper lower semicontinuous ϕ-convex function of order two $V : R^n \to R \cup \{+\infty\}$ such that

$$F(x, y) \subset \partial_F V(y), \quad \forall (x, y) \in M.$$

Finally, by a *viable solution* to problem (1.1) we mean an absolutely continuous function $x(.) : [0,T] \to R^n$ with an absolutely continuous derivative $x'(.)$ such that $x(0) = x_0, x'(0) = y_0$,

$$x''(t) \in F(x(t), x'(t)) \quad a.e. \ ([0,T])$$

and

$$(x(t), x'(t)) \in M, \quad \forall t \in [0,T].$$

3. The main result

In order to prove our main result we need the following lemma.

Lemma 3.1 ([9]). *Assume that Hypothesis* 2.4 *is verified and let* $M_0 \subset M$ *be a compact subset such that* $(x_0, y_0) \in M_0$. *Then, for every* $k \in N$ *there exist* $h_k^0 \in (h_0(k), \frac{1}{k}]$ *and* $u_k^0 \in R^n$ *such that*

$$x_0 + h_k^0 y_0 + \frac{(h_k^0)^2}{2} u_k^0 \in M_1,$$

$$(x_0, y_0, u_k^0) \in graph(F) + \frac{1}{k}(B \times B \times B).$$

Our main result is the following:

Theorem 3.2. *Consider* $F : M \to \mathcal{P}(R^n)$ *and* $V : R^n \to R \cup \{\infty\}$ *satisfying Hypothesis* 2.4. *Then, for every* $(x_0, y_0) \in graph(K_M(.))$ *there exist* $T > 0$ *and* $x(.) : [0,T] \to R^n$ *a viable solution to problem* (1.1).

Proof. Let $(x_0, y_0) \in graph(K_M(.))$. Since M is locally compact, there exists $r > 0$ such that $M_0 = M \cap \overline{B}_r(x_0, y_0)$ is a compact set. Moreover, by the upper semicontinuity of F and Proposition 1.1.3 in [1], the set

$$F(\overline{B}_r(x_0, y_0)) = \cup_{(x,y) \in \overline{B}_r(x_0,y_0)} F(x,y)$$

is compact, hence there exists $L > 0$ such that

$$\sup\{\|v\|;\ v \in F(x,y);\ (x,y) \in \overline{B}_r(x_0, y_0)\} \le L < +\infty.$$

Let ϕ_V be the continuous function appearing in Definition 2.2.

Since $V(.)$ is continuous on $D(V)$ (e.g. [7]), by possibly decreasing r one can assume that for all $y \in B_r(y_0) \cap D(V)$

$$|V(y) - V(y_0)| \leq 1.$$

Set

$$S := \sup\{\phi_v(y_1, y_2, z_1, z_2); y_i \in \overline{B}_r(y_0), z_i \in [V(y_0) - 1, V(y_0) + 1], i = 1, 2\},$$

$$T = \min\left\{\frac{r}{2(L+1)}, \sqrt{\frac{r}{L+1}}, \frac{r}{2(\|y_0\|+1)}\right\}.$$

By Lemma 3.1, for $(x_0, y_0) \in M_0$ there exist $h_k^0 \in (h_0(k), \frac{1}{k}]$ and $u_k^0 \in R^n$ such that

$$x_0 + h_k^0 y_0 + \frac{(h_k^0)^2}{2} u_k^0 \in M_1,$$
$$(x_0, y_0, u_k^0) \in graph(F) + \frac{1}{k}(B \times B \times B).$$

We define

$$x_k^1 := x_0 + h_k^0 y_0 + \frac{(h_k^0)^2}{2} u_k^0,$$
$$y_k^1 := y_0 + h_k^0 u_0^k.$$

According to the choice of T and if $h_k^0 < T$ we have

$$\|x_k^1 - x_0\| \leq h_k^0 \|y_0\| + \frac{(h_k^0)^2}{2} \|u_k^0\|$$
$$\leq h_k^0 \|y_0\| + \frac{(h_k^0)^2}{2}(L+1) < r,$$
$$\|y_k^1 - y_0\| \leq h_k^0 \|u_0^k\| \leq h_k^0 (L+1) < r.$$

Hence $(x_k^1, y_k^1) \in M_0$ and applying again Lemma 3.1 we find $h_k^1 \in (h_0(k), \frac{1}{k}]$ and $u_k^1 \in R^n$ such that

$$x_k^1 + h_k^1 y_k^1 + \frac{(h_k^1)^2}{2} u_k^1 \in M_1,$$
$$(x_k^1, y_k^1, u_k^1) \in graph(F) + \frac{1}{k}(B \times B \times B).$$

By recurrence, for each $k \in N$, there exist $m_k \in N^*$ and $h_k^p, x_k^p, y_k^p, u_k^p$ such that for every $p = 2, \ldots, m_k - 1$ we have:

(a) $\sum_{j=0}^{m_k-1} h_k^j \leq T < \sum_{j=0}^{m_k} h_k^j$,

(b) $x_k^p = x_k^0 + (\sum_{i=0}^{p-1} h_k^i) y_0 + \frac{1}{2}\sum_{i=0}^{p-1}(h_k^i)^2 u_k^i + \sum_{i=0}^{p-2}\sum_{j=i+1}^{p-1} h_k^i h_k^j u_k^i$,

(c) $y_k^p = y_k^0 + \sum_{i=0}^{p-1} h_k^i u_k^i$,

(d) $(x_k^p, y_k^p) \in M_0$,

(e) $(x_k^p, y_k^p, u_k^p) \in graph(F) + \frac{1}{k}(B \times B \times B)$.

Assume that $h_k^q, x_k^q, y_k^q, u_k^q$ have been constructed for $q \leq p$ satisfying (a) – (e) and we construct $h_k^{p+1}, x_k^{p+1}, y_k^{p+1}, u_k^{p+1}$.

By Lemma 3.1, since $(x_k^p, y_k^p) \in M_0$ there exist $h_k^p \in (h_0(k), \frac{1}{k}]$ and $u_k^p \in R^n$ such that

$$x_k^p + h_k^p y_k^p + \frac{(h_k^p)^2}{2} u_k^p \in M_1,$$
$$(x_k^p, y_k^p, u_k^p) \in graph(F) + \frac{1}{k}(B \times B \times B).$$

If $h_k^0 + h_k^1 + \ldots + h_k^p \geq T$, then we set $m_k = p$. If $h_k^0 + h_k^1 + \ldots + h_k^p < T$ we define

$$x_k^{p+1} := x_k^p + h_k^p y_k^p + \frac{(h_k^p)^2}{2} u_k^p,$$
$$y_k^{p+1} := y_k^p + h_k^p u_p^k.$$

From (b) and (c) it follows

$$\begin{aligned} x_k^{p+1} &= x_k^p + h_k^p y_k^p + \frac{(h_k^p)^2}{2} u_k^p \\ &= x_k^0 + \Big(\sum_{i=0}^{p} h_k^i\Big) y_0 + \frac{1}{2}\sum_{i=0}^{p}(h_k^i)^2 u_k^i + \sum_{i=0}^{p-1}\sum_{j=i+1}^{p} h_k^i h_k^j u_k^i, \\ y_k^{p+1} &:= y_k^p + h_k^p u_p^k = y_k^0 + \sum_{i=0}^{p} h_k^i u_k^i. \end{aligned}$$

We deduce that

$$\|x_k^{p+1} - x_0\| \leq \Big(\sum_{i=0}^{p} h_k^i\Big)\|y_0\| + \frac{L+1}{2}\Big(\sum_{i=0}^{p} h_k^i\Big)^2 < r,$$
$$\|y_k^{p+1} - y_0\| \leq \sum_{i=0}^{p} h_k^i \|u_0^i\| \leq (L+1)\Big(\sum_{i=0}^{p} h_k^i\Big) < r,$$

hence $(x_k^{p+1}, y_k^{p+1}) \in M_0$.

Let us note that this iterative process is finite because $h_k^p \in (h_0(k), \frac{1}{k}]$, implies the existence of $m_k \in N$ such that

$$h_k^0 + h_k^1 + \ldots + h_k^{m_k-1} \leq T < h_k^0 + h_k^1 + \ldots + h_k^{m_k}.$$

By (e), for every $k \in N$ and any $p \in \{0, 1, \ldots, m_k\}$ there exists $(a_k^p, b_k^p, v_k^p) \in graph(F)$ such that

$$\|x_k^p - a_k^p\| < \frac{1}{k}, \|y_k^p - b_k^p\| < \frac{1}{k}, \|u_k^p - v_k^p\| < \frac{1}{k}.$$

It follows that

$$\begin{aligned}
\|x_k^p\| &\leq \|x_k^p - x_0\| + \|x_0\| \leq 1 + \|x_0\|, \\
\|y_k^p\| &\leq \|y_k^p - x_0\| + \|y_0\| \leq 1 + \|y_0\|, \\
\|u_k^p\| &\leq \|u_k^p - v_k^p\| + \|v_k^p\| \leq 1 + L.
\end{aligned}$$

For every $k \in N^*$, by the upper semicontinuity of $F(.)$ in $(x_k^{p-1}, y_k^{p-1}) \in M_0$, there exists $\delta = \delta(k)$ such that $\|x - x_k^{p-1}\| < \delta$, $\|y - y_k^{p-1}\| < \delta$ implies $F(x,y) \subset F(x_k^{p-1}, y_k^{p-1}) + \frac{1}{k}B$. One may take $\delta \geq \frac{1}{k}$, hence $\|a_k^{p-1} - x_k^{p-1}\| < \delta$, $\|b_k^{p-1} - y_k^{p-1}\| < \delta$ and then

$$\begin{aligned}
&F(a_k^{p-1}, b_k^{p-1}) \subset F(x_k^{p-1}, y_k^{p-1}) + \frac{1}{k}B \\
&F(a_k^{p-1}, b_k^{p-1}) + \frac{1}{k}B \subset F(x_k^{p-1}, y_k^{p-1}) + \frac{2}{k}B \qquad (3.1)\\
&u_k^{p-1} \in F(x_k^{p-1}, y_k^{p-1}) + \frac{2}{k}B
\end{aligned}$$

We define $t_k^p = h_k^0 + h_k^1 + \ldots + h_k^{p-1}, t_k^0 = 0$. Obviously, $\forall\, k \in N$, $p \in \{1, \ldots, m_k\}$

$$t_k^p - t_k^{p-1} < \frac{1}{k} \quad \text{and} \quad t_k^{m_k} \leq T < t_k^{m_k+1}.$$

For $k \geq 1, p \in \{1, \ldots, m_k\}$ we define $I_k^p := [t_k^{p-1}, t_k^p]$ and

$$x_k(t) = x_k^{p-1} + (t - t_k^{p-1})y_k^{p-1} + \frac{1}{2}(t - t_k^{p-1})^2 u_k^{p-1}, \quad t \in I_k^p.$$

Hence

$$x_k'(t) = y_k^{p-1} + (t - t_k^{p-1})u_k^{p-1}, \quad t \in I_k^p.$$
$$x_k''(t) = u_k^{p-1}, \quad t \in I_k^p,$$

thus, for all $t \in [0,T]$

$$\begin{aligned} &\|x_k''(t)\| \le \|u_k^{p-1}\| < L+1, \\ &\|x_k'(t)\| \le \|y_k^{p-1}\| + (t - t_k^p)\|u_k^p\| < \|y_0\| + L + 2, \qquad (3.2) \\ &\|x_k(t)\| \le \|x_0\| + \|y_0\| + L + 3. \end{aligned}$$

On the other hand,

$$\|x_k(t) - x_k^p\| \le \frac{1}{k}(\|y_0\| + L + 2), \quad \forall t \in [0,T],$$
$$\|x_k'(t) - y_k^p\| \le \frac{1}{k}(L+1), \quad \forall t \in [0,T],$$

hence from (e) we find that

$$(x_k(t), x_k'(t), x_k''(t)) \in graphF + \epsilon(k)(B \times B \times B),$$

where $\epsilon(k) \to 0$ as $k \to \infty$.

Then by (3.2) we find that $x_k''(.)$ is bounded in $L^2([0,T],R^n)$, $x_k(.)$, $x_k'(.)$ are bounded in $C([0,T],R^n)$, and equi-Lipschitzean; therefore, applying Theorem 0.3.4 in [1] we obtain the existence of a subsequence (again denoted by $x_k(.)$) and an absolutely continuous function $x(.) : [0,T] \to R^n$ such that

$$\begin{aligned} &x_k(.) \quad \text{converges uniformly to} \quad x(.), \\ &x_k'(.) \quad \text{converges uniformly to} \quad x'(.), \\ &x_k''(.) \quad \text{converges weakly in } L^2([0,T],R^n) \text{ to} \quad x''(.). \end{aligned}$$

Taking into account Hypothesis 2.4 and Theorem 1.4.1 in [1] we obtain that

$$x''(t) \in coF(x(t), x'(t)) \subset \partial_F V(x'(t)) \quad a.e.\ ([0,T]).$$

Since the mapping $x(.)$ is absolutely continuous and $x''(t) \in \partial_F V(x'(t))$ almost everywhere on $[0,T]$, we apply Theorem 2.2 in [4] and we deduce that there exists $T_1 > 0$ such that the mapping $t \to V(x'(t))$ is absolutely continuous on $[0, \min\{T, T_1\}]$ and

$$(V(x'(t)))' = < x''(t), x''(t) > \quad a.e.\ [0, \min\{T, T_1\}]. \tag{3.3}$$

Without loss of generality we may assume that $T = \min\{T, T_1\}$.
Using the properties of the mapping $V(.)$, the definition of S and (3.3) we find $b_k^{p-1} \in B$ such that

$$u_k^{p-1} - \frac{2}{k} b_k^{p-1} \in F(x_k(t), x'_k(t)) \subset \partial_F V(x'_k(t))$$

and such that for every $t \in I_k^{p-1}$ we have

$$\begin{aligned} & V(x'_k(t_k^p)) - V(x'_k(t_k^{p-1})) \geq \langle u_k^{p-1} - \frac{2}{k}\, b_k^{p-1}, \int_{t_k^{p-1}}^{t_k^p} x''_k(t) dt \rangle \\ & - \phi_V(x'_k(t_k^p), x'_k(t_k^{p-1}), V(x'_k(t_k^p)), V(x'_k(t_k^{p-1})) \\ & . \Big(1 + \|u_k^{p-1} - \frac{2}{k} b_k^{p-1}\|^2\Big) \|x'_k(t_k^p) - x'_k(t_k^{p-1})\|^2. \end{aligned}$$

If $T \in (t_k^{m_k - 1}, t_k^{m_k}]$, we obtain

$$\begin{aligned} & V(x'_k(T)) - V(x'_k(t_k^{m_k-1})) \geq \langle u_k^{m_k-1} - \frac{2}{k}\, b_k^{m_k-1}, \int_{t_k^{m_k-1}}^{T} x''_k(t) dt \rangle \\ & - \phi_V(x'_k(T), x'_k(t_k^{m_k-1}), V(x'_k(T)), V(x'_k(t_k^{m_k-1})) \\ & . \Big(1 + \|u_k^{m_k-1} - \frac{2}{k} b_k^{m_k-1}\|^2\Big) \|x'_k(T) - x'_k(t_k^{m_k-1})\|^2. \end{aligned}$$

By adding on p the last inequalities we get

$$V(x'_k(T)) - V(y_0) \geq \int_0^T \|x''_k(t)\|^2 dt + a(k) + b(k),$$

where

$$\begin{aligned} a(k) &= -\sum_{p=1}^{m_k} \frac{2}{k}\, \langle b_k^{p-1}, (t_k^p - t_k^{p-1}) u_k^{p-1} \rangle \\ b(k) &= -\sum_{p=1}^{m_k} \phi_V(x'_k(t_k^p), x'_k(t_k^{p-1})_k, V(x'_k(t_k^p)), V(x'_k(t_k^{p-1})) \\ & . \Big(1 + \|u_k^{p-1} - \frac{2}{k} b_k^{p-1}\|^2\Big) \|x'_k(t_k^p) - x'_k(t_k^{p-1})\|^2. \end{aligned}$$

On the other hand, one has

$$\begin{aligned}|a(k)| &\leq \frac{2}{k}\sum_{p=1}^{m_k}\|b_k^{p-1}\|.\|\int_{t_k^{p-1}}^{t_k^p} x_k''(t)dt\| \\ &\leq \frac{2}{k}\int_0^T \|x_k''(s)\|ds \leq \frac{1}{k}\,2T(L+1)\end{aligned}$$

and

$$\begin{aligned}&|b(k)| \leq \sum_{p=1}^{m_k} S(1+L^2)\sum_{p=1}^{m_k}\|\int_{t_k^{p-1}}^{t_k^p} x_k''(t)dt\|^2 \\ &\leq S(1+L^2)\sum_{p=1}^{m_k}\frac{1}{k}\int_{t_k^{p-1}}^{t_k^p}\|x_k''(t)\|^2dt \leq S(1+L^2)\sum_{p=1}^{m_k}\frac{1}{k}\int_0^T \|x_k''(t)\|^2dt \\ &\leq \frac{1}{k}S(1+L^2)T(L+1)^2\end{aligned}$$

and thus $\lim_{k\to\infty} a(k) = \lim_{k\to\infty} b(k) = 0$. We deduce that

$$V(x_k'(T)) - V(y_0) \geq \limsup_{k\to\infty}\int_0^T \|x_k''(t)\|^2dt\,. \tag{3.4}$$

Using (3.3) we infer that

$$\limsup_{k\to\infty}\int_0^T \|x_k''(t)\|^2dt \leq \int_0^T \|x''(t)\|^2dt$$

and, since $\{x_k''(.)\}_k$ converges weakly in $L^2([0,T],R^n)$ to $x''(.)$, by the lower semicontinuity of the norm in $L^2([0,T],R^n)$ (e.g. Proposition III.30 in [3]) we obtain that

$$\lim_{k\to\infty}\int_0^T \|x_k''(t)\|^2dt = \int_0^T \|x''(t)\|^2dt$$

i.e., $x_k''(.)$ converges strongly in $L^2([0,T],R^n)$. Hence, there exists a subsequence (still denoted) $x_k''(.)$ that converges pointwise to $x''(.)$.

On the other hand, from Hypothesis 2.4 $graph(F)$ is closed and it follows that

$$\lim_{k\to\infty} d((x_k(t),x_k'(t),x_k''(t)),graph(F)) = 0.$$

Thus

$$x''(t) \in F(x(t), x'(t)) \quad a.e. \ \ ([0,T]).$$

It remains to prove that $(x(t), x'(t)) \in M$, $\forall t \in [0,T]$. By the definition of $x_k(.)$ and (3.2) we have

$$\|x_k(t) - x_k^p\| < \frac{1}{k}\,(\|y_0\| + L + 2),$$
$$\|x_k'(t) - y_k^p\| < \frac{1}{k}\,(L+1)$$

and thus $\lim_{k\to\infty} d((x_k(t), x_k'(t)), (x_k^p, y_k^p)) = 0$. But $(x_k^p, y_k^p) \in M_0$, $\forall n \in N^*$, so we have that

$$d((x(t), x'(t)), M_0) \leq d((x(t), x'(t)), (x_k(t), x_k'(t)))$$
$$+\, d((x_k(t), x_k'(t)), (x_k^p, y_k^p)) + d((x_k^p, y_k^p), M_0)$$

and letting $k \to \infty$ we find that

$$d((x(t), x'(t)), M_0) = 0.$$

Finally, using the fact that M_0 is closed we get that $(x(t), x'(t)) \in M_0$, $\forall t \in [0,T]$ which completes the proof.

Remark 3.3. If $V(.) : R^n \to R$ is a proper lower semicontinuous convex function, then (e.g. [7]) $\partial_F V(x) = \partial V(x)$, where $\partial V(.)$ is the subdifferential in the sense of convex analysis of $V(.)$, and Theorem 3.2 yields the result in [9].

References

[1] J.P. Aubin and A. Cellina, Differential Inclusions, Springer, Berlin 1984.

[2] A. Auslender and J. Mechler, *Second order viability problems for differential inclusions*, J. Math. Anal. Appl. **181** (1984), 205–218.

[3] H. Brezis, *Analyse fonctionelle, théorie et applications*, Masson, Paris 1983.

[4] T. Cardinali, G. Colombo, F. Papalini and M. Tosques, *On a class of evolution equations without convexity*, Nonlinear Anal. **28** (1996), 217–234.

[5] A. Cernea, *Existence of viable solutions for a class of nonconvex differential inclusions*, J. Convex Anal., submitted.

[6] B. Cornet and B. Haddad, *Théoreme de viabilité pour inclusions differentielles du second order*, Israel J. Math. **57** (1987), 225–238.

[7] M. Degiovanni, A. Marino and M. Tosques, *Evolution equations with lack of convexity*, Nonlinear Anal. **9** (1995), 1401–1443.

[8] T.X.D. Ha and M. Marques, *Nonconvex second order differential inclusions with memory*, Set-valued Anal. **5** (1995), 71–86.

[9] V. Lupulescu, *Existence of solutions to a class of second order differential inclusions*, Cadernos de Matematica, Aveiro Univ., CM01/I-11.

[10] L. Marco and J.A. Murillo, *Viability theorems for higher-order differential inclusions*, Set-valued Anal. **6** (1998), 21–37.

[11] M. Tosques, *Quasi-autonomus parabolic evolution equations associated with a class of non linear operators*, Ricerche Mat. **38** (1989), 63–92.

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