SET-VALUED STOCHASTIC INTEGRALS AND STOCHASTIC INCLUSIONS IN A PLANE

WŁADYSŁAW SOSULSKI

Institute of Mathematics
University of Zielona Góra
Podgórna 50, 65–246 Zielona Góra, Poland

Abstract

We present the concepts of set-valued stochastic integrals in a plane and prove the existence of a solution to stochastic integral inclusions of the form

\[ z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v})dudv + \int_0^s \int_0^t G_{u,v}(z_{u,v})dw_{u,v}. \]

Keywords: stochastic inclusions in the plane, set-valued random field, two–parameter stochastic process, weak compactness.

2000 Mathematics Subject Classification: 93E03, 93C30.

1. Introduction

In this paper, we shall consider a stochastic inclusion in a plane of the form:

\[ z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v})dudv + \int_0^s \int_{\mathbb{D}} G_{u,v}(z_{u,v})dw_{u,v}. \]

The results of the paper generalize some problems dealing with stochastic integral equations in a plane of the form:

\[ x_{s,t} = x + \int_0^s \int_0^t a(u,v,x_{u,v})du dv + \int_0^s \int_0^t b(u,v,x_{u,v})dudv. \]
that have been investigated by Tudor [8], Panomarenco [5] and others. The first integral in equation (2) is a stochastic integral in a plane considered with respect to the Wiener-Yeh process. It was defined by Cairoli [1], Panomarenco [5], Tsarenco [7]. Its generalization with respect to two parameters martingale is given by Cairoli and Walsh in [2].

In this paper, we introduce the stochastic integral in a plane for set-valued mappings taking values from space Comp ($R^n$) of all nonempty closed subsets of $n$-dimensional Euclidean space $R^n$ with respect to the Wiener-Yeh process and we establish some of its properties. The space Comp ($R^n$) is considered with the Hausdorff metric $h$ defined in the usual way, i.e., $h(A, B) = \max\{h(A, B), h(B, A)\}$, for $A, B \in \text{Comp} (R^n)$ where $h(A, B) = \sup\{\text{dist} (a, B) : a \in A\}$ and $h(B, A) = \sup\{\text{dist} (b, A) : b \in B\}$.

We assume, as given, a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{s,t})_{s,t\geq 0}, P)$ where a filtration $(\mathcal{F}_{s,t})_{s,t\geq 0}$ is assumed to satisfy: $\mathcal{F}_{s,t} \subset \mathcal{F}_{u,v}$ for $(s, t) \leq (u, v)$ i.e., for $s \leq u$ and $t \leq v$. We define $\mathcal{F}_{s,t}^1 = \mathcal{F}_{s,\infty} = \bigcup_v \mathcal{F}_{s,v}$ and $\mathcal{F}_{s,t}^2 = \mathcal{F}_{\infty,t} = \bigcup_u \mathcal{F}_{u,t}$. We put $R^2_+ = [0, \infty) \times [0, \infty)$ and $\beta^2_+$ denotes the Borel $\sigma$-algebra on $R^2_+$. We consider set-valued two parameter processes (set-valued random fields) $(F_{s,t})_{s,t\geq 0}$ and $(G_{s,t})_{s,t\geq 0}$ that are assumed to be nonanticipative and such that

\[
\int_0^\infty \int_0^\infty \|F_{s,t}\|^2 dsdt < \infty \text{ and }
\]

\[
\int_0^\infty \int_0^\infty \|G_{s,t}\|^2 dsdt < \infty \text{ a.s.,}
\]

where $\|A\| := \sup\{|a| : a \in A\}$ for a nonempty set $A \subset R^n$. Next, using the method which has been used by Kisielewicz in [4], we investigate the existence of solutions to stochastic inclusion (1).

### 2. Basic definitions and notations

Throughout the paper, we shall assume that filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{s,t})_{s,t\geq 0}, P)$ satisfies the following conditions:

(i) if $(s, t) \leq (s', t')$ then $\mathcal{F}_{s,t} \subset \mathcal{F}_{s',t'}$;

(ii) $\mathcal{F}_{0,0}$ contains all null sets of $\mathcal{F}$;

(iii) for each $(s, t), \mathcal{F}_{s,t} = \bigcap_{(u,v)>(s,t)} \mathcal{F}_{u,v}$;

(iv) for each $(s, t), \mathcal{F}^1_{s,t}$ and $\mathcal{F}^2_{s,t}$ are conditionally independent given $\mathcal{F}_{s,t}$. 
We write \((s, t) \leq (s', t')\) if \(s \leq s'\) and \(t \leq t'\). As usual, we shall consider a set \(R^2_+ \times \Omega\) as a measurable space with the product \(\sigma\)-algebra \(B^2_+ \otimes \mathcal{F}\).

An \(n\)-dimensional two parameter stochastic process \(z\) (random field) understood as a function \(z : R^2_+ \times \Omega \rightarrow R^n\) with \(\mathcal{F}\)-measurable sections \(z_{s,t}\), each \(s, t \geq 0\), is denoted by \((z_{s,t})_{s,t \geq 0}\). It is measurable if \(z\) is \(B^2_+ \otimes \Omega\)-measurable. The process \((z_{s,t})_{s,t \geq 0}\) is \(\mathcal{F}_{s,t}\)-adapted or adapted if \(z_{s,t}\) is \(\mathcal{F}_{s,t}\)-measurable for \(s, t \geq 0\). Every measurable and adapted process is called nonanticipative. In what follows, the Banach space \((L^2(R^2_+ \times \Omega, \beta^2_+ \otimes \mathcal{F}, dsdt \otimes P), \| \cdot \|_2)\), where \(dsdt\) denotes Lebesgue measure on \(R^2\), will be denoted by \(L^2_+\). Similarly, the Banach spaces \((L^2(\Omega, \mathcal{F}, P, R^n), \| \cdot \|)\) and \((L^2(\Omega, \mathcal{F}_{s,t}, P, R^n), \| \cdot \|)\) are denoted by \(L^2(\mathcal{F})\) and \(L^2(\mathcal{F}_{s,t})\), respectively.

Let \(\mathcal{M}^2(\mathcal{F}_{s,t})\) denote the family of all (equivalence classes of) \(n\)-dimensional nonanticipative processes \((f_{s,t})_{s,t \geq 0}\) such that \(\int_0^\infty \int_0^\infty |f_{s,t}|^2 dsdt < \infty\), a.s.

We shall also consider a subspace \(\mathcal{L}^2(\mathcal{F}_{s,t})\) of \(\mathcal{M}^2(\mathcal{F}_{s,t})\) defined by

\[
\mathcal{L}^2(\mathcal{F}_{s,t}) = \left\{ (f_{s,t})_{s,t \geq 0} \in \mathcal{M}^2(\mathcal{F}_{s,t}) : E \int_0^\infty \int_0^\infty |f_{s,t}|^2 dsdt < \infty \right\}.
\]

It is a closed subset of the Banach space \(L^2_n\). Finally, by \(M_n(\mathcal{F}_{s,t})\) we denote a space of all (equivalence classes of) \(n\)-dimensional \(\mathcal{F}_{s,t}\)-measurable mappings. Throughout, by \((w_{s,t})_{s,t \geq 0}\) we mean two parameter \(\mathcal{F}_{s,t}\)-Brownian motion, i.e., a continuous Gaussian process such that \(E(w_{s,t}) = 0\) and \(E(w_{s,t} \cdot w_{s',t'}) = \min(s, s') \cdot \min(t, t')\) for every \(s, s', t, t'\). Given \(g \in \mathcal{M}^2(\mathcal{F}_{s,t})\), by \((\int_0^s \int_0^t g_{u,v} dw_{u,v})_{s,t \geq 0}\) we denote its stochastic integral with respect to an \(\mathcal{F}_{s,t}\)-Brownian motion \((w_{s,t})_{s,t \geq 0}\). Let us denote by \(D\) the family of all \(n\)-dimensional \(\mathcal{F}_{s,t}\)-adapted continuous processes \((z_{s,t})_{s,t \geq 0}\) such that \(E \sup_{s,t \geq 0} |z_{s,t}|^2 < \infty\). The space \(D\) is considered as a normed space with the norm \(|\cdot|\) defined by \(|x| = \|\sup_{s,t \geq 0} |x_{s,t}|\|_{L^2}\) for \(x = (x_{s,t})_{s,t \in D}\), where \(\| \cdot \|_{L^2}\) is a norm of \(L^2(\Omega, \mathcal{F}, P, R)\). It can be verified that \((D, |\cdot|)\) is a Banach space. In what follows, we shall deal with upper and lower semicontinuous valued mappings. Recall that a set valued mapping \(\mathcal{R}\) with nonempty values in a topological space \((Y, T_Y)\) is said to be upper (lower) semicontinuous [u.s.c. (l.s.c.)] on a topological space \((X, T_X)\) if \(\mathcal{R}^{-}(C) := \{x \in X : \mathcal{R}(x) \cap C \neq \emptyset\}\) (resp. \(\mathcal{R}^{+}(C) := \{x \in X : \mathcal{R}(x) \subset C\}\)) is a closed subset of \(X\) for every closed set \(C \subset Y\). In particular, for \(\mathcal{R}\) defined on a metric space \((X, d)\) with values in \(\text{Comp}(R^n)\), it is equivalent (see [4]) to \(\lim_{n \rightarrow \infty} \mathcal{R}(\mathcal{R}(x_n), \mathcal{R}(x)) = 0\) (\(\lim_{n \rightarrow \infty} \mathcal{R}(\mathcal{R}(x), \mathcal{R}(x_n)) = 0\)) for every \(x \in X\).
and every sequence \((x_n)\) of \(X\) converging to \(x\). In what follows, we shall need the following well-known (see [4]) fixed point and continuous selection theorems.

**Theorem 1** (Schauder, Tikhonov). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(f\) a continuous mapping of \(K\) into itself. Then \(f\) has fixed point in \(K\).

**Theorem 2** (Kakutani, Fan). Let \((X, \mathcal{F}_X)\) be a locally convex topological Hausdorff space, \(K\) a nonempty compact convex subset of \(X\) and \(CCl(K)\) a family of all nonempty closed convex subsets of \(K\). If \(K : K \to CCl(K)\) is u.s.c. on \(K\), then there exists \(x \in K\) such that \(x \in K(x)\).

**Theorem 3** (Michael). Let \((X, \mathcal{F}_X)\) be a paracompact space and let \(K\) be a set-valued mapping from \(X\) to a Banach space \((Y, \|\cdot\|)\) whose values are closed and convex. Suppose, further \(K\) is l.s.c. on \(X\). Then there is a continuous function \(f : X \to Y\) such that \(f(x) \in K(x)\), for each \(x \in X\).

3. Set-valued stochastic integrals in the plane

Given measure space \((X, \beta, m)\), a set-valued function \(R : X \to \text{Comp} (\mathbb{R}^n)\) is said to be \(\beta\)-measurable if \(R^{-}(C) \in \beta\) for every closed \(C \subset \mathbb{R}^n\). For such a multifunction we define integrals subtrajectory (see [6]) as a set

\[
S^p(R) = \{g \in L^p(X, \beta, m, \mathbb{R}^n) : g(x) \in R(x) \text{ a.e.} \}.
\]

It is clear that for nonemptiness of \(S^p(R)\) we have to assume more than \(\beta\)-measurability of \(R\). In what follows, we shall assume that \(\beta\)-measurable set-valued function \(R : X \to \text{Comp} (\mathbb{R}^n)\) is \(p\)-integrably bounded, \(p \geq 1\), i.e., that a real-valued mapping \(X \ni x \to \|R(x)\| \in \mathbb{R}_+\) belongs to \(L^p(X, \beta, m, \mathbb{R}_+)\) where \(\|A\| := \sup\{|a| : a \in A\}\) for \(A \in \text{Comp}(\mathbb{R}^n)\). It can be verified that a \(\beta\)-measurable set-valued mapping \(R : X \to \text{Comp} (\mathbb{R}^n)\) is \(p\)-integrably bounded \(p \geq 1\), if and only if \(S^p(R)\) is nonempty and bounded in \(L^p(X, \beta, m, \mathbb{R}^n)\) (see [4]). Finally, it is easy to see that \(S^p(R)\) is decomposable, i.e., such that \(1|A|f_1 + 1|X\setminus A||f_2 \in S^p(R)\) for \(A \in \beta\) and \(f_1, f_2 \in S^p(R)\). We have the following general result dealing with properties of subtrajectory integrals (see [4], [5]).

**Proposition 4.** Let \(R : X \to \text{Comp} (\mathbb{R}^n)\) be \(\beta\)-measurable and \(p\)-integrably bounded, \(p \geq 1\). The set \(S^p(R)\) is a nonempty bounded and closed subset
of $L^p(X, \beta, m, R^n)$. Moreover, if $R$ takes on convex values, then $S^p(R)$ is convex and weakly compact in $L^p(X, \beta, m, R^n)$.

Let $G = (G_{s,t})_{s,t \geq 0}$ be a set-valued two parameter stochastic process with values in $\text{Comp}(R^n)$, i.e., a family of $\mathcal{F}$-measurable set-valued mappings $G_{s,t} : \Omega \to \text{Comp}(R^n), s, t \geq 0$. We call $G$ measurable if it is $\beta_2^2 \otimes \mathcal{F}$-measurable. Similarly, $G$ is said to be $\mathcal{F}_{s,t}$-adapted or adapted if $G_{s,t}$ is $\mathcal{F}_{s,t}$-measurable for each $s, t \geq 0$. A measurable and adapted set-valued two parameter stochastic process is called nonanticipative. Denote by $M^2_{s\to\upsilon}(\mathcal{F}_{s,t})$ a family of all nonanticipative set-valued processes $G = (G_{s,t})_{s,t \geq 0}$ such that

$$\int_0^\infty \int_0^\infty \|G_{s,t}\|^2 \, ds \, dt < \infty, \text{ a.s.}$$

Immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [4]) it follows that for every $F, G \in M^2_{s\to\upsilon}(\mathcal{F}_{s,t})$ their subtrajectory integrals

$$S^2(F) = f \in \{M^2(\mathcal{F}_{s,t} : f_{s,t}(\omega) \in F_{s,t}(\omega), ds \, dt \times P - \text{a.e.}\}$$

and

$$S^2(G) = \{g \in M^2(\mathcal{F}_{s,t}) : g_{s,t}(\omega) \in G_{s,t}(\omega), ds \, dt \times P - \text{a.e.}\},$$

are nonempty. Indeed, let $\Sigma = \{Z \in B_2^2 \otimes \mathcal{F} : Z_{s,t} \in \mathcal{F}_{s,t}, \text{ each } s, t \geq 0\}$, where $Z_{s,t}$ denotes a section of $Z$ determined by $s, t \geq 0$. It is a $\sigma$-algebra on $R_+^2 \times \Omega$ and function $f : R_+^2 \times \Omega \to R^n$ (a multifunction $F : R_+^2 \times \Omega \to \text{Comp}(R^n)$) is nonanticipative if and only if it is $\Sigma$-measurable. Therefore, by the Kuratowski and Ryll-Nardzewski measurable selection theorem, every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in M^2_{s\to\upsilon}(\mathcal{F}_{s,t})$ such selectors belong to $M^2(\mathcal{F}_{s,t})$. Finally, denote

$$\mathcal{L}^2_{s\to\upsilon}(\mathcal{F}_{s,t}) = \left\{ G \in M_{s\to\upsilon}(\mathcal{F}_{s,t} : E \int_0^\infty \int_0^\infty \|G_{s,t}\|^2 \, ds \, dt < \infty \right\}.$$

Given set-valued two parameter processes

$$F = (F_{s,t})_{s,t \geq 0} \in M^2_{s\to\upsilon}(\mathcal{F}_{s,t}) \text{ and } G = (G_{s,t})_{s,t \geq 0} \in M^2_{s\to\upsilon}(\mathcal{F}_{s,t})$$

by their stochastic integrals we mean families $(\int_0^s \int_0^t F_{u,v} \, du \, dv)_{s,t \geq 0}$ and $(\int_0^s \int_0^t G_{u,v} \, du \, dv)_{s,t \geq 0}$ of subsets of $M_n(\mathcal{F}_{s,t})$, defined by

$$\int_0^s \int_0^t F_{u,v} \, du \, dv := \left\{ \int_0^s \int_0^t f_{u,v} \, du \, dv : f \in S^2(F) \right\}.$$
and
\[ \int_0^s \int_0^t G_{u,v} \, dw_{u,v} := \left\{ \int_0^s \int_0^t g_{u,v} \, dw_{u,v} : g \in S^2(G) \right\}. \]

Immediately, from the above definitions the following simple results follow.

**Theorem 5.** Let \( F, G \in \mathcal{M}^{s}_{a-v}(\mathcal{F}_{s,t}) \). Then

(i) \( \int_0^s \int_0^t F_{u,v} \, du \, dv \) and \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) are nonempty subsets of \( M_n(\mathcal{F}_{s,t}) \), each \( s, t \geq 0 \). They are convex if \( F \) and \( G \) take on convex values.

(ii) If \( G \in L^2_{a-v}(\mathcal{F}_{s,t}) \), then \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) is a nonempty subset of \( L^2(\mathcal{F}_{s,t}) \), each \( s \geq 0, t \geq 0 \),

(iii) If \( F, G \in L^2_{a-v}(\mathcal{F}_{s,t}) \) take on convex values then \( \int_0^s \int_0^t F_{u,v} \, du \, dv \) and \( \int_0^s \int_0^t G_{u,v} \, dw_{u,v} \) are nonempty convex and weakly compact subsets of \( L^2(\mathcal{F}_{s,t}) \), each \( s \geq 0, t \geq 0 \).

**Proof.** It is clear that (i) follows immediately from the definitions of set-valued stochastic integrals. To verify (ii) let us observe that the operator \( I_{s,t} \) defined by the formula \( I_{s,t}g = \int_0^s \int_0^t g_{u,v} \, dw_{u,v} \) is a linear continuous mapping from \( L^2(\mathcal{F}_{s,t}) \) into \( L^2(\mathcal{F}_{s,t}) \). By the properties of stochastic integrals in the plane (see [6]) one has \( E|I_{s,t}|^2 = \|g\|^2 \), each \( s, t \geq 0 \) and \( g \in L^2(\mathcal{F}_{s,t}) \), where \( \| \cdot \|_2 \) denotes the norm of \( L^2(\mathcal{F}_{s,t}) \). Therefore, \( I_{s,t} \) maps closed subsets of \( L^2(\mathcal{F}_{s,t}) \) into closed subsets of \( L^2(\mathcal{F}_{s,t}) \). Thus (ii) is satisfied. To verify (iii), it suffices only to observe that for every \( s, t \geq 0 \), all sets \( S^2(F) \) and \( S^2(G) \) are convex and weakly compact in \( L^2(\mathcal{F}_{s,t}) \). Hence, \( J_{s,t}(S^2(F)) \) and \( J_{s,t}(S^2(G)) \) (where \( J_{s,t} \) is defined by \( J_{s,t}f = \int_0^s \int_0^t f_{u,v} \, du \, dv \)) are convex and weakly compact subsets of \( L^2(\mathcal{F}_{s,t}) \) because \( J_{s,t} \) and \( I_{s,t} \) are linear and continuous for fixed \( s, t \geq 0 \). Then (iii) also holds.

\[ \square \]

**4. Stochastic inclusions in the plane**

Let \( F = \{(F_{s,t}(z))_{s,t \geq 0} : z \in R^n\} \) and \( G = \{(G_{s,t}(z))_{s,t \geq 0} : z \in R^n\} \). Assume \( F \) and \( G \) are such that \( (F_{s,t}(z))_{s,t \geq 0} \in \mathcal{M}^2_{a-v}(\mathcal{F}_{s,t}) \) and \( (G_{s,t}(z))_{s,t \geq 0} \in \mathcal{M}^2_{a-v}(\mathcal{F}_{s,t}) \) for each \( z \in R^n \). By a stochastic inclusion in the plane we mean the relation

\[ z_{s,t} \in z_{0,0} + \int_0^s \int_0^t F_{u,v}(x_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}. \]
Every two parameter stochastic process \((z_{s,t})_{s,t \geq 0} \in D\) such that \(F_{s,t}(z_{s,t}) \in \mathcal{M}_{s-v}(\mathcal{F})\) and \(G_{s,t}(z_{s,t}) \in \mathcal{M}_{s-v}(\mathcal{F})\) satisfying a.s. the relation (3) is said to be a global solution to (3). Suppose \(F\) and \(G\) satisfy the following conditions \((C_1)\)

(i) \(F = \{(F_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n\}\) and \(G = \{(G_{s,t}(z))_{s,t \geq 0} : z \in \mathbb{R}^n\}\) are such that mappings \(R^2_+ \times \Omega \times \mathbb{R}^n \ni (s,t,w,z) \rightarrow F_{s,t}(z) \in \text{Cl}(\mathbb{R}^n)\) and \(R^2_+ \times \Omega \times \mathbb{R}^n \ni (s,t,w,z) \rightarrow G_{s,t}(z) \in \text{Cl}(\mathbb{R}^n)\) are \(\beta^2_+ \otimes \Omega \otimes \beta^n\) measurable.

(ii) \((F_{s,t}(z))_{s,t \geq 0}\) and \((G_{s,t}(z))_{s,t \geq 0}\) are uniformly square-integrably bounded, i.e.,

\[
\sup_{z \in \mathbb{R}^n} \|F_{s,t}(z)\| \in L^2_1 \quad \text{and} \quad \sup_{z \in \mathbb{R}^n} \|G_{s,t}(z)\| \in L^2_1.
\]

Now define a linear continuous mapping \(\Phi\) on \(L^2_1 \times L^2_1\) taking \(\Phi(f,g) = (I_{s,t}f + J_{s,t}g)_{s,t \geq 0}\) for \((f,g) \in L^2_1 \times L^2_1\). It is clear that \(\Phi\) maps \(L^2_1 \times L^2_1\) into \(D\). Let \(\varphi = \varphi_{s,t} \in D\) be given. For \(F\) and \(G\) satisfying \((C_1)\) define a set-valued mapping \(K\) on \(D\) by setting

\[
K(z) = \varphi + \Phi(S^2 F_{s,t}(z))_{s,t \geq 0} \times S^2(G_{s,t}(z))_{s,t \geq 0}
\]

for \(z = (z_{s,t})_{s,t \geq 0} \in D\).

Let \(F\) and \(G\) satisfy conditions \((C_1)\) and the following condition \((C_2)\) : set-valued functions

\[
D \ni z \rightarrow (F_{s,t}(z))_{s,t \geq 0}(\omega) \subset \mathbb{R}^n \quad \text{and} \quad D \ni z \rightarrow (G_{s,t}(z))_{s,t \geq 0}(\omega) \subset \mathbb{R}^n
\]

are w-w.s.u.s.c. on \(D\), i.e., for every \(z \in D\) and every sequence \((z_n)\) of \((D,\|\cdot\|_1)\) converging weakly to \(z\), one has

\[
\mathcal{T} \int_A \int_A \int_A \int_A \left( F_{s,t} \left( z_{s,t}^{(n)} \right) \right) dsdtdP, \int_A \int_A \int_A \int_A \left( G_{s,t} \left( z_{s,t}^{(n)} \right) \right) dsdtdP \to 0
\]

and

\[
\mathcal{T} \int_A \int_A \int_A \int_A \left( G_{s,t} \left( z_{s,t}^{(n)} \right) \right) dsdtdP, \int_A \int_A \int_A \int_A \left( G_{s,t} \left( z_{s,t}^{(n)} \right) \right) dsdtdP \to 0,
\]

for every \(A \in \beta^2_+ \otimes \Omega\).

**Lemma 6.** Assume \(F\) and \(G\) take on convex values and satisfy \((C_1)\) and \((C_2)\). Then a set-valued mapping \(K\) is u.s.c. as a multifunction defined on a locally convex topological Hausdorff space \((D, \sigma(D,D^*))\) with nonempty values in \((D,\sigma(D,D^*))\).
Proof. Let $C$ be a nonempty weakly closed subset of $D$ and select a sequence $(z^n)$ of $\mathcal{K}^{-}(C)$ weakly converging to $z \in D$. There is a sequence $(y^n)$ of $C$ such that $y^n \in \mathcal{K}(z^n)$ for $n = 1, 2, \ldots$. By the uniform square-integrable boundedness of $F$ and $G$, there is a convex weakly compact subset $B \subset \mathcal{L}_n^2 \times \mathcal{L}_n^2$ such that $\mathcal{K}(z_n) \subset \Phi(B)$. Therefore, $y^n \in \Phi(B)$ for $n = 1, 2, \ldots$ which, by the weak compactness of $\Phi(B)$ implies the existence of a subsequence $(y^k)$ of $(y^n)$ weakly converging to $y \in \Phi(B)$. We have $y^k \in \mathcal{K}(z^k)$ for $k = 1, 2, \ldots$. Let $(f^k, g^k) \in S^2(F_{s,t}(z^k)) \times S^2(G_{s,t}(z^k))$ be such that $\Phi(f^k, g^k) = y^k$, for each $k = 1, 2, \ldots$. We have of course $(f^k, g^k) \in B$. Therefore, there is a subsequence, say again $(f^k, g^k)$ of $(f^k, g^k)$ weakly converging in $\mathcal{L}_n^2 \times \mathcal{L}_n^2$ to $(f, g) \in B$. Now, for every $A \in \beta_n^2 \otimes \Omega$ one obtains

$$\text{dist} \left( \int_A f_{s,t} \ dsdt \Phi(z) \ dsdt \right) \leq \int_A \left( f_{s,t} - f^k_{s,t} \right) \ dsdt \Phi(z)$$

$$+ \text{dist} \left( \int_A f^k_{s,t} \ dsdt \Phi(z) \ dsdt \right)$$

$$+ \text{dist} \left( \int_A F_{s,t}(z^k) \ dsdt \right)$$

$$+ \text{dist} \left( \int_A F_{s,t}(z) \ dsdt \right).$$

Therefore, $f \in S^2(F_{s,t}(z_{s,t}), \bar{s}, \bar{t})$. Quite similarly, we also get $g \in S^2(G_{s,t}(z_{s,t}), \bar{s}, \bar{t})$. Thus, $\varphi + \Phi(f, g) \in \mathcal{K}(z)$, which implies $y \in \mathcal{K}(z)$. On the other hand, we also have $y \in C$, because $C$ is weakly closed. Therefore, $z \in \mathcal{K}^{-}(C)$. Now the result follows immediately from Eberlein and Šmulian’s Theorem.

Theorem 7. If $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_2)$, then there is $z \in D$ such that

$$z_{s,t} \in \varphi_{s,t} + \int_0^t \int_0^s f_{u,v}(z_{u,v}) \ du dv + \int_0^t \int_0^s G_{u,v}(z_{u,v}) \ dw_{u,v}.$$ 

Proof. Let

$$\{B = (f, g) \in \mathcal{L}_n^2 \times \mathcal{L}_n^2 : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\|\}$$

and put $K = \varphi + \Phi(B)$. It is clear that $K$ is a nonempty convex weakly compact subset of $D$ such that $\mathcal{K}(z) \subset K$ for $z \in D$. By (iii) of Theorem 5
\( K(z) \) is a convex and weakly compact subset of \( D \), for each \( z \in D \). By Lemma 6 \( K \) is u.s.c. on a locally convex topological Hausdorff space \((D, \sigma(D, D^*))\). Therefore, by the Kakutani and Fan fixed point theorem there exists \( z \in K \) such that \( z \in K(z) \), i.e.,

\[
z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v}.
\]

Assume now that \( F \) and \( G \) satisfy conditions (C1) and the following condition (C3) \( F \) and \( G \) are such that set-valued functions:

\[
D \ni z \rightarrow (F_{s,t}(z))_{s,t \geq 0}(\omega) \subset \mathbb{R}^n \quad \text{and} \quad D \ni z \rightarrow (G_{s,t}(z))_{s,t \geq 0}(\omega) \subset \mathbb{R}^n
\]

are s.-w.s.l.s.c. on \( D \), i.e., for every \( z \in D \) and every sequence \((z_n)\) of \((D, |\cdot|)\) converging weakly to \( z \) one has

\[
\overline{h}[(F_{s,t}(z))_{s,t \geq 0}(w), (F_{s,t}(z_n))_{s,t \geq 0}(\omega)] \rightarrow 0
\]

and

\[
\overline{h}[(G_{s,t}(z))_{s,t \geq 0}(w), (G_{s,t}(z_n))_{s,t \geq 0}(\omega)] \rightarrow 0 \ \text{a.e.}
\]

**Lemma 8.** Assume \( F \) and \( G \) take on convex values and satisfy (C1) and (C3). Then a set-valued mapping \( K \) is l.s.c. as a multifunction defined on a locally convex topological Hausdorff space \((D, \sigma(D, D^*))\).

**Proof.** Let \( C \) be a nonempty weakly closed subset of \( D \) and \((z^{(n)})\) a sequence of \( \mathcal{R}_-(C) \) weakly converging to \( z \in D \). Select arbitrary \( u \in K(z) \) and suppose \((f, g) \in S^2(F_{s,t}(z))_{s,t \geq 0} \times S^2(G_{s,t}(z))_{s,t \geq 0}\) is such that \( u = \Phi(f, g) \).

Let \((f^{(n)}, g^{(n)}) \in S^2(F_{s,t}(z^{(n)}))_{s,t \geq 0} \times S^2(G_{s,t}(z^{(n)}))_{s,t \geq 0}\) be such that

\[
|f_{s,t}(\omega) - f_{s,t}^{(n)}(\omega)| = \text{dist} \left( f_{s,t}(\omega), \left(F_{s,t}(z^{(n)})\right)(\omega) \right)
\]

and

\[
|g_{s,t}(\omega) - g_{s,t}^{(n)}(\omega)| = \text{dist} \left( g_{s,t}(\omega), \left(G_{s,t}(z^{(n)})\right)(\omega) \right)
\]

on \( R^2_+ \times \Omega \), for each \( n = 1, 2, \ldots \). By virtue of (C3) one gets \(|f_{s,t}(\omega) - f_{s,t}^{(n)}(\omega)| \rightarrow 0\) and \(|g_{s,t}(\omega) - g_{s,t}^{(n)}(\omega)| \rightarrow 0\) a.e., as \( n \rightarrow \infty \). Hence by (C1) we can see that a sequence \((u^{(n)})\), defined by \( u^{(n)} = \Phi(f^{(n)}, g^{(n)}) \) weakly converges to \( u \). But \( u^{(n)} \in K(z^{(n)}) \subset C \) for \( n = 1, 2, \ldots \) and \( C \) is weakly closed. Then \( u \in C \), which implies \( K(z) \subset C \). Thus \( z \in K_-(C) \).
Theorem 9. If $F$ and $G$ take on convex values and satisfy $(C_1)$ and $(C_3)$ then stochastic integral inclusion (1) admits a solution.

Proof. Let

$$\mathcal{B} = \{(f, g) \in L^2_n \times L^2_n : |f_{s,t}(\omega)| \leq \|F_{s,t}(\omega)\|, |g_{s,t}(\omega)| \leq \|G_{s,t}(\omega)\|\}$$

and put $K = \varphi + \Phi(\mathcal{B})$. It is clear that $K$ is a nonempty convex weakly compact subset of $D$ such that $K(z) \subset K$ for $z \in D$. By virtue of Lemma 8, $K$ is l.s.c. as a set-valued mapping from a paracompact space $D$ considered with its relative topology induced by a weak topology $\sigma(D, D^*)$ on $D$ into a Banach space $(D, \| \cdot \|)$. By (iii) of Theorem 5, $K(z)$ is a closed and convex subset of $D$, for each $z \in K$. Therefore, by Michael’s theorem, there is a continuous selection $k : K \to D$ for $K$. But $K(K) \subset K$. Then $K$ maps $K$ into itself and is continuous with respect to the relative topology on $K$, defined above. Therefore, by the Schauder and Tikhonov fixed point theorem, there is $z \in K$ such that $z = k(z) \in K(z)$, i.e.,

$$z_{s,t} \in \varphi_{s,t} + \int_0^s \int_0^t F_{u,v}(z_{u,v}) \, du \, dv + \int_0^s \int_0^t G_{u,v}(z_{u,v}) \, dw_{u,v} \text{ a.s.}$$

References


Received 10 October 2001
Revised 18 November 2001