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A TIKHONOV-TYPE THEOREM FOR ABSTRACT PARABOLIC DIFFERENTIAL INCLUSIONS IN BANACH SPACES†

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Abstract

We consider a class of singularly perturbed systems of semilinear parabolic differential inclusions in infinite dimensional spaces. For such a class we prove a Tikhonov-type theorem for a suitably defined subset of the set of all solutions for $\varepsilon \geq 0$, where ε is the perturbation parameter. Specifically, assuming the existence of a Lipschitz selector of the involved multivalued maps we can define a nonempty subset $Z_L(\varepsilon)$ of the solution set of the singularly perturbed system. This subset is the set of the Hölder continuous solutions defined in $[0, d], d > 0$

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with prescribed exponent and constant L. We show that $Z_L(\varepsilon)$ is uppersemicontinuous at $\varepsilon = 0$ in the $C[0,d] \times C[\delta,d]$ topology for any $\delta \in (0,d]$.

Keywords: singular perturbations, differential inclusions, analytic semigroups, multivalued compact operators, Lipschitz selections.

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1. Introduction

In this paper, we consider the problem of extending a Tikhonov-type result for a system of singularly perturbed semilinear parabolic inclusions of the form

$$(1)\quad \begin{cases} x'(t) \in Ax(t) + \psi_1(t, x(t)) + b_{12}(x(t))y(t) \\ \varepsilon y'(t) \in By(t) + \psi_2(t, x(t)) + b_{21}(x(t))y(t) + b_{22}y(t), \;\; t \in [0,d], \end{cases}$$

$$(2)\quad x(0) = x_0, \quad y(0) = y_0,$$

where: A and B are generators of the analytic semigroups of linear operators e^{At} and e^{Bt}, acting in separable Banach spaces E_1 and E_2 with E_2^* satisfying the Radon-Nikodym condition (see [3]); ε is a small positive parameter; $x_0 \in D(A)$; $y_0 \in D(B)$; ψ_i, $i = 1,2$, are nonlinear multivalued operators, b_{12}, b_{21}, b_{22} are singlevalued operators. All the assumptions will be made precise in the next Section.

In [4], [9], [10], the uppersemicontinuity in the uniform topology at $\varepsilon = 0$ of a suitable defined subset of the set of solution pairs (x, y) of a singularly perturbed system of differential inclusions, was established in the case of finite dimensional spaces.

This paper represents an attempt to obtain in infinite dimensional spaces a result similar to that obtained in [4] and [10]. Specifically, we provide conditions under which for system (1) – (2) we can obtain an analog of the classical Tikhonov theorem.

The behaviour of solutions $(x_\varepsilon, y_\varepsilon)$ as $\varepsilon \to 0$ for systems of differential inclusions in infinite dimensional Banach spaces was considered in [1] and [6]. To describe this behaviour the crucial point is the choice of the topology for the convergence of $(x_\varepsilon, y_\varepsilon)$ as $\varepsilon \to 0$. The aim is to obtain the uppersemicontinuity of the solution map $\varepsilon \to Z(\varepsilon)$ at $\varepsilon = 0$. In [1] convergence with respect to x-variable in the space $C([0,d],E_1)$ and the weak convergence in $L^1([0,d],E_2)$ with respect to y-variable were considered. With this choice it was possible to show the uppersemicontinuity of the solution map at $\varepsilon = 0$ by means of the introduction of a suitable measure of noncompactness and an application of topological degree theory for condensing operators in locally convex spaces. When the uniform convergence is considered for the y-variable on $[\delta, d]$, $\delta \in (0,d]$, then the map $\varepsilon \to Z(\varepsilon)$ is not in general uppersemicontinuous at $\varepsilon = 0$, even in the linear finite-dimensional case (see [5]). Therefore the full analog of the Tikhonov theorem for differential inclusions cannot be obtained. In the sequel, the spaces $C([0,d],E_i)$ and $L^1([0,d],E_i)$ will also be simply denoted by $C[0,d]$ and $L^1[0,d]$ when no confusion will arise.

In this paper, we consider a particular class of nonlinear singular perturbation systems where the fast variable y appears affinely and the involved multivalued maps depend only on the slow variable x. The crucial assumption is that the multivalued maps have a Lipschitz selection, since this allows us to define a nonempty subset $Z_L(\varepsilon), \varepsilon \geq 0$, of the solution map $Z(\varepsilon)$ consisting of solution pairs (x,y) defined in $[0,d], d > 0$, which are Hölder continuous of prescribed exponent and constant. For this subset we can prove in Theorem 3.1 the uppersemicontinuity at $\varepsilon = 0$ in the $C[0,d] \times C[\delta,d]$ topology for any $\delta \in (0,d]$.

The system (1) under consideration can also be viewed as a control process where the control y, following the approach proposed for nonlinear control problems in finite dimensional spaces in [2], is designed by means of a singularly perturbed equation depending on the dynamics of the state through a suitably defined function s which represents the objective of the control action. The uniform convergence of the pair state-control as $\varepsilon \to 0$ is proved by means of the classical theory of singularly perturbed systems. Furthermore, we observe that, from the control theory point of view, the presence in (1) of multivalued maps can model a deterministic uncertainty.

The paper is organized as follows. In Section 2, we formulate the assumptions on (1) – (2) under which we can prove two preliminary lemmas. Finally, in Section 3, we state and prove the main result: Theorem 3.1.

2. Assumptions and preliminary results

We consider the Cauchy problem for a system of singularly perturbed differential inclusions of the following form

$$(1)\qquad \begin{cases} x'(t) \in Ax(t) + \psi_1(t, x(t)) + b_{12}(x(t))y(t) \\ \varepsilon y'(t) \in By(t) + \psi_2(t, x(t)) + b_{21}(x(t))y(t) + b_{22}y(t), \ t \in [0, d], \end{cases}$$

$$(2)\qquad x(0) = x_0, \quad y(0) = y_0,$$

where A and B are generators of analytic semigroups of linear operators e^{At} and e^{Bt}, acting in separable Banach spaces E_1 and E_2 with E_2^* satisfying the Radon-Nikodym property (see [3]), and ε is a small positive parameter. The operators A^{-1} and B^{-1} are assumed to be completely continuous, $x_0 \in D(A)$, $y_0 \in D(B)$, ψ_i, $i = 1, 2$, are nonlinear multivalued operators, b_{12}, b_{21}, b_{22} are singlevalued operators satisfying suitable conditions which will be specified in the sequel. For $\varepsilon = 0$ we have the reduced system

$$(1^*)\qquad \begin{cases} x'(t) \in Ax(t) + \psi_1(t, x(t)) + b_{12}(x(t))y(t) \\ 0 \in By(t) + \psi_2(t, x(t)) + b_{21}(x(t))y(t) + b_{22}y(t), \end{cases}$$

$$(2^*)\qquad x(0) = x_0.$$

Following [1] we say that $(x_\varepsilon, y_\varepsilon)$ is a solution to (1) – (2) on $[0, d]$ if $x_\varepsilon, y_\varepsilon$ are continuous functions defined on the interval $[0, d]$ with values in E_1 and E_2 respectively satisfying the inclusions

$$x_\varepsilon(t) \in \{g_1(t) : g_1(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}[f_1(s) + b_{12}(x_\varepsilon(s))y_\varepsilon(s)]ds,$$
$$(3)\qquad f_1(s) \in \psi_1(s, x_\varepsilon(s)) \ \text{ for a.a. } \ s \in [0, d]\}, \quad t \in [0, d];$$

$$y_\varepsilon(t) \in \{g_2(t) : g_2(t)$$
$$= e^{\frac{1}{\varepsilon}Bt}y_0 + \frac{1}{\varepsilon}\int_0^t e^{\frac{1}{\varepsilon}B(t-s)}[f_2(s) + b_{21}(x_\varepsilon(s))y_\varepsilon(s) + b_{22}y_\varepsilon(s)]ds,$$
$$(4)\qquad f_2(s) \in \psi_2(s, x_\varepsilon(s)) \ \text{ for a.a. } \ s \in [0, d]\}, \quad t \in [0, d].$$

The reduced system at $\varepsilon = 0$ corresponding to the system (3) – (4) is given by

$$x^0(t) \in \{g_1(t) : g_1(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}[f_1(s) + b_{12}(x^0(s))y^0(s)]ds,$$
$$(3^*)\qquad f_1(s) \in \psi_1(s, x^0(s)) \ \text{ for a.a } \ s \in [0, d]\}, \quad t \in [0, d];$$

$$y^0(t) \in \{g_0(t) : g_0(t) = -B^{-1}[f_0(t) + b_{21}(x^0(t))y^0(t) + b_{22}y^0(t)],$$

$$f_0(t) \in \psi_2(t, x^0(t)) \text{ for a.a. } t \in [0,d]\}, \quad t \in [0,d]. \tag{4*}$$

Let $Z(\varepsilon)$, $\varepsilon > 0$, be the set of solutions to the system (3) – (4) and let $Z(0)$ be the set of solutions to the system (3^*) – (4^*).

We recall some preliminary results from the theory of analytic semigroups which we use in the sequel, (see e.g. [8]).

Proposition 2.1. *The closed operator A having dense domain, is the infinitesimal generator of the analytic semigroup* e^{At} *if and only if the resolvent set of this operator contains a half-plane* $Re\lambda \leq \sigma_0$ *and the resolvent satisfies the inequality*

$$\|(\lambda I - A)^{-1}\| \leq C(1+ \mid \lambda \mid)^{-1}$$

for some $C > 0$.

If A is the infinitesimal generator of the analytic semigroup e^{At}, then

$$e^{At} = -\frac{1}{2\pi i}\int_{\Pi(\beta,\sigma)} e^{\lambda t}(\lambda I - A)^{-1}d\lambda, \quad t > 0,$$

where $\Pi(\beta, \sigma)$ consists of two rays

$$\lambda = \sigma + \rho e^{-i\beta} \text{ and } \lambda = \sigma + \rho e^{i\beta}, \quad \sigma \leq \sigma_0, \quad \arcsin\frac{1}{C} < \beta < \frac{\pi}{2}.$$

If $\sigma_0 < 0$, then the negative fractional powers of A are defined by the formula

$$A^{-\alpha} = -\frac{1}{2\pi i}\int_{\Pi(\beta,\sigma)} \lambda^{-\alpha}(\lambda I - A)^{-1}d\lambda, \quad 0 < \alpha < 1.$$

The operator

$$A^{\alpha}e^{At} = -\frac{1}{2\pi i}\int_{\Pi(\beta,\sigma)} \lambda^{\alpha}e^{\lambda t}(\lambda I - A)^{-1}d\lambda, \quad t > 0,$$

satisfies the estimate

$$\|A^{\alpha}e^{At}\| \leq Ct^{-\alpha}.$$

From this inequality one obtains

$$\|A^{-\alpha}(e^{At} - I)\| \leq Ct^{-\alpha}.$$

We assume the following conditions.

A_0) There exists a positive constant d_2 such that

$$\|e^{Bt}\| \leq e^{-d_2 t}$$

for any $t \geq 0$.

A_1) Let $\bar{\psi}_i(t,x) = \psi_i(t, A^{-\alpha}x),\ \alpha \in (0,1),\ i = 1,2$. For every $x \in E_1$ and for every $t \in [0,d]$ the set $\bar{\psi}_i(t,x)$ is nonempty, compact and convex. For a.a. $t \in [0,d]$ the operators $\bar{\psi}_i(t,\cdot)$, $i = 1,2$, are upper semicontinuous.

A_2) There exists a positive constant ρ such that

$$\|\bar{\psi}_i(t,x)\| \leq \rho(1 + \|x\|),\ i = 1,2,$$

for every $x \in E_1$ and for every $t \in [0,d]$. Here $\|D\| = \sup_{x\in D} \|x\|_{E_i}$, for any bounded set $D \subset E_i$.

A_3) There exists a selector $\bar{f}_i : [0,d]\times E_1 \longrightarrow E_i$ of the map $\bar{\psi}_i(t,x)$ satisfying the Lipschitz condition

$$\|\bar{f}_i(t_1,x_1) - \bar{f}_i(t_2,x_2)\| \leq k[\|x_1 - x_2\|_{E_1} + \mid t_1 - t_2 \mid],\ i = 1,2,$$

for any $x_j \in E_1$ and any $t_j \in [0,d],\ j = 1,2$.

We now formulate the assumptions on the nonlinear operators b_{12} and b_{21} defined on E_1 with values in $L(E_2,E_i)$, $i = 1,2$, respectively. Here $L(E_2,E_i)$ denotes the space of bounded linear operators acting from E_2 to E_i.

A_4) There exist positive constants $\sigma,\ \gamma,\ p$ such that

(i) $$\|b_{12}(A^{-\alpha}x)\| \leq \sigma,$$

(ii) $$\|b_{21}(A^{-\alpha}x)\| \leq \gamma,$$

(iii) $$\|b_{12}(A^{-\alpha}x_1) - b_{12}(A^{-\alpha}x_2)\| \leq p\|x_1 - x_2\|$$

for every $x_1, x_2 \in E_1$. We assume that b_{21} also satisfies (iii).

A_5) There exists a positive constant β such that the linear operator b_{22}: $E_2 \longrightarrow E_2$ satisfies

$$\|b_{22}\| \leq \beta.$$

A_6) Finally, we assume the following

$$\beta + \gamma < d_2.$$

Let $\delta(\varepsilon)$ be a function satisfying the following conditions

$$\delta(\varepsilon) \ \rightarrow \ 0 \text{ for } \varepsilon \ \rightarrow \ 0 \quad \text{and} \quad \delta(\varepsilon) \geq \frac{\varepsilon(\ln(1/\varepsilon))}{d_2 - \gamma - \beta}.$$

Definition 2.1. Fix $\theta \in (0,1)$. We consider the subset $Z_L(\varepsilon)$ of the set $Z(\varepsilon)$ defined in the following way:

$$Z_L(\varepsilon) = \{(x,y) \in Z(\varepsilon) : x, y \text{ satisfy a Hölder condition on } [0,d] \text{ and } [\delta(\varepsilon), d] \\ \text{respectively with exponent } \theta(1-\alpha) \text{ and constant } L\}.$$

We also consider the subset $Z_L(0)$ of the set $Z(0)$ defined in the following way:

$$Z_L(0) = \{(x,y) \in Z(0) : x, y \text{ satisfy a Hölder condition on } [0,d] \\ \text{with exponent } \theta(1-\alpha) \text{ and constant } L\}.$$

By the change of variable $x_\varepsilon(t) = A^{-\alpha}\tilde{x_\varepsilon}(t)$, $(3) - (4)$ and $(3^*) - (4^*)$ take the form

$$\tilde{x_\varepsilon}(t) \in \{g_1(t) : g_1(t) = e^{At}A^\alpha x_0 + \int_0^t A^\alpha e^{A(t-s)}[f_1(s) + b_{12}(A^{-\alpha}\tilde{x_\varepsilon}(s))y_\varepsilon(s)]ds,$$

$$(3') \qquad f_1(s) \in \psi_1(s, A^{-\alpha}\tilde{x_\varepsilon}(s)) \ \text{ for a.a. } \ s \in [0,d]\},$$

$$y_\varepsilon(t) \in \{g_2(t) : g_2(t)$$

$$(4') \qquad = e^{\frac{1}{\varepsilon}Bt}y_0 + \frac{1}{\varepsilon}\int_0^t e^{\frac{1}{\varepsilon}B(t-s)}[f_2(s) + b_{21}(A^{-\alpha}\tilde{x_\varepsilon}(s))y_\varepsilon(s) + b_{22}y_\varepsilon(s)]ds,$$

$$f_2(s) \in \psi_2(s, A^{-\alpha}\tilde{x_\varepsilon}(s)) \ \text{ for a.a. } \ s \in [0,d]\},$$

and

$$\tilde{x}(t) \in \{g_1(t) : g_1(t) = e^{At}A^\alpha x_0 + \int_0^t A^\alpha e^{A(t-s)}[f_1(s) + b_{12}(A^{-\alpha}\tilde{x}(s))y(s)]ds,$$

$$(3'') \qquad f_1(s) \in \psi_1(s, A^{-\alpha}\tilde{x}(s)) \text{ for a.a. } s \in [0,d]\},$$

$$y(t) \in \{g_0(t) : g_0(t) = -B^{-1}[f_0(t) + b_{21}(A^{-\alpha}\tilde{x}(t))y(t) + b_{22}y(t)],$$

$$(4'') \qquad f_0(t) \in \psi_2(t, A^{-\alpha}\tilde{x}(t)) \text{ for a.a. } t \in [0,d]\},\ t \in [0,d].$$

Therefore, if $(\tilde{x_\varepsilon}, y_\varepsilon)$ is a solution to the system $(3') - (4')$ and $(\tilde{x}, y)$ is a solution to the system $(3'') - (4'')$, then $(x_\varepsilon, y_\varepsilon)$, with $x_\varepsilon(t) = A^{-\alpha}\tilde{x_\varepsilon}(t)$, is a solution to the system $(3) - (4)$ and (x, y), with $x(t) = A^{-\alpha}\tilde{x}(t)$, is a solution to the system $(3^*) - (4^*)$.

We denote by $\hat{Z}(\varepsilon)$ the set of solutions to the system $(3') - (4')$ and by $\hat{Z}(0)$ the set of solutions to the system $(3'') - (4'')$. Let us consider the subset $\hat{Z}_l(\varepsilon)$ of the set $\hat{Z}(\varepsilon)$ defined as follows

$$\hat{Z}_l(\varepsilon) = \{(x, y) \in \hat{Z}(\varepsilon) : x, y \text{ satisfy a Hölder condition on } [0,d] \text{ and } [\delta(\varepsilon), d] \\ \text{respectively with exponent } \theta(1-\alpha) \text{ and constant } l\}.$$

We consider also a subset $\hat{Z}_l(0)$ of the set $\hat{Z}(0)$:

$$\hat{Z}_l(0) = \{(x, y) \in \hat{Z}(0) : x, y \text{ satisfy a Hölder condition on } [0,d] \\ \text{with exponent } \theta(1-\alpha) \text{ and constant } l\}.$$

Observe that if we prove the existence of a constant $l > 0$ such that the map $\varepsilon \to \hat{Z}_l(\varepsilon)$ is uppersemicontinuous, then if we take $L = \|A^{-\alpha}\|l$ the map $\varepsilon \to Z_L(\varepsilon)$ is also uppersemicontinuous. Therefore, we can deal with the solution set $\hat{Z}_l(\varepsilon)$.

We have the following result.

Lemma 2.1. *Assume that the conditions $A_0 - A_6$ are satisfied for some $\alpha \in (0,1)$. Then the solutions to the system $(3') - (4')$ (if any) are uniformly bounded with respect to ε.*

Proof. Let us estimate the norm of $y_\varepsilon(t)$. We have

$$\|y_\varepsilon(t)\| \le \|e^{\frac{1}{\varepsilon}Bt}y_0\| + \frac{1}{\varepsilon}\int_0^t \|e^{\frac{1}{\varepsilon}B(t-s)}\| \; [\; \|f_2(s)\| + \|b_{21}(A^{-\alpha}\tilde{x_\varepsilon}(s))y_\varepsilon(s)\|$$

$$+ \|b_{22}y_\varepsilon(s)\| \;]\; ds \le e^{-\frac{1}{\varepsilon}d_2 t}\|y_0\| + \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}d_2(t-s)} \;[\; \rho(1+\|\tilde{x_\varepsilon}(s)\|)$$

$$+ (\gamma+\beta)\|y_\varepsilon(s)\| \;]ds.$$

By the Gronwall Lemma we have

$$\|y_\varepsilon(t)\| \le e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t}\|y_0\| + \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(t-s)}\rho(1+\|\tilde{x_\varepsilon}(s)\|)ds \tag{5}$$

Using the estimates

$$\|A^\alpha e^{At}\| \le \frac{C(\alpha)}{t^\alpha} \;(0<t<\infty),$$

$$\|e^{At}\| \le C(d_1)e^{d_1 t},$$

where $C(\alpha)$, d_1 and $C(d_1)$ are some constants, from $(3')$ we obtain

$$\|\tilde{x_\varepsilon}(t)\| \le C(d_1)e^{d_1 t}\|A^\alpha x_0\| + \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho(1+\|\tilde{x_\varepsilon}(s)\|)ds$$

$$+ \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma\|y_\varepsilon(s)\|ds\,.$$

From (5) we have

$$\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma\|y_\varepsilon(s)\|ds \le \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}\|y_0\|ds$$

$$+\frac{1}{\varepsilon}\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma\int_0^s e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-\tau)}\rho(1+\|\tilde{x_\varepsilon}(\tau)\|)d\tau\; ds \le \frac{C(\alpha)}{1-\alpha}\sigma\|y_0\|d^{1-\alpha}$$

$$+\frac{C(\alpha)}{1-\alpha}\sigma\rho\frac{1}{d_2-\gamma-\beta}d^{1-\alpha} + \frac{1}{\varepsilon}C(\alpha)\sigma\rho\int_0^t\int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-\tau)}}{(t-s)^\alpha}\|\tilde{x_\varepsilon}(\tau)\|d\tau\; ds.$$

Therefore,

$$\|\tilde{x_\varepsilon}(t)\| \le C(d_1)e^{d_1 t}\|A^\alpha x_0\| + \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho(1+\|\tilde{x_\varepsilon}(s)\|)ds + \frac{C(\alpha)}{1-\alpha}\sigma\|y_0\|d^{1-\alpha}$$

$$+ \frac{C(\alpha)}{1-\alpha}\sigma\rho\frac{1}{d_2-\gamma-\beta}d^{1-\alpha} + \frac{1}{\varepsilon}C(\alpha)\sigma\rho\int_0^t\int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-\tau)}}{(t-s)^\alpha}\|\tilde{x_\varepsilon}(\tau)\|d\tau\ ds.$$

We introduce the following equivalent norm in the space of continuous functions

$$|\ \tilde{x_\varepsilon}\ |_H = \max_{t\in[0,d]} e^{-Ht}\|\tilde{x_\varepsilon}(t)\|,$$

where H is a positive constant that will be chosen in the sequel. Therefore,

$$e^{-Ht}\|\tilde{x_\varepsilon}(t)\| \le e^{-Ht}[C(d_1)e^{d_1 t}\|A^\alpha x_0\| + \frac{C(\alpha)}{1-\alpha}\rho d^{1-\alpha} + \frac{C(\alpha)}{1-\alpha}\sigma\|y_0\|d^{1-\alpha}$$

$$+ \frac{C(\alpha)}{1-\alpha}\sigma\rho\frac{1}{d_2-\gamma-\beta}d^{1-\alpha}] + e^{-Ht}\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho e^{Hs}e^{-Hs}\|\tilde{x_\varepsilon}(s)\|ds$$

$$+ \frac{1}{\varepsilon}C(\alpha)\sigma\rho e^{-Ht}\int_0^t\int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-\tau)}}{(t-s)^\alpha}e^{H\tau}e^{-H\tau}\|\tilde{x_\varepsilon}(\tau)\|d\tau\ ds. \tag{6}$$

Let $h \in (0,d)$ be arbitrary fixed. We consider the first integral

$$e^{-Ht}\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho e^{Hs}e^{-Hs}\|\tilde{x_\varepsilon}(s)\|ds \le e^{-Ht}\ |\ \tilde{x_\varepsilon}\ |_H \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho e^{Hs}ds.$$

For $t \ge h$ we have

$$e^{-Ht}\ |\ \tilde{x_\varepsilon}\ |_H \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho e^{Hs}ds \le e^{-Ht}\ |\ \tilde{x_\varepsilon}\ |_H \int_0^{t-h} \frac{C(\alpha)}{h^\alpha}\rho e^{Hs}ds$$

$$+ e^{-Ht}\ |\ \tilde{x_\varepsilon}\ |_H \int_{t-h}^t \frac{C(\alpha)}{(t-s)^\alpha}\rho e^{Ht}ds \le |\ \tilde{x_\varepsilon}\ |_H\ C(\alpha)\rho\frac{1}{h^\alpha H}$$

$$+ |\ \tilde{x_\varepsilon}\ |_H\ C(\alpha)\rho\frac{h^{1-\alpha}}{1-\alpha} = |\ \tilde{x_\varepsilon}\ |_H\ C(\alpha)\rho\left[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}\right].$$

For $t < h$ we obtain

$$e^{-Ht} \mid \tilde{x_\varepsilon} \mid_H \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho e^{Hs} ds \leq \mid \tilde{x_\varepsilon} \mid_H \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho ds$$

$$= \mid \tilde{x_\varepsilon} \mid_H \frac{C(\alpha)}{1-\alpha}\rho t^{1-\alpha} < \mid \tilde{x_\varepsilon} \mid_H \frac{C(\alpha)}{1-\alpha}\rho h^{1-\alpha}.$$

Therefore,

$$e^{-Ht} \mid \tilde{x_\varepsilon} \mid_H \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\rho e^{Hs} ds \leq \mid \tilde{x_\varepsilon} \mid_H C(\alpha)\rho \left[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}\right],$$

for every $t \in [0, d]$.

We consider now the second integral in (6)

$$\frac{1}{\varepsilon}C(\alpha)\sigma\rho e^{-Ht} \int_0^t \int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-\tau)}}{(t-s)^\alpha} e^{H\tau} e^{-H\tau} \|\tilde{x_\varepsilon}(\tau)\| d\tau \, ds$$

$$\leq \frac{1}{\varepsilon}C(\alpha)\sigma\rho e^{-Ht} \mid \tilde{x_\varepsilon} \mid_H \int_0^t \int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-\tau)}}{(t-s)^\alpha} e^{H\tau} d\tau \, ds.$$

After the change of variable $s - \tau = \xi$ we obtain

$$\int_0^t \int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-\tau)}}{(t-s)^\alpha} e^{H\tau} d\tau \, ds = \int_0^t \int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)\xi}}{(t-s)^\alpha} e^{H(s-\xi)} d\xi \, ds$$

$$\leq \int_0^t \frac{e^{Hs}}{(t-s)^\alpha} \int_0^s e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)\xi} d\xi \, ds \leq \frac{\varepsilon}{d_2-\gamma-\beta} \int_0^t \frac{e^{Hs}}{(t-s)^\alpha} ds$$

$$\leq \frac{\varepsilon}{d_2-\gamma-\beta} e^{Ht} \left[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}\right].$$

Therefore,

$$\frac{1}{\varepsilon}C(\alpha)\sigma\rho e^{-Ht} \int_0^t \int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-\tau)}}{(t-s)^\alpha} \|\tilde{x_\varepsilon}(\tau)\| d\tau \, ds$$

$$\leq C(\alpha)\sigma\rho \frac{1}{d_2-\gamma-\beta} \mid \tilde{x_\varepsilon} \mid_H \left[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}\right].$$

Hence,

$$
\begin{aligned}
&| \tilde{x_\varepsilon} |_H \leq C_1 + | \tilde{x_\varepsilon} |_H \, C(\alpha)\rho \left[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}\right] \\
&+ | \tilde{x_\varepsilon} |_H \, C(\alpha)\sigma\rho \frac{1}{d_2-\gamma-\beta}\left[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}\right] \\
&= C_1 + | \tilde{x_\varepsilon} |_H \, C_2 \left[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}\right],
\end{aligned}
$$

and so

$$
| \tilde{x_\varepsilon} |_H \left(1 - C_2\left[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}\right]\right) \leq C_1.
$$

Now, we choose h in such way that

$$
\frac{h^{1-\alpha}}{1-\alpha} < \frac{1}{2C_2}.
$$

Furthermore, we choose H such that

$$
\frac{1}{h^\alpha}\frac{1}{H} < \frac{1}{2C_2}.
$$

Then

$$
| \tilde{x_\varepsilon} |_H \leq \frac{C_1}{1 - C_2[\frac{1}{h^\alpha}\frac{1}{H} + \frac{h^{1-\alpha}}{1-\alpha}]}.
$$

Since the introduced norm is equivalent to the standard norm in the space of continuous functions on $[0,d]$, there exists a constant $C_3 > 0$ such that

$$
| \tilde{x_\varepsilon} |_{C[0,d]} \leq C_3.
$$

Further, from (5)

$$
\|y_\varepsilon(t)\| \leq \|y_0\| + \frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(t-s)}\rho(1+C_3)ds \leq \|y_0\| + \frac{\rho(1+C_3)}{d_2-\gamma-\beta} = C_4.
$$

Therefore, for every solution $(\tilde{x_\varepsilon}, y_\varepsilon)$ to the system $(3') - (4')$ the estimates

$$
| \tilde{x_\varepsilon} |_{C[0,d]} \leq C_3, \tag{7}
$$

$$
| y_\varepsilon |_{C[0,d]} \leq C_4 \tag{8}
$$

hold. ■

According to assumption A_3) there exists a selection $\bar{f}_i : [0,d] \times E_1 \to E_i$ of $\bar{\psi}_i(t,x), i = 1,2$, satisfying the Lipschitz condition. Let us consider the following system

$$x_\varepsilon(t) = e^{At}A^\alpha x_0 + \int_0^t A^\alpha e^{A(t-s)}[\bar{f}_1(s,x_\varepsilon(s)) + b_{12}(A^{-\alpha}x_\varepsilon(s))y_\varepsilon(s)]ds, \tag{9}$$

$$\begin{aligned} y_\varepsilon(t) &= e^{\frac{1}{\varepsilon}Bt}y_0 \\ &+\tfrac{1}{\varepsilon}\int_0^t e^{\frac{1}{\varepsilon}B(t-s)}\left[\bar{f}_2(s,x_\varepsilon(s)) + b_{21}(A^{-\alpha}x_\varepsilon(s))y_\varepsilon(s) + b_{22}y_\varepsilon(s)\right]ds. \end{aligned} \tag{10}$$

For every $\varepsilon > 0$ fixed, the system (9) – (10) has a solution that is also a solution to the system ($3'$) – ($4'$), (see for example [7]). We denote this solution by $(\bar{x_\varepsilon}, \bar{y_\varepsilon})$.

We have the following.

Lemma 2.2. *Assume that the conditions $A_0 - A_6$ are satisfied for some $\alpha \in (0,1)$. Then for every $\varepsilon_0 > 0$ and for every $\theta \in (0,1)$ there exists a constant $C(\theta)$ such that for every $\varepsilon \in (0,\varepsilon_0]$ the solution $(\bar{x_\varepsilon},\ \bar{y_\varepsilon})$ to the system ($3'$) – ($4'$) satisfies the following property*

$$\|\bar{x_\varepsilon}(t+\tau) - \bar{x_\varepsilon}(t)\| \le C(\theta)\tau^{\theta(1-\alpha)}, \tag{11}$$

$$\|\bar{y_\varepsilon}(t+\tau) - \bar{y_\varepsilon}(t)\| \le C(\theta)\tau^{\theta(1-\alpha)}\left(1 + \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t}\right), \tag{12}$$

for every $\tau \in [0,d]$ and for every $t \in [0,d-\tau]$.

Proof. We begin by the estimate

$$\begin{aligned} \|\bar{y_\varepsilon}(t+\tau) - \bar{y_\varepsilon}(t)\| &\le \|e^{\frac{1}{\varepsilon}B(t+\tau)}y_0 - e^{\frac{1}{\varepsilon}Bt}y_0\| \\ &+\frac{1}{\varepsilon}\Big\|\int_0^{t+\tau} e^{\frac{1}{\varepsilon}B(t+\tau-s)}[\bar{f}_2(s,\bar{x_\varepsilon}(s)) + b_{21}(A^{-\alpha}\bar{x_\varepsilon}(s))\bar{y_\varepsilon}(s) + b_{22}\bar{y_\varepsilon}(s)]ds \\ &-\int_0^t e^{\frac{1}{\varepsilon}B(t-s)}\left[\bar{f}_2(s,\bar{x_\varepsilon}(s)) + b_{21}(A^{-\alpha}\bar{x_\varepsilon}(s))\bar{y_\varepsilon}(s) + b_{22}\bar{y_\varepsilon}(s)\right]ds\Big\|. \end{aligned}$$

Let us estimate the second norm in the last expression

$$\frac{1}{\varepsilon}\Big\|\int_0^{t+\tau} e^{\frac{1}{\varepsilon}B(t+\tau-s)}\left[\bar f_2(s,\bar{x_\varepsilon}(s)) + b_{21}(A^{-\alpha}\bar{x_\varepsilon}(s))\bar{y_\varepsilon}(s) + b_{22}\bar{y_\varepsilon}(s)\right] ds$$
$$-\int_0^{t} e^{\frac{1}{\varepsilon}B(t-s)}\left[\bar f_2(s,\bar{x_\varepsilon}(s)) + b_{21}(A^{-\alpha}\bar{x_\varepsilon}(s))\bar{y_\varepsilon}(s) + b_{22}\bar{y_\varepsilon}(s)\right] ds\Big\|$$
$$= \frac{1}{\varepsilon}\Big\|\int_0^{\tau} e^{\frac{1}{\varepsilon}B(t+\tau-s)}\left[\bar f_2(s,\bar{x_\varepsilon}(s)) + b_{21}(A^{-\alpha}\bar{x_\varepsilon}(s))\bar{y_\varepsilon}(s) + b_{22}\bar{y_\varepsilon}(s)\right] ds$$
$$+\int_\tau^{t+\tau} e^{\frac{1}{\varepsilon}B(t+\tau-s)}\left[\bar f_2(s,\bar{x_\varepsilon}(s)) + b_{21}(A^{-\alpha}\bar{x_\varepsilon}(s))\bar{y_\varepsilon}(s) + b_{22}\bar{y_\varepsilon}(s)\right] ds$$
$$-\int_0^{t} e^{\frac{1}{\varepsilon}B(t-s)}\left[\bar f_2(s,\bar{x_\varepsilon}(s)) + b_{21}(A^{-\alpha}\bar{x_\varepsilon}(s))\bar{y_\varepsilon}(s) + b_{22}\bar{y_\varepsilon}(s)\right] ds\Big\| = P$$

In the second integral we make a change of variable $\xi = s - \tau$.

$$P \le \frac{1}{\varepsilon}M_2\int_0^{\tau} e^{-\frac{d_2}{\varepsilon}(t+\tau-s)}ds + \frac{1}{\varepsilon}\Big\|\int_0^{t} e^{\frac{1}{\varepsilon}B(t-\xi)}\,[\;\bar f_2(\xi+\tau,\bar{x_\varepsilon}(\xi+\tau))$$
$$+ b_{21}(A^{-\alpha}\bar{x_\varepsilon}(\xi+\tau))\bar{y_\varepsilon}(\xi+\tau) + b_{22}\bar{y_\varepsilon}(\xi+\tau)\;]\,d\xi - \int_0^{t} e^{\frac{1}{\varepsilon}B(t-s)}[\bar f_2(s,\bar{x_\varepsilon}(s))$$
$$+ b_{21}(A^{-\alpha}\bar{x_\varepsilon}(s))\bar{y_\varepsilon}(s) + b_{22}\bar{y_\varepsilon}(s)]ds\Big\| \le \frac{1}{\varepsilon}M_2\int_0^{\tau} e^{-\frac{d_2}{\varepsilon}(t+\tau-s)}ds$$
$$+ \frac{1}{\varepsilon}\int_0^{t} e^{-\frac{1}{\varepsilon}d_2(t-s)}[k(\|\bar{x_\varepsilon}(s+\tau) - \bar{x_\varepsilon}(s)\| + \tau) + C_4p\|\bar{x_\varepsilon}(s+\tau) - \bar{x_\varepsilon}(s)\|$$
$$+ \gamma\|\bar{y_\varepsilon}(s+\tau) - \bar{y_\varepsilon}(s)\| + \beta\|\bar{y_\varepsilon}(s+\tau) - \bar{y_\varepsilon}(s)\|]ds \le \frac{1}{\varepsilon}M_2\int_0^{\tau} e^{-\frac{d_2}{\varepsilon}(t+\tau-s)}ds$$
$$+ \frac{1}{\varepsilon}\int_0^{t} e^{-\frac{1}{\varepsilon}d_2(t-s)}[G\|\bar{x_\varepsilon}(s+\tau) - \bar{x_\varepsilon}(s)\| + k\tau + (\gamma+\beta)\|\bar{y_\varepsilon}(s+\tau) - \bar{y_\varepsilon}(s)\|]ds,$$

where $G = k + C_4p$. Let

$$v(s) = \|\bar{x_\varepsilon}(s+\tau) - \bar{x_\varepsilon}(s)\|$$

and let

$$u(s) = \|\bar{y_\varepsilon}(s+\tau) - \bar{y_\varepsilon}(s)\|.$$

Then

$$u(t) \leq \|e^{\frac{1}{\varepsilon}Bt}y_0 - y_0\|e^{-\frac{1}{\varepsilon}d_2 t} + \frac{1}{\varepsilon}M_2 \int_0^\tau e^{-\frac{d_2}{\varepsilon}(t+\tau-s)}ds$$

$$+\frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}d_2(t-s)}[Gv(s) + k\tau + (\gamma+\beta)u(s)]ds.$$

Let $Y_0 = \|e^{\frac{1}{\varepsilon}B\tau}y_0 - y_0\|$. By the Gronwall Lemma we have

$$u(t) \leq e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t}Y_0 + e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t}\frac{M_2}{\varepsilon}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2(t+\tau-s)}ds$$

$$+\frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(t-s)}[Gv(s) + k\tau]ds. \tag{13}$$

Further,

$$v(t) \leq \|e^{A(t+\tau)}A^\alpha x_0 - e^{At}A^\alpha x_0\| + \|\int_0^{t+\tau} A^\alpha e^{A(t+\tau-s)}[\bar{f}_1(s, A^{-\alpha}\bar{x}_\varepsilon(s))$$

$$+ b_{12}(A^{-\alpha}\bar{x}_\varepsilon(s))\bar{y}_\varepsilon(s)]ds - \int_0^t A^\alpha e^{A(t-s)}[\bar{f}_1(s, A^{-\alpha}\bar{x}_\varepsilon(s))$$

$$+ b_{12}(A^{-\alpha}\bar{x}_\varepsilon(s))\bar{y}_\varepsilon(s)]ds\| \leq \|e^{A\tau}A^\alpha x_0 - A^\alpha x_0\|C(d_1)e^{d_1 t}$$

$$+ \|\int_0^\tau A^\alpha e^{A(t+\tau-s)}[\bar{f}_1(s, A^{-\alpha}\bar{x}_\varepsilon(s)) + b_{12}(A^{-\alpha}\bar{x}_\varepsilon(s))\bar{y}_\varepsilon(s)]ds\|$$

$$+ \|\int_\tau^{t+\tau} A^\alpha e^{A(t+\tau-s)}[\bar{f}_1(s, A^{-\alpha}\bar{x}_\varepsilon(s)) + b_{12}(A^{-\alpha}\bar{x}_\varepsilon(s))\bar{y}_\varepsilon(s)]ds$$

$$- \int_0^t A^\alpha e^{A(t-s)}[\bar{f}_1(s, A^{-\alpha}\bar{x}_\varepsilon(s)) + b_{12}(A^{-\alpha}\bar{x}_\varepsilon(s))\bar{y}_\varepsilon(s)]ds\|$$

$$\leq \|e^{A\tau}A^\alpha x_0 - A^\alpha x_0\|C(d_1)e^{d_1 t} + M_1\int_0^\tau \frac{C(\alpha)}{(\tau-s)^\alpha}C(d_1)e^{d_1 t}ds$$

$$+ \|\int_0^t A^\alpha e^{A(t-\xi)}[\bar{f}_1(\xi+\tau, A^{-\alpha}\bar{x}_\varepsilon(\xi+\tau)) + b_{12}(A^{-\alpha}\bar{x}_\varepsilon(\xi+\tau))\bar{y}_\varepsilon(\xi+\tau)]d\xi$$

$$- \int_0^t A^\alpha e^{A(t-s)}[\bar{f}_1(s, A^{-\alpha}\bar{x}_\varepsilon(s)) + b_{12}(A^{-\alpha}\bar{x}_\varepsilon(s))\bar{y}_\varepsilon(s)]ds\|$$

$$\leq \|e^{A\tau}A^\alpha x_0 - A^\alpha x_0\| C(d_1)e^{d_1 t} + M_1 \tfrac{C(\alpha)}{1-\alpha} C(d_1) e^{d_1 t}\tau^{1-\alpha}$$

$$+ \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}[Gv(s) + k\tau + \sigma u(s)]ds.$$

Let us consider

$$\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma u(s)ds.$$

We have that

$$\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma u(s)ds = \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s} Y_0 ds$$

$$+\frac{M_2}{\varepsilon}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2(\tau-m)}dm \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s} e^{-\frac{1}{\varepsilon}d_2 s} ds$$

$$+\frac{1}{\varepsilon}\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma \int_0^s e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-m)}[Gv(m) + k\tau]dm\ ds$$

$$\leq \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s} Y_0 ds + \frac{M_2}{\varepsilon}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2(\tau-m)}dm$$

$$\cdot \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s} ds + \frac{C(\alpha)}{1-\alpha}\sigma\ \frac{1}{d_2-\gamma-\beta} d^{1-\alpha} k\tau$$

$$+\frac{1}{\varepsilon}C(\alpha)\sigma G \int_0^t \int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-m)}}{(t-s)^\alpha} v(m)dm\ ds.$$

Here, the second integral has the form

$$\frac{M_2}{\varepsilon}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2(\tau-m)}dm \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s} ds = M_2 C(\alpha)\sigma T(t,\tau),$$

where

$$T(t,\tau) = \frac{1}{\varepsilon}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2(\tau-m)}dm \int_0^t \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}}{(t-s)^\alpha} ds.$$

We make the change of variable $\tau - m = \xi$. Let $t \geq \varepsilon^\theta$. Then

$$T(t,\tau) = \frac{1}{\varepsilon}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi \int_0^t \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}}{(t-s)^\alpha}ds$$

$$= \frac{1}{\varepsilon}\int_0^{\varepsilon^\theta} \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}}{(t-s)^\alpha}ds \int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi + \frac{1}{\varepsilon}\int_{\varepsilon^\theta}^t \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}}{(t-s)^\alpha}ds \int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi$$

$$\leq \frac{1}{\varepsilon}\int_0^{\varepsilon^\theta} \frac{1}{(\varepsilon^\theta-s)^\alpha}ds \int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi + \frac{1}{\varepsilon}\int_{\varepsilon^\theta}^t \frac{e^{-\frac{1}{\varepsilon^{1-\theta}}(d_2-\gamma-\beta)}}{(t-s)^\alpha}ds \int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi$$

$$= \frac{1}{1-\alpha}\frac{1}{\varepsilon^{1-\theta(1-\alpha)}}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi + \frac{1}{\varepsilon}\frac{1}{1-\alpha}d^{1-\alpha}e^{-\frac{1}{\varepsilon^{1-\theta}}(d_2-\gamma-\beta)}\tau$$

$$\leq \frac{1}{1-\alpha}\frac{1}{\varepsilon^{1-\theta(1-\alpha)}}\left(\int_0^\tau 1 d\xi\right)^{\theta(1-\alpha)}\left(\int_0^\tau \left(e^{-\frac{1}{\varepsilon}d_2\xi}\right)^{\frac{1}{1-\theta(1-\alpha)}}d\xi\right)^{1-\theta(1-\alpha)} + \tilde{C}(\theta)\tau$$

$$\leq \frac{1}{1-\alpha}\frac{1}{\varepsilon^{1-\theta(1-\alpha)}}\tau^{\theta(1-\alpha)}\left[\frac{\varepsilon(1-\theta(1-\alpha))}{d_2}\right]^{1-\theta(1-\alpha)} + \tilde{C}(\theta)\tau$$

$$= \tilde{\tilde{C}}(\theta)\tau^{\theta(1-\alpha)} + \tilde{C}(\theta)\tau.$$

Let $t < \varepsilon^\theta$. Then

$$T(t,\tau) \leq \frac{1}{\varepsilon}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi \int_0^t \frac{1}{(t-s)^\alpha}ds = \frac{1}{\varepsilon}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi \frac{1}{1-\alpha}t^{1-\alpha}$$

$$\leq \frac{1}{1-\alpha}\frac{1}{\varepsilon^{1-\theta(1-\alpha)}}\int_0^\tau e^{-\frac{1}{\varepsilon}d_2\xi}d\xi \leq \tilde{\tilde{C}}(\theta)\tau^{\theta(1-\alpha)}.$$

Furthermore,

$$\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}Y_0 ds$$

$$= \int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}ds \left\|\int_0^\tau \frac{1}{\varepsilon}Be^{\frac{1}{\varepsilon}Bm}y_0 dm\right\|$$

$$= \frac{1}{\varepsilon}C(\alpha)\sigma\int_0^t \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}}{(t-s)^\alpha}ds \int_0^\tau e^{-\frac{1}{\varepsilon}d_2 m}\|By_0\|dm$$

$$= \frac{1}{\varepsilon}C(\alpha)\sigma\|By_0\|\int_0^t \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)s}}{(t-s)^\alpha}ds \int_0^\tau e^{-\frac{1}{\varepsilon}d_2 m}dm$$

$$= C(\alpha)\sigma\|By_0\|T(t,\tau) \leq C(\tau^{\theta(1-\alpha)} + \tau).$$

Hence,

$$\int_0^t \frac{C(\alpha)}{(t-s)^\alpha}\sigma u(s)ds \leq C_5(\tau^{\theta(1-\alpha)}+\tau)$$
$$+\frac{1}{\varepsilon}C(\alpha)\sigma G\int_0^t\int_0^s \frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-m)}}{(t-s)^\alpha}v(m)dm\ ds.$$

Let $q>0$ be a constant. We estimate $\mid v\mid_q$.

$$e^{-qt}v(t)\leq C(d_1)e^{d_1t}\|e^{A\tau}A^\alpha x_0-A^\alpha x_0\|e^{-qt}+M_1\frac{C(\alpha)}{1-\alpha}C(d_1)e^{d_1t}e^{-qt}\tau^{1-\alpha}$$
$$+e^{-qt}\int_0^t\frac{C(\alpha)}{(t-s)^\alpha}[Ge^{qs}e^{-qs}v(s)+k\tau]ds+C_5(\tau^{\theta(1-\alpha)}+\tau)e^{-qt}$$
$$+\frac{1}{\varepsilon}C(\alpha)\sigma Ge^{-qt}\int_0^t\int_0^s\frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-m)}}{(t-s)^\alpha}e^{qm}e^{-qm}v(m)dm\ ds$$
$$\leq C(d_1)e^{d_1t}\left\|\int_0^\tau Ae^{As}A^\alpha x_0ds\right\|+M_1\frac{C(\alpha)}{1-\alpha}C(d_1)e^{d_1t}\tau^{1-\alpha}$$
$$+e^{-qt}\mid v\mid_q\int_0^t\frac{C(\alpha)}{(t-s)^\alpha}Ge^{qs}ds+\frac{C(\alpha)}{1-\alpha}d^{1-\alpha}k\tau+C_5(\tau^{\theta(1-\alpha)}+\tau)$$
$$+\frac{1}{\varepsilon}C(\alpha)\sigma Ge^{-qt}\mid v\mid_q\int_0^t\int_0^s\frac{e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(s-m)}}{(t-s)^\alpha}e^{qm}dm\ ds$$
$$\leq C(d_1)e^{d_1t}\|Ax_0\|\int_0^\tau\frac{C(\alpha)}{s^\alpha}ds+M_1\frac{C(\alpha)}{1-\alpha}C(d_1)e^{d_1t}\tau^{1-\alpha}$$
$$+\mid v\mid_q C(\alpha)G\left[\frac{1}{j^\alpha}\frac{1}{q}+\frac{j^{1-\alpha}}{1-\alpha}\right]+\frac{C(\alpha)}{1-\alpha}d^{1-\alpha}k\tau+C_5(\tau^{\theta(1-\alpha)}+\tau)$$
$$+\mid v\mid_q C(\alpha)\sigma G\frac{1}{d_2-\gamma-\beta}\left[\frac{1}{j^\alpha}\frac{1}{q}+\frac{j^{1-\alpha}}{1-\alpha}\right]$$
$$\leq\mid v\mid_q C(\alpha)G\Big(1+\frac{\sigma}{d_2-\gamma-\beta}\Big)\left[\frac{1}{j^\alpha}\frac{1}{q}+\frac{j^{1-\alpha}}{1-\alpha}\right]+C_6\Big(\tau+\tau^{1-\alpha}+\tau^{\theta(1-\alpha)}\Big),$$

where j is an arbitrary number from interval $(0,d)$. Hence,

$$\mid v\mid_q\leq C_7\left[\frac{1}{j^\alpha}\frac{1}{q}+\frac{j^{1-\alpha}}{1-\alpha}\right]\mid v\mid_q+C_6(\tau+\tau^{1-\alpha}+\tau^{\theta(1-\alpha)}).$$

We choose j such that

$$\frac{j^{1-\alpha}}{1-\alpha} < \frac{1}{2C_7}.$$

Furthermore, we choose q in such a way

$$\frac{1}{j^\alpha}\frac{1}{q} < \frac{1}{2C_7}.$$

Then

$$1 - C_7\left[\frac{1}{j^\alpha}\frac{1}{q} + \frac{j^{1-\alpha}}{1-\alpha}\right] > 0$$

and

$$\mid v \mid_q \leq \frac{C_6\tau^{\theta(1-\alpha)}(\tau^{1-\theta(1-\alpha)} + \tau^{(1-\alpha)(1-\theta)} + 1)}{1 - C_7[\frac{1}{j^\alpha}\frac{1}{q} + \frac{j^{1-\alpha}}{1-\alpha}]}.$$

Finally, we have

$$\|\bar{x_\varepsilon}(t+\tau) - \bar{x_\varepsilon}(t)\| \leq C_8\tau^{\theta(1-\alpha)}$$

for every $\tau \in [0,d]$ and for every $t \in [0, d-\tau]$. By substituting (11) in (13) we obtain

$$u(t) \leq \frac{1}{\varepsilon}\|By_0\|\tau e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t} + \frac{1}{\varepsilon}M_2\tau e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t}$$

$$+\frac{1}{\varepsilon}\int_0^t e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)(t-s)}[GC_8\tau^{\theta(1-\alpha)} + k\tau]ds$$

$$\leq \frac{1}{\varepsilon}\|By_0\|\tau e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t} + \frac{1}{\varepsilon}M_2 e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t} + C_9\tau^{\theta(1-\alpha)}.$$

Thus,

$$\|\bar{y_\varepsilon}(t+\tau) - \bar{y_\varepsilon}(t)\| \leq C_{10}\tau^{\theta(1-\alpha)}\left(1 + \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}(d_2-\gamma-\beta)t}\right).$$

Denoting $C(\theta) = \max\{C_8, C_{10}\}$ we get (11), (12). Inequalities (11) and (12) mean that the functions $\bar{x_\varepsilon}$ satisfy Hölder condition on $[0,d]$ with exponent $\theta(1-\alpha)$ and constant $C(\theta)$ and the functions $\bar{y_\varepsilon}$ satisfy Hölder condition on $[\delta(\varepsilon), d]$ with exponent $\theta(1-\alpha)$ and constant $l = 2C(\theta)$. ■

3. Main result

We have the following main result.

Theorem 3.1. *Assume that the conditions $A_0 - A_6$ are satisfied for some $\alpha \in (0,1)$. Then for every $\varepsilon_0 > 0$ there exists a constant $L > 0$ such that the map $\varepsilon \longrightarrow Z_L(\varepsilon)$ is nonempty-valued for every $\varepsilon \in [0,\varepsilon_0]$ and upper semicontinuous at $\varepsilon = 0$ in $C[0,d] \times C[\delta,d]$ for every $\delta \in (0,d]$.*

Proof. In Lemma 2.2 we proved that the set $\hat{Z}_l(\varepsilon)$ is nonempty for all $\varepsilon \in (0,\varepsilon_0]$. We shall prove now that the set $\hat{Z}_l(0)$ is nonempty. Let $\varepsilon_k \in (0,\varepsilon_0]$, $\varepsilon_k \to 0$ and (x_k, y_k) be a corresponding sequence of solutions to $(3') - (4')$ satisfying condition (11) and (12). We extend in a continuous way y_k on the interval $[0,2d]$ assuming that it is constant outside of $(0,d)$.

Since $\delta(\varepsilon_k) \geq \frac{\varepsilon_k(ln(1/\varepsilon_k))}{d_2-\gamma-\beta}$, from (12) we obtain

$$\|y_k(\delta(\varepsilon_k)+\tau) - y_k(\delta(\varepsilon_k))\| \leq 2C(\theta)\tau^{\theta(1-\alpha)}$$

for every $\tau \in [0,d]$. We consider functions $\tilde{y}_k$, defined in the following way

$$\tilde{y}_k(t) = \begin{cases} y_k(t) & \delta(\varepsilon_k) \leq t \leq 2d \\ y_k(\delta(\varepsilon_k)) & 0 \leq t < \delta(\varepsilon_k). \end{cases}$$

Then from (12) it follows that for every $t, \tau \in [0,d]$

$$\|\tilde{y}_k(t+\tau) - \tilde{y}_k(t)\| \leq 2C(\theta)\tau^{\theta(1-\alpha)}.$$

Hence, the sequence $(x_k, \tilde{y}_k)$ satisfies an Hölder condition on $[0,d]$. Now we shall prove the relative compactness of the sets $\{x_k(t^*)\}$ and $\{\tilde{y}_k(t^*)\}$ for every fixed $t^* \in [0,d]$.

$$x_k(t^*) = e^{At^*}A^\alpha x_0 + \int_0^{t^*} A^\alpha e^{A(t^*-s)}[f_k^1(s) + b_{12}(A^{-\alpha}x_k(s))y_k(s)]ds.$$

We first verify the relative compactness of the set

$$\left\{ \int_0^{t^*} A^\alpha e^{A(t^*-s)} \left[f_k^1(s) + b_{12}(A^{-\alpha}x_k(s))y_k(s)\right] ds \right\}.$$

Let α_0 such that $\alpha + \alpha_0 < 1$. Then

$$\int_0^{t^*} A^\alpha e^{A(t^*-s)} \left[f_k^1(s) + b_{12}(A^{-\alpha}x_k(s))y_k(s)\right] ds$$
$$= A^{-\alpha_0} \int_0^{t^*} A^{\alpha+\alpha_0} e^{A(t^*-s)} \left[f_k^1(s) + b_{12}(A^{-\alpha}x_k(s))y_k(s)\right] ds.$$

Since the operator $A^{-\alpha_0}$ is completely continuous it is sufficient to prove boundedness of the set

$$\left\{ \int_0^{t^*} A^{\alpha+\alpha_0} e^{A(t^*-s)} \left[f_k^1(s) + b_{12}(A^{-\alpha} x_k(s)) y_k(s) \right] ds \right\}.$$

For this we evaluate

$$\begin{aligned}
&\left\| \int_0^{t^*} A^{\alpha+\alpha_0} e^{A(t^*-s)} \left[f_k^1(s) + b_{12}(A^{-\alpha} x_k(s)) y_k(s) \right] ds \right\| \\
&\leq \int_0^{t^*} \| A^{\alpha+\alpha_0} e^{A(t^*-s)} \| \| f_k^1(s) \| ds \\
&+ \int_0^{t^*} \| A^{\alpha+\alpha_0} e^{A(t^*-s)} \| \| b_{12}(A^{-\alpha} x_k(s)) \| \| y_k(s) \| ds \\
&\leq \int_0^{t^*} \frac{C(\alpha+\alpha_0)}{(t^*-s)^{\alpha+\alpha_0}} K_1 ds + \int_0^{t^*} \frac{C(\alpha+\alpha_0)}{(t^*-s)^{\alpha+\alpha_0}} \sigma K_2 ds \leq \frac{\hat{C} d^{1-(\alpha+\alpha_0)}}{1-(\alpha+\alpha_0)}.
\end{aligned}$$

We consider now $\{\tilde{y}_k(t^*)\}$.

$$\tilde{y}_k(t^*) = e^{\frac{1}{\varepsilon_k} B t^*} y_0 + \frac{1}{\varepsilon_k} \int_0^{t^*} B^{-\alpha} B^{\alpha} e^{\frac{1}{\varepsilon_k} B(t^*-s)} [f_k^2(s) + b_{21}(A^{-\alpha} x_k(s)) + b_{22} y_k(s)] ds.$$

Let us prove the boundedness of the set

$$\left\{ \frac{1}{\varepsilon_k} \int_0^{t^*} B^{\alpha} e^{\frac{1}{\varepsilon_k} B(t^*-s)} \left[f_k^2(s) + b_{21}(A^{-\alpha} x_k(s)) + b_{22} y_k(s) \right] ds \right\}.$$

In the sequel, we use the following estimate (see [7])

$$\| B^{\alpha} e^{Bt} \| \leq \frac{C(\alpha)}{t^{\alpha}} \| e^{Bt} \|^{1-\frac{\alpha}{n}},$$

where $0 < \alpha < 1$ and n is a positive integer greater than α. For definiteness we take $n = 2$. Consider

$$\begin{aligned}
&\left\| \frac{1}{\varepsilon_k} \int_0^{t^*} B^\alpha e^{\frac{1}{\varepsilon_k} B(t^*-s)} \left[f_k^2(s) + b_{21}(A^{-\alpha} x_k(s)) + b_{22} y_k(s) \right] ds \right\| \\
&\leq C \int_0^{t^*} \frac{1}{\varepsilon_k} \left\| B^\alpha e^{\frac{1}{\varepsilon_k} B(t^*-s)} \right\| ds \leq C \int_0^{t^*} \frac{1}{\varepsilon_k} \frac{C(\alpha)}{(t^*-s)^\alpha} {\varepsilon_k}^\alpha \left\| e^{\frac{1}{\varepsilon_k} B(t^*-s)} \right\|^{1-\frac{\alpha}{2}} ds \\
&\leq C \int_0^{t^*} \frac{1}{\varepsilon_k} \frac{C(\alpha)}{(t^*-s)^\alpha} {\varepsilon_k}^\alpha e^{-\frac{(1-\frac{\alpha}{2})}{\varepsilon_k} d_2 (t^*-s)} ds \\
&= CC(\alpha) \int_0^{t^*-\varepsilon_k} \frac{1}{\varepsilon_k} \frac{{\varepsilon_k}^\alpha}{(t^*-s)^\alpha} e^{-\frac{(1-\frac{\alpha}{2})}{\varepsilon_k} d_2 (t^*-s)} ds \\
&+ CC(\alpha) \int_{t^*-\varepsilon_k}^{t^*} \frac{1}{\varepsilon_k} \frac{{\varepsilon_k}^\alpha}{(t^*-s)^\alpha} e^{-\frac{(1-\frac{\alpha}{2})}{\varepsilon_k} d_2 (t^*-s)} ds \\
&\leq CC(\alpha) \int_0^{t^*-\varepsilon_k} \frac{1}{\varepsilon_k} \frac{{\varepsilon_k}^\alpha}{\varepsilon_k^\alpha} e^{-\frac{(1-\frac{\alpha}{2})}{\varepsilon_k} d_2 (t^*-s)} ds + CC(\alpha) \int_{t^*-\varepsilon_k}^{t^*} \frac{1}{\varepsilon_k} \frac{{\varepsilon_k}^\alpha}{(t^*-s)^\alpha} ds \\
&\leq \frac{CC(\alpha)}{(1-\frac{\alpha}{2}) d_2} + \frac{CC(\alpha)}{1-\alpha}.
\end{aligned}$$

Thus we have proved the relative compactness of the set $\{y_k(t^*)\}$, with $k \in N$ such that $\varepsilon_k < t^*$. Since $\varepsilon_k \to 0$ as $k \to \infty$ and t^* is fixed, the set $\{\tilde{y}_k(t^*)\}$ differs from the set $\{y_k(t^*)\}$ by a finite number of elements and so $\{\tilde{y}_k(t^*)\}$ is also relatively compact. From Ascoli-Arzela theorem it follows that the set $\{(x_k, \tilde{y}_k)\}$ is relatively compact in $C[0,d] \times C[0,d]$. Without loss of generality, we can assume that the sequence $(x_k, \tilde{y}_k)$ converges to some $(x^0, y^0) \in C[0,d] \times C[0,d]$. It is clear that the functions x^0 and y^0 also satisfy a Hölder condition with exponent $\theta(1-\alpha)$ and constant l.

Let us prove now that $(x^0, y^0) \in \hat{Z}_l(0)$. One can show (see [3]) that the sequences $\{f_k^i\}$, $i = 1, 2,$ are uniformly bounded and weakly compact in $L^1([0,d], E_i)$ and $f_k^i \overset{weakly}{\longrightarrow} f_0^i$, where $f_0^i(s) \in \psi_i(s, A^{-\alpha} x^0(s))$ for a.a. $s \in [0,d]$.

We consider the sequence $\{x_k(t)\}$ and show that $\int_0^t A^\alpha e^{A(t-s)} f_k^1(s) ds$ converges weakly in $L^1[0,d]$ to $\int_0^t A^\alpha e^{A(t-s)} f_0^1(s) ds$. Let $\varphi \in (L^1([0,d], E_1))^*$ and $\lambda > 0$. Let us denote by $g_k(t) = \int_0^t A^\alpha e^{A(t-s)} f_k^1(s) ds$ and $g_0(t) = \int_0^t A^\alpha e^{A(t-s)}\ f_0^1(s) ds$. Let $\mu \in [0,d]$ be a fixed number. We put

$$\bar{\varphi}_k(t) = \begin{cases} 0 & t \le \mu, \\ \int_0^{t-\mu} A^\alpha e^{A(t-s)} f_k^1(s)ds + \int_{t-\mu}^t A^\alpha e^{A(t-s)} f_k^1(s)ds & \mu < t \le d, \end{cases}$$

$$\bar{\bar{\varphi}}_k(t) = \begin{cases} \int_0^t A^\alpha e^{A(t-s)} f_k^1(s)ds & t \le \mu, \\ 0 & \mu < t \le d, \end{cases}$$

$$\bar{\varphi}_0(t) = \begin{cases} 0 & t \le \mu, \\ \int_0^{t-\mu} A^\alpha e^{A(t-s)} f_0^1(s)ds + \int_{t-\mu}^t A^\alpha e^{A(t-s)} f_0^1(s)ds & \mu < t \le d, \end{cases}$$

$$\bar{\bar{\varphi}}_0(t) = \begin{cases} \int_0^t A^\alpha e^{A(t-s)} f_0^1(s)ds & t \le \mu, \\ 0 & \mu < t \le d. \end{cases}$$

Then $g_k(t) = \bar{\varphi}_k(t) + \bar{\bar{\varphi}}_k(t)$ and $g_0(t) = \bar{\varphi}_0(t) + \bar{\bar{\varphi}}_0(t)$. Further

$$|< \varphi, g_k - g_0 >|=|< \varphi, \bar{\varphi}_k - \bar{\varphi}_0 + \bar{\bar{\varphi}}_k - \bar{\bar{\varphi}}_0 >|\le|< \varphi, \bar{\varphi}_k - \bar{\varphi}_0 >| + |< \varphi, \bar{\bar{\varphi}}_k - \bar{\bar{\varphi}}_0 >| \,.$$

Let us consider the second term

$$|< \varphi, \bar{\bar{\varphi}}_k - \bar{\bar{\varphi}}_0 >|\le \|\varphi\|\|\bar{\bar{\varphi}}_k - \bar{\bar{\varphi}}_0\|_{L^1[0,d]}.$$

For $t < \mu$ we have

$$\left\|\int_0^t A^\alpha e^{A(t-s)}(f_k^1(s) - f_0^1(s))ds\right\| \le \int_0^t \frac{C(\alpha)}{(t-s)^\alpha} \left\|f_k^1(s) - f_0^1(s)\right\| ds \le \frac{MC(\alpha)}{1-\alpha} t^{1-\alpha} \le \frac{MC(\alpha)}{1-\alpha}\mu^{1-\alpha}.$$

We choose a μ satisfying the following inequality

$$\frac{MC(\alpha)}{1-\alpha}\mu^{1-\alpha} < \frac{\lambda}{3d\|\varphi\|}.$$

Then,

$$\|\bar{\bar{\varphi}}_k - \bar{\bar{\varphi}}_0\|_{L^1[0,d]} = \int_0^d \|\bar{\bar{\varphi}}_k(t) - \bar{\bar{\varphi}}_0(t)\|dt < \int_0^d \frac{\lambda}{3d\|\varphi\|}dt = \frac{\lambda}{3\|\varphi\|}. \tag{14}$$

We put

$$m_k(t) = \begin{cases} 0 & t \le \mu, \\ \displaystyle\int_0^{t-\mu} A^\alpha e^{A(t-s)} f_k^1(s)ds & \mu < t \le d, \end{cases}$$

$$n_k(t) = \begin{cases} 0 & t \le \mu, \\ \displaystyle\int_{t-\mu}^{t} A^\alpha e^{A(t-s)} f_k^1(s)ds & \mu < t \le d, \end{cases}$$

$$m_0(t) = \begin{cases} 0 & t \le \mu, \\ \displaystyle\int_0^{t-\mu} A^\alpha e^{A(t-s)} f_0^1(s)ds & \mu < t \le d, \end{cases}$$

$$n_0(t) = \begin{cases} 0 & t \le \mu, \\ \displaystyle\int_{t-\mu}^{t} A^\alpha e^{A(t-s)} f_0^1(s)ds & \mu < t \le d. \end{cases}$$

Then $\bar\varphi_k(t) = m_k(t) + n_k(t)$ and $\bar\varphi_0(t) = m_0(t) + n_0(t)$. Therefore

$$|< \varphi, \bar\varphi_k - \bar\varphi_0 >| \le |< \varphi, m_k - m_0 >| + |< \varphi, n_k - n_0 >|,$$

$$|< \varphi, n_k - n_0 >| \le \|\varphi\| \|n_k - n_0\|_{L^1[0,d]}.$$

It is easy to verify that

$$\left\| \int_{t-\mu}^{t} A^\alpha e^{A(t-s)} (f_k^1(s) - f_0^1(s))ds \right\| \le \frac{MC(\alpha)}{1-\alpha} \mu^{1-\alpha}.$$

Thus,

$$\|n_k - n_0\|_{L^1[0,d]} < \frac{\lambda}{3\|\varphi\|}. \tag{15}$$

Let us consider $|< \varphi, m_k - m_0 >|$. Since the linear operator $\int_0^{t-\mu} A^\alpha e^{A(t-s)}$ is continuous in the strong topology of $L^1[0,d]$, it is continuous in the weak topology, i.e. $\int_0^{t-\mu} A^\alpha e^{A(t-s)}(f_k^1(s) - f_0^1(s))ds \to 0$ weakly in $L^1[0,d]$. Therefore, there exists $k = k(\lambda)$ such that for all $k \ge K$

$$|< \varphi, m_k - m_0 >| < \frac{\lambda}{3}. \tag{16}$$

It follows from (14), (15), (16) that $|< \varphi, g_k - g_0 >| < \lambda$, for $k \ge K$.

Therefore, the weak convergence of $\int_0^t A^\alpha e^{A(t-s)} f_k^1(s)ds$ to $\int_0^t A^\alpha e^{A(t-s)} f_0^1(s)$ is proved.

Now we shall prove that $\{b_{12}(A^{-\alpha}x_k(s))y_k(s)\}$ weakly converges to $b_{12}(A^{-\alpha}x^0(s))y^0(s)$ in $L^1[0,d]$. It was shown above that for all $t^* \in [0,d]$ the set $\{y_k(t^*)\}$ is relatively compact. It follows from the Distel principle (see [3]) that the sequence $\{y_k\}$ is weakly compact in $L^1([0,d],E_2)$. Let $\{y_k\}$ converges to g_* weakly in $L^1([0,d],E_2)$. We shall show that $\{\tilde{y}_k\}$ also weakly converges to g_* in the space $L^1([0,d],E_2)$. Since the space E_2^* satisfies the Radon-Nikodym property, $(L^1([0,d],E_2))^* = L^\infty([0,d],E_2^*)$ (see [3]). Let ψ be an arbitrary linear bounded functional defined on $L^1([0,d],E_2)$ and evaluate

$$|<\psi, \tilde{y}_k - g_*>| \le |<\psi, \tilde{y}_k - y_k>| + |<\psi, y_k - g_*>| \, .$$

$$\tilde{y}_k(t) - y_k(t) = \begin{cases} y_k(\delta(\varepsilon_k)) - e^{\frac{1}{\varepsilon_k}Bt} y_0 - \frac{1}{\varepsilon_k}\displaystyle\int_0^t e^{-\frac{1}{\varepsilon_k}B(t-s)} [f_k^2(s) & \\ \qquad + b_{21}(A^{-\alpha}x_k(s))y_k(s) + b_{22}y_k(s)]ds & t < \delta(\varepsilon_k), \\ \qquad 0 & t \ge \delta(\varepsilon_k). \end{cases}$$

$$|<\psi, \tilde{y}_k(t) - y_k(t)>| \le \int_0^{\delta(\varepsilon_k)} \|\psi(t)\|_{E_2^*} \|\tilde{y}_k(t) - y_k(t)\| dt \le C\delta(\varepsilon_k) \overset{k\to\infty}{\longrightarrow} 0.$$

Therefore, $|<\psi, \tilde{y}_k - g_*>| \to 0$ for $k \to \infty$ and $\tilde{y}_k \overset{weakly}{\longrightarrow} g_*$. But $\tilde{y}_k \overset{C[0,d]}{\longrightarrow} y^0$, hence $y_k \overset{weakly}{\longrightarrow} y^0$.

Let $\varphi \in (L^1([0,d],E_1))^*$. We observe

$$\begin{aligned} &|<\varphi, b_{12}(A^{-\alpha}x_k)y_k - b_{12}(A^{-\alpha}x^0)y^0>| \\ &\le |<(b_{12}(A^{-\alpha}x_k))^*\varphi - (b_{12}(A^{-\alpha}x^0))^*\varphi, y_k>| \\ &+ |<(b_{12}(A^{-\alpha}x^0))^*\varphi, y_k - y^0>| \, . \end{aligned}$$

The second term tends to zero in virtue of the weak convergence of y_k to y^0.

We consider now the first term

$$\begin{aligned} &|<(b_{12}(A^{-\alpha}x_k) - b_{12}(A^{-\alpha}x^0))^*\varphi, y_k>| \\ &= |<\varphi, (b_{12}(A^{-\alpha}x_k) - b_{12}(A^{-\alpha}x^0))y_k>| \\ &\le \|\varphi\|_{(L^1([0,d],E_1))^*} \|(b_{12}(A^{-\alpha}x_k) - b_{12}(A^{-\alpha}x^0))y_k\|_{L^1([0,d],E_1)}. \end{aligned}$$

Moreover,

$$\|(b_{12}(A^{-\alpha}x_k) - b_{12}(A^{-\alpha}x^0))y_k\|_{L^1([0,d],E_1)}$$

$$= \int_0^d \|(b_{12}(A^{-\alpha}x_k(s)) - b_{12}(A^{-\alpha}x^0(s)))y_k(s)\|ds$$

$$\leq Cp\int_0^d \|x_k(s) - x^0(s)\|ds \leq Cpd \mid x_k - x^0 \mid_{C[0,d]} \to 0 \text{ for } k \to \infty.$$

Thus, x_k weakly converges in $L^1([0,d], E_1)$ to

$$e^{At}A^{\alpha}x_0 + \int_0^t A^{\alpha}e^{A(t-s)}f_0^1(s)ds + \int_0^t A^{\alpha}e^{A(t-s)}b_{12}(A^{-\alpha}x^0(s))y^0(s)ds.$$

On the other hand, $x_k \to x^0$ in $C[0,d]$. Therefore,

$$x^0(t) = e^{At}A^{\alpha}x_0 + \int_0^t A^{\alpha}e^{A(t-s)}f_0^1(s)ds + \int_0^t A^{\alpha}e^{A(t-s)}b_{12}(A^{-\alpha}x^0(s))y^0(s)ds.$$

We consider now the sequence $\{y_k\}$. It is not difficult to show that $b_{21}(A^{-\alpha}x_k)y_k$ weakly converges to $b_{21}(A^{-\alpha}x^0)y^0$ and $b_{22}y_k$ weakly converges to $b_{22}y^0$ in $L^1((0,d),E_2)$. Hence $f_k^2 + b_{21}(A^{-\alpha}x_k)y_k + b_{22}y_k$ weakly converges to $f_0^2 + b_{21}(A^{-\alpha}x^0)y^0 + b_{22}y^0$ in $L^1((0,d),E_2)$.

In [1] it was shown that from weak convergence in $L^1([0,d],E_2)$ of a sequence $\{g_k\}$ to g_0 it follows the weak convergence in $L^1([0,d],E_2)$ of the sequence $\frac{1}{\varepsilon_k}\int_0^t e^{\frac{1}{\varepsilon_k}B(t-s)}g_k(s)ds$ to $-B^{-1}g_0(t)$. Consequently,

$$e^{\frac{1}{\varepsilon_k}Bt}y_0 + \frac{1}{\varepsilon_k}\int_0^t e^{\frac{1}{\varepsilon_k}B(t-s)}[f_k^2(s) + b_{21}(A^{-\alpha}x_k(s))y_k(s) + b_{22}y_k(s)]ds$$

$$\xrightarrow{weakly} -B^{-1}[f_0^2(t) + b_{21}(A^{-\alpha}x^0(t))y^0(t) + b_{22}y^0(t)].$$

Then $\tilde{y}_k(t) \xrightarrow{weakly} -B^{-1}[f_0^2(t) + b_{21}(A^{-\alpha}x^0(t))y^0(t) + b_{22}y^0(t)]$.

On the other hand, $\tilde{y}_k \xrightarrow{C[0,d]} y^0$. Therefore,

$$y^0(t) = -B^{-1}[f_0^2(t) + b_{21}(A^{-\alpha}x^0(t))y^0(t) + b_{22}y^0(t)].$$

Thus, $(x^0, y^0) \in \hat{Z}_l(0)$. Note that every solution to the system $(3') - (4')$ satisfying $(11), (12)$ belongs to $\hat{Z}_l(\varepsilon)$.

We turn to the proof of the upper semicontinuity at the point $\varepsilon = 0$ of the map $\varepsilon \to \hat{Z}_l(\varepsilon)$. Let $\delta \in (0, d]$. It is easy to show that the set $\hat{Z}_l(\varepsilon)$ is compact in $C[0, d] \times C[\delta, d]$ for every $\varepsilon \geq 0$. By contradiction we suppose that $\hat{Z}_l(\cdot)$ is not upper semicontinuous at $\varepsilon = 0$ in $C[0, d] \times C[\delta, d]$, then there exists $\nu > 0$ and sequences $\varepsilon_k \to 0$, $z_k \in \hat{Z}_l(\varepsilon_k)$ such that the $C[0, d] \times C[\delta, d]$ distance from z_k, $k \in N$, to $\hat{Z}_l(0)$ is greater than ν. Let $z_k = (x_k, y_k)$. Using the sequence $\{(x_k, y_k)\}$ we form the sequence $\{(x_k, \tilde{y}_k)\}$ as above. By repeating the same arguments as before we obtain that $(x_k, \tilde{y}_k) \overset{C[0,d]}{\longrightarrow} (x^0, y^0) \in \hat{Z}_l(0)$, then $\tilde{y}_k \to y^0$ in $C[\delta, d]$.

Since $\delta(\varepsilon_k) \to 0$ as $k \to \infty$ we can find K such that for $k > K$ we have $\delta(\varepsilon_k) < \delta$ and hence $y_k = \tilde{y}_k$ on $[\delta, d]$. Therefore $x_k \overset{C[0,d]}{\longrightarrow} x^0$, $y_k \overset{C[\delta,d]}{\longrightarrow} y^0$, which is a contradiction. This completes the proof of the Theorem. ■

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