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ON INITIAL VALUE PROBLEMS FOR A CLASS OF FIRST ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS

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Abstract

We investigate the existence of solutions to first order initial value problems for differential inclusions subject to impulsive effects. We shall rely on a fixed point theorem for condensing maps to prove our results.

Keywords and phrases: impulsive initial value problem, set-valued map, condensing map, fixed point.

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1. Introduction

The present paper is devoted to the study of the existence of solutions to initial value problems for first order differential inclusions subject to impulsive effects.

The theory of impulsive differential equations appears naturally in the description of physical and biological phenomena that are subjected to instantaneous changes at some time instants called moments. This theory has received much attention in recent years, see for instance, Lakshmikantham, Bainov and Simeonov [14], Smoilenko and Perestyuk [20], Pierson Gorez [19], E. Liz [17] and D. Franco [8], Frigon and O'Regan [9], [11], Liz and Nieto [16], and Yujun and Erxin [21]. However, very few results are available for impulsive differential inclusions (see for example Benchohra and Boucherif [4], [5], Erbe and Krawcewicz [7], Frigon and O'Regan [10]).

The fundamental tools used in the existence proofs of all above mentioned works are essentially the fixed point arguments, nonlinear alternative of the Leray-Schauder type, degree theory, topological transversality theorem or the monotone iterative technique combined with upper and lower solutions.

Let $J := [0, T]$, $0 < T < \infty$ and consider a set $J' := \{t_1, t_2, \ldots, t_m\} \subset J$ with $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$.

Our objective is to establish existence results for the following problem

$$y' \in F(t, y), \quad t \in J \backslash J' \tag{1.1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \ k = 1, \ldots, m, \tag{1.2}$$

$$y(0) = y_0, \tag{1.3}$$

where $y_0 \in \mathbb{R}$, $I_k \in C(\mathbb{R}, \mathbb{R})$ for each $k = 1, 2, \ldots, m$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively and $F : J \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is a set valued map given by

$$F(t, y) := [\phi(t, y), \psi(t, y)] \ \text{ for all } \ (t, y) \in J \times \mathbb{R}, \tag{1.4}$$

where the functions ϕ, $\psi : J \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfy conditions that will be specified later. We should point out that such set valued map was also proposed in [6]. If $F(t, y)$ is a nonempty compact convex subset of $\mathbb{R}$, then

$F(t,y)$ is a compact interval and hence of the form indicated in (1.4). Of course, this includes the case when F is single-valued and $\phi(t,y) = \psi(t,y) = F(t,y)$.

In this paper, we give two existence results to (1.1) – (1.3). In our results, we do not assume any type of monotonicity condition on $I_k,\ k = 1,\dots,m$, which is usually the situation in the literature.

We use a fixed point approach to establish our existence results. In particular, we use a fixed point theorem for condensing maps as used by Martelli ([18]).

2. Preliminaries

In this section, we introduce the basic definitions and notations which will be used in the remainder.

Consider a function $f : J \times \mathbb{R} \to \mathbb{R}$, and $z \in \mathbb{R}$.
$f(t,.)$ is lower semi-continuous (lsc for short) at z if $f(t,z) \leq \liminf_{x\to z} f(t,x)$.
$f(t,.)$ is upper semi-continuous (usc for short) at z if $f(t,z) \geq \limsup_{x\to z} f(t,x)$.

Note that $f(t,.)$ is usc if and only if $-f(t,.)$ is lsc. f is said to be of type $\mathcal{M}$ if for every measurable function $y : J \to \mathbb{R}$, the function $t \longmapsto f(t,y(t))$ is measurable. A typical example of such a function is a Carathéodory function (see [12]). $C(J,\mathbb{R})$ is the Banach space of continuous functions $y : J \longrightarrow \mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\} \quad \text{for all} \quad y \in C(J,\mathbb{R}).$$

$AC(J,\mathbb{R})$ is the space of all absolutely continuous functions $y : J \longrightarrow \mathbb{R}$. For such functions the derivative y' exists almost everywhere.

$L^2(J,\mathbb{R})$ denotes the Banach space of Lebesgue measurable functions $y : J \longrightarrow \mathbb{R}$ for which $\int_0^T |y(t)|^2 dt < +\infty$, with the norm

$$\|y\|_{L^2} = \Big(\int_0^T |y(t)|^2 dt\Big)^{1/2} \quad \text{for all} \quad y \in L^2(J,\mathbb{R}).$$

$H^1(J,\mathbb{R})$ denotes the Banach space of functions $y : J \longrightarrow \mathbb{R}$ which are absolutely continuous and whose derivative y' is an element of $L^2(J,\mathbb{R})$ with the norm

$$\|y\|_{H^1} = \|y\|_{L^2} + \|y'\|_{L^2} \quad \text{for all } y \in H^1(J,\mathbb{R}).$$

In order to define the solution to (1.1) – (1.3) we shall consider the following spaces. $\Omega = \{y : [0,T] \longrightarrow \mathbb{R} : y$ is continuous on $J\backslash J'$, $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k) = y(t_k^-)$, $k = 1,\dots,m$ }. Evidently, Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t\in J} |y(t)|.$$

Let $\Omega^1 := \Omega \cap H^1(J,\mathbb{R})$. For each $y \in \Omega^1$ we let $\|y\| = \|y\|_{H^1}$. Hence Ω^1 is a Banach space.

Definition 1. By a solution to (1.1) – (1.3), we mean a function $y \in \Omega_0^1 := \{y \in \Omega^1 : y(0) = y_0\}$ that satisfies the differential inclusion

$$y'(t) \in F(t, y(t)) \text{ almost everywhere on } J\backslash J',$$

and for each $k = 1,\dots,m$ the function y satisfies the equations $y(t_k^+) = I_k(y(t_k^-))$.

Let $(X, \|.\|)$ be a Banach space. A set valued map $G : X \longrightarrow 2^X$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for any bounded subset B of X (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq N$.

G is said to be completely continuous if $G(B) = \cup_{x\in B} G(x)$ is relatively compact for every bounded subset $B \subseteq X$. G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following, $cc(X)$ denotes the set of all nonempty compact convex subsets of X.

An upper semi-continuous map $G : X \longrightarrow 2^X$ is said to be condensing if for any bounded subset $N \subseteq X$ with $\alpha(N) > 0$, we have $\alpha(G(N)) < \alpha(N)$, where α denotes the Kuratowski measure of noncompactness (see [2], [3], [18]).

We remark that a compact map is the simplest example of a condensing map. For more details on set valued maps see for instance Aubin-Frankowska [1], Deimling [6], Hu and Papageorgiou [13].

The following result, which is a generalization of the classical Schaeffer's theorem to set-valued mappings, is crucial in the proof of our main results.

Lemma 1 [18]**.** *Let X be a Banach space and $G : X \longrightarrow cc(X)$ a condensing map. If the set*

$$M := \{y \in X : \lambda y \in G(y) \ \ \textit{for some} \ \ \lambda > 1\}$$

is bounded, then G has a fixed point.

Let $F(t, y) := [\phi(t, y), \psi(t, y)]$. If ϕ and ψ are of type $\mathcal{M}$, then the set-valued map F is called of type $\mathcal{M}$.

Lemma 2 (Proposition VI.1. p. 40 [12])**.** *Assume that F is of type $\mathcal{M}$ and for each $k \geq 0$, there exists $\phi_k \in L^2(J, \mathbb{R})$ such that*

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \leq \phi_k(t) \ \ \textit{for} \ \ |y| \leq k.$$

Then the operator $\mathcal{F} : C(J, \mathbb{R}) \longrightarrow 2^{L^2(J,\mathbb{R})}$ defined by

$$\mathcal{F}y := \{h : J \longrightarrow \mathbb{R} \ \textit{measurable:} \ h(t) \in F(t, y(t)) \ \textit{a.e.} \ t \in J\}$$

is well defined, u.s.c., bounded on bounded sets in $C(J, \mathbb{R})$ and has convex values.

3. Main result

In this section, we state and prove our main result. For that purpose we shall assume that the functions ϕ and ψ that define the set valued map F, satisfy

(H1) ϕ and ψ are functions of type $\mathcal{M}$;

(H2) $\phi(t, .)$ is lsc and $\psi(t, .)$ is usc, with $\phi(t, y) \leq \psi(t, y)$ for all $(t, y) \in J \times \mathbb{R}$;

(H3) there exists $\theta : [0, \infty) \to (0, \infty)$ continuous such that $1/\theta \in L^2_{loc}([0, \infty))$ and

$$\max\{|\phi(t, y)|, |\psi(t, y)|\} \leq \theta(|y|) \ \ \text{for all} \ \ t \in J.$$

The first result induced by these asumptions is the following.

Proposition 1. *Suppose that the conditions* (H1) *and* (H2) *are satisfied. Then the set-valued map F is of type $\mathcal{M}$ and $F(t, .)$ is usc with compact convex values.*

Proof. The fact that F is of type $\mathcal{M}$ follows from the definition of functions of type $\mathcal{M}$ and (H1) (see [12] p. 8). Also, note that for each $y \in \mathbb{R}$ $F(t,y) = [\phi(t,y), \psi(t,y)]$ is a closed interval in $\mathbb{R}$. It follows from (H2) that $F(t,.)$ has compact and convex values (see [6], p. 5). ∎

Let $G : C(J,\mathbb{R}) \to L^2(J,\mathbb{R})$ be the set-valued map defined by

$$(Gy)(t) := F(t, y(t)) \quad \text{for all} \quad t \in J$$

that is

$$Gy := \{u : J \to \mathbb{R} \text{ measurable}: \; u(t) \in F(t,y(t)) \text{ for almost all } t \in J\}.$$

Proposition 2. *If the asumptions* (H1) – (H3) *are satisfied, then the set-valued operator G is well defined, usc, with convex values, and bounded on bounded subsets of $C(J,\mathbb{R})$.*

Proof. See proof of Proposition II.7. p. 16 in [12].

Our first result reads as follows.

Theorem 1. *Let $t_0 = 0, t_{m+1} = T$. Suppose that, in addition to* (H1), (H2) *and* (H3), *the following condition is satisfied*

(C1) *There exist $\{r_j\}_{j=0}^m$ and $\{s_j\}_{j=0}^m$ such that*

(i) $s_0 \le y_0 \le r_0$

(ii) $s_{j+1} \le \min\limits_{[s_j,r_j]} I_{j+1}(y) \le \max\limits_{[s_j,r_j]} I_{j+1}(y) \le r_{j+1}, \; j = 0,1,\ldots,m-1;$

(iii) $\phi(t,s_k) \ge 0$ *and* $\psi(t,r_k) \le 0$ *for all* $t \in [t_k, t_{k+1}], \; k = 0,\ldots,m.$

Then the impulsive initial value problem (1.1) – (1.3) *has at least one solution.*

Proof. This proof will be given in several steps.

Step 1. We restrict our attention to the problem on $[0,t_1]$, that is the initial value problem

$$\left\{\begin{array}{l} y'(t) \in F(t,y(t)), \quad t \in [(0,t_1)] \\ y(0) = y_0. \end{array}\right. \tag{3.1}$$

Define a modified set valued map F_1 relative to the pair (s_0, r_0) by

$$F_1(t,y) := \begin{cases} [\phi(t,s_0), \psi(t,s_0)] & \text{if} \quad y < s_0 \\ [\phi(t,y), \psi(t,y)] & \text{if} \quad s_0 \le y \le r_0 \\ [\phi(t,r_0), \psi(t,r_0)] & \text{if} \quad y > r_0 \end{cases}$$

and for all $t \in [0, t_1]$.

Consider the modified initial value problem

$$\begin{cases} y'(t) \in F_1(t, y(t)), \quad t \in [(0, t_1)] \\ y(0) = y_0. \end{cases} \tag{3.2}$$

Transform this problem into a fixed point problem. For this, consider the operators

$$L : H^1([0,t_1], \mathbb{R}) \longrightarrow L^2([0,t_1], \mathbb{R}) \text{ defined by } Ly = y',$$

$$j : H^1([0,t_1], \mathbb{R}) \longrightarrow C([0,t_1], \mathbb{R}) \text{ defined by } jy = y,$$

the completely continuous imbedding, and $G_1 : C([0,t_1], \mathbb{R}) \longrightarrow 2^{L^2([0,t_1],\mathbb{R})}$ defined by

$$G_1 y := \Big\{ u : [0,t_1] \longrightarrow \mathbb{R} \text{ measurable} : \ u(t) \in F_1(t, y(t)) \text{ for a.e. } t \in [0,t_1] \Big\}.$$

We can easily show that L is one-to-one and onto with a bounded inverse L^{-1}. It follows from Lemma 2 that G_1 is well defined, usc, bounded on bounded subsets of $C([0,t_1], \mathbb{R})$ and has convex values. It is clear that the solutions to problem (3.2) are solutions of the fixed point inclusion $y \in H_1(y)$ and vice-versa, where the set valued map H_1 is given by $H_1 = L^{-1} G_1 j$. Note that H_1 is compact (because G_1 is bounded on bounded subsets of $C([0,t_1), \mathbb{R})$ and j is completely continuous), usc and has convex values. Consequently H_1 is a condensing map.

Step 2. Consider the set

$$U_1 := \{ y \in C([0,t_1], \mathbb{R}) : \lambda y \in H_1(y) \ \text{ for some } \ \lambda > 1 \}.$$

Let $y \in U_1$. Then $\lambda y \in H_1(y)$, for some $\lambda > 1$. Hence, y satisfies the initial value problem

$$(3.3) \qquad \begin{cases} \lambda y'(t) \in (G_1 y)(t), & t \in [(0, t_1)] \\ y(0) = y_0. \end{cases}$$

This problem is equivalent to

$$\lambda y(t) \in \Big\{ y_0 + \int_0^t h(s)ds;\ h(s) \in F_1(s, y(s)),\ 0 \leq s \leq t \leq t_1 \Big\}.$$

This shows that for some $\lambda > 1$ and $h \in G_1 y$

$$y(t) = \lambda^{-1} y_0 + \lambda^{-1} \int_0^t h(s)ds,\ 0 \leq t \leq t_1.$$

Thus,

$$|y(t)| \leq |y_0| + \|h\|_{L^2} \text{ for all } t \in [0, t_1].$$

Now, since $h(t) \in F_1(t, y(t))$, if follows from the definition of $F_1(t, y)$ and the assumption (H3) that there exists a positive constant h_0 such that $\|h\|_{L^2} \leq h_0$. In fact

$$h_0 = \max\{|r_0|, |s_0|, \sup_{s_0 \leq y \leq r_0} |\theta(y)|\}.$$

Therefore, we have

$$|y(t)| \leq h_0 + |y_0| \text{ for all } t \in [0, t_1].$$

This yields

$$\|y\|_\infty \leq h_0 + |y_0|,$$

which shows that the set U_1 is bounded. It follows from the Lemma 1 that the set-valued map H_1 has a fixed point, which is a solution to our problem (3.2).

Step 3. We shall show that the solution y to (3.2) satisfies

$$s_0 \leq y(t) \leq r_0 \quad \text{for all} \quad t \in [0, t_1].$$

First, we prove that

$$s_0 \leq y(t) \quad \text{for all} \quad t \in [0, t_1].$$

Suppose on the contrary that there exist $\sigma_1, \sigma_2 \in [0, t_1]$, $\sigma_1 < \sigma_2$ such that $y(\sigma_1) = s_0$ and

$$s_0 > y(t) \quad \text{for all} \quad t \in (\sigma_1, \sigma_2).$$

This implies that

$$F_1(t, y(t)) = [\phi(t, s_0), \psi(t, s_0)] \text{ for all } t \in (\sigma_1, \sigma_2).$$

Hence, we have

$$y'(t) \in [\phi(t, s_0), \psi(t, s_0)],$$

which implies that

$$y'(t) \geq \phi(t, s_0) \text{ for all } t \in (\sigma_1, \sigma_2).$$

It follows that for all $t \in (\sigma_1, \sigma_2)$

$$y(t) \geq y(\sigma_1) + \int_{\sigma_1}^{t} \phi(s, s_0) ds.$$

Since $\phi(t, s_0) \geq 0$ for $t \in [0, t_1]$ we get

$$0 > y(t) - s_0 \geq \int_{\sigma_1}^{t} \phi(s, s_0) ds \geq 0 \text{ for all } t \in (\sigma_1, \sigma_2)$$

which is a contradiction. Thus $s_0 \leq y(t)$ for $t \in [0, t_1]$. Similarly, we can show that $y(t) \leq r_0$ for $t \in [0, t_1]$. Hence

$$s_0 \leq y(t) \leq r_0 \quad \text{for all} \quad t \in [0, t_1].$$

But for all $y \in [s_0, r_0]$ we have

$$F_1(t,y) = [\phi(t,y), \psi(t,y)] = F(t,y) \text{ for all } t \in [0,t_1].$$

This implies that y is a solution to the initial value problem (3.1). Denote this solution by y_1.

Step 4. Consider now the problem

$$\begin{cases} y' \in F_2(t,y), \quad t \in [(t_1,t_2)], \\ y(t_1^+) = I_1((y_1(t_1^-)), \end{cases} \tag{3.4}$$

where the set valued map F_2 is given by

$$F_2(t,y) := \begin{cases} [\phi(t,s_1), \psi(t,s_1)] & \text{if } \ y < s_1 \\ [\phi(t,y), \psi(t,y)] & \text{if } \ s_1 \le y \le r_1 \\ [\phi(t,r_1), \psi(t,r_1)] & \text{if } \ y > r_1 \end{cases}$$

Proceeding as in the above three steps we show that any solution of the problem (3.4) is a fixed point of the set valued map H_2 defined by $H_2 y := L^{-1} G_2 j y$ where $L^{-1} : L^2([t_1,t_2],\mathbb{R}) \to H^1([t_1,t_2],\mathbb{R})$, $j : H^1([t_1,t_2],\mathbb{R}) \longrightarrow C([t_1,t_2],\mathbb{R})$ is the completely continuous embedding, and

$$G_2 y := \Big\{ u : [t_1,t_2] \longrightarrow \mathbb{R} \text{ measurable}: \ u(t) \in F_2(t, y(t)) \text{ for a.e. } t \in [t_1,t_2] \Big\}.$$

Similarly, we can show that the set

$$U_2 := \{ y \in C([t_1,t_2],\mathbb{R}) : \lambda y \in H_2(y) \ \text{ for some } \ \lambda > 1 \}$$

is bounded. We again apply Lemma 1 to show that H_2 has a fixed point, which we denote by y_2, and so is a solution to problem (3.4) on the interval $[t_1,t_2]$.

We now show that $s_1 \le y_2(t) \le r_1$ for all $t \in [t_1,t_2]$. Since $y_1(t_1^-) \in [s_0,r_0]$ condition (C1) (ii) implies that

$$s_1 \le I_1(y_1(t_1^-)) \le r_1.$$

Hence

$$s_1 \leq y(t_1^+) \leq r_1.$$

Also, condition (C1) (iii) implies that

$$s_1 \leq y_2(t) \leq r_1 \quad \text{for} \quad t \in [t_1, t_2],$$

and hence y_2 is a solution to

$$\begin{cases} y' \in F(t, y), & t \in [(t_1, t_2)], \\ y(t_1^+) = I_1(y_1(t_1^-)). \end{cases}$$

Step 5. We continue the above process and construct solutions y_k on $[t_{k-1}, t_k]$, for $k = 3, \ldots, m+1$, to

$$\begin{cases} y' \in F(t, y), & t \in [(t_{k-1}, t_k)], \\ y(t_{k-1}^+) = I_{k-1}(y_{k-1}(t_{k-1}^-)), \end{cases}$$

with $s_{k-1} \leq y_k(t) \leq r_{k-1}$ for $t \in [t_{k-1}, t_k]$. Then

$$y(t) = \begin{cases} y_1(t) & t \in [0, t_1] \\ y_2(t) & t \in [t_1, t_2] \\ \cdot & \\ \cdot & \\ \cdot & \\ y_{m+1}(t) & t \in [t_m, T] \end{cases}$$

is a solution to (1.1) – (1.3). This completes the proof of the theorem. ■

Using the same reasoning as that used above we can obtain the following result.

Theorem 2. *Let* $t_0 = 0, t_{m+1} = T$. *Suppose that, in addition to* (H1), (H2) *and* (H3), *the following condition is satisfied*

(C2) *There exist functions* $\{r_j\}_{j=0}^m$, $\{s_j\}_{j=0}^m$ *continuous on* $[t_j, t_{j+1}]$ *such that*

(i) $s_0(t) \le y_0 \le r_0(t)$ *for each* $t \in [0, t_1]$*;*

(ii) $s_j(t) \le r_j(t)$ *for each* $t \in [t_j, t_{j+1}],\ j = 1, \ldots, m$*;*

(iii) $s_{j+1}(t_{j+1}^+) \le \min_{[s_j(t_{j+1}^-), r_j(t_{j+1}^-)]} I_{j+1}(y) \le \max_{[s_j(t_{j+1}^-, r_j(t_{j+1}^-)]} I_{j+1}(y) \le r_{j+1}(t_{j+1}^+),\ j = 0, 1, \ldots, m-1$;

(iv) $\int_{\alpha_j}^{\beta_j} \phi(t, s_j(t))dt \ge s_j(\beta_j) - s_j(\alpha_j)$ *and* $\int_{\alpha_j}^{\beta_j} \psi(t, r_j(t))dt \le r_j(\beta_j) - r_j(\alpha_j)$ *with* $t_j \le \alpha_j < \beta_j \le t_{j+1},\ j = 0, \ldots, m.$

Then the impulsive initial value problem (1.1) – (1.3) *has at least one solution.*

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