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NONLINEAR MULTIVALUED BOUNDARY VALUE PROBLEMS *

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Abstract

In this paper, we study nonlinear second order differential inclusions with a multivalued maximal monotone term and nonlinear boundary conditions. We prove existence theorems for both the convex and nonconvex problems, when $\mathrm{dom}A \neq \mathbb{R}^N$ and $\mathrm{dom}A = \mathbb{R}^N$, with A being the maximal monotone term. Our formulation incorporates as special cases the Dirichlet, Neumann and periodic problems. Our tools come from multivalued analysis and the theory of nonlinear monotone operators.

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1. Introduction

In this paper, we deal with the following multivalued boundary value problem:

$$(1)\quad \left\{\begin{array}{r} \left(\|x'(t)\|^{p-2}x'(t)\right)' \in A(x(t)) + F(t,x(t),x'(t)) \text{ a.e. on } T=[0,b] \\ (\varphi(x'(0)), -\varphi(x'(b))) \in \xi(x(0),x(b)), 2 \le p < \infty. \end{array}\right\}$$

Here $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map, $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a multifunction and $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $\varphi(r) = \|r\|^{p-2}r$. In a recent paper (see Bader-Papageorgiou [2]), the authors studied (1) with F single-valued and proved two existence theorems. This work extends the results investigated in Bader-Papageorgiou [2] to the multivalued boundary problem. At the same time it extends the set-valued results of Erbe-Krawcewicz [7], Frigon [9], Halidias-Papageorgiou [11] and Kandilakis-Papageorgiou [14], where the inclusion is semilinear (i.e. $p = 2$) and $A = 0$. Moreover, our boundary conditions are general nonlinear boundary conditions, which get a unified treatment of the classical boundary value problems, such as the Dirichlet problem, the Neumann problem and the periodic problem. Finally, we should also mention the recent works on problems involving the one dimensional p-Laplacian, by Boccardo-Drabek-Giachetti-Kucera [3], Dang-Oppenheimer [4], Del Pino-Elgueta-Manasevich [5], Del Pino-Manasevich-Murua [6], Fabry-Fayyad [8], Guo [10], and Manasevich-Mawhin [15]. We point out that in all these works F is single-valued, $A = 0$, the boundary conditions are among the classical ones (Dirichlet, Neumann and periodic) and with the exception of Manasevich-Mawhin, they all deal with the scalar problem (i.e. $N = 1$).

Our approach is based on notions and results from multivalued analysis and from the theory of nonlinear operators of monotone type. They lead to an eventual application of a generalized version of the Leray-Schauder alternative principle, proved recently by Bader [1]. In Section 2, we recall the basic definitions and facts from multivalued analysis and the theory of monotone operators, which we will need in the sequel. Our main sources are the books of Hu-Papageorgiou [13] and Zeidler [17].

2. Mathematical preliminaries

Let (Ω, Σ) be a measurable space and X a seperable Banach space. We introduce the following notations: $P_{f(c)}(X) = \{A \subseteq X : A$ is nonempty, closed (and convex)} and $P_{(w)k(c)}(X) = \{A \subseteq X : A$ is nonempty, (weakly-) compact (and convex)$\}$. A multifunction $F : \Omega \to P_f(X)$ is said to be *measurable*, if for all $x \in X, \omega \mapsto d(x, F(\omega))$ is measurable. Also we say that $F : \Omega \to 2^X \setminus \{\emptyset\}$ is *graph measurable*, if $\mathrm{Gr}F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions, measurability implies graph measurability, while the converse is true if Σ is complete (i.e. $\Sigma = \hat{\Sigma} =$ the universal σ-field). Recall that if μ is a measure on Σ and Σ is μ-complete, then $\Sigma = \hat{\Sigma}$. Now let (Ω, Σ, μ) be a finite measure space. Given a multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ and $1 \leq p \leq \infty$, we introduce the set $S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega)\ \ \mu\text{-a.e.}\}$. In general, this set may be empty. It is easy to check that if $\omega \mapsto \inf\{\|x\| : x \in F(\omega)\} \in L^p(\Omega)$, then $S_F^p \neq \emptyset$.

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z \setminus \{\emptyset\}$ is said to be *lower semicontinuous* (lsc for short) (resp. *upper semicontinuous* (usc for short)), if for all $C \subseteq Z$ closed, then the set $G^+(C) = \{y \in Y : G(y) \subseteq C\}$(resp. $G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$) is closed in Y. An usc multifunction G has a closed graph (i.e. $\mathrm{Gr}G = \{(y, z) \in Y \times Z : z \in G(y)\}$ is closed), while the converse is true if G is locally compact. Also if Z is a metric space, then G is lsc if and only if for every $y_n \to y$ in Y, we have $G(y) \subseteq \underline{\lim} G(y_n) = \{z \in Z : \lim d(z, G(y_n)) = 0\} = \{z \in Z : z = \lim z_n, z_n \in G(y_n), n \geq 1\}$.

Let X be a reflexive Banach space and X^* its dual. A map $A : D \subseteq X \to 2^{X^*}$ is said to be *monotone*, if for all $(x, x^*), (y, y*) \in \mathrm{Gr}A$, we have $(x^* - y^*, x - y) \geq 0$ (by $(\cdot, \cdot)$ we denote the duality brackets for the pair (X, X^*)). When $(x^* - y^*, x - y) = 0$ implies that $x = y$, then we say that A is *strictly monotone*. The map A is said to be *maximal monotone*, if $(x^* - y^*, x - y) \geq 0$ for all $(x, x^*) \in \mathrm{Gr}A$, imply that $(y, y*) \in \mathrm{Gr}A$. So according to this definition the graph of A is maximal with respect to the inclusion among the graphs of all monotone maps from X into 2^{X^*}. It is easy to see that a maximal monotone map A has a *demiclosed* graph, i.e. $\mathrm{Gr}A$ is sequentially closed in $X \times X_w^*$ or in $X_w \times X^*$ (here by X_w and X_w^* we denote the spaces X and X^* furnished with their respective weak topologies). If $A : X \to X^*$ is every where defined, single-valued map, we say that A is *demicontinuous*, if $x_n \to x$ in X implies that $A(x_n) \xrightarrow{w} A(x)$ in X^*. A map $A : X \to X^*$ which is monotone and demicontinuous, is maximal monotone. Also a map $A : D \subseteq X \to 2^{X^*}$ is said to be *coercive*, if $D \subseteq X$ is bounded or if D is unbounded and $\frac{\inf[(x^*, x) : x^* \in A(x)]}{\|x\|} \to \infty$ as $\|x\| \to \infty, x \in D$. A maximal monotone, coercive map is surjective.

Let Y, Z be Banach spaces and $K : Y \to Z$ a generally nonlinear map. We say

(a) K is *completely continuous*, if $y_n \xrightarrow{w} y$ in Y implies $K(y_n) \to K(y)$ in Z;

(b) K is *compact*, if K is continuous and maps bounded sets into relatively compact sets.

In general, these are two distinct notions. However, if Y is reflexive, then complete continuity implies compactness. Moreover, if Y is reflexive and K is linear, then the two notions are equivalent.

Finally, we will need the following generalization of the Leray- Schauder principle. Let X, Y be Banach spaces, $G : X \to P_{wkc}(Y)$ an usc multifunction from X into Y_w and $K : Y \to X$ a completely continuous map. We set $\Phi = K \circ G$. We have the following alternative principle (see Bader [1]):

Proposition 1. *If X, Y and Φ are as above and Φ is compact, then the set*

$$S = \{x \in X : x \in \lambda\Phi(x) \textit{ for some } 0 < \lambda < 1\}$$

is unbounded or otherwise Φ has a fixed point.

3. An auxiliary problem

In this section, we consider the following"regular" approximation to problem (1):

$$(2) \quad \left\{ \begin{array}{rcl} \left(\|x'(t)\|^{p-2}x'(t)\right)' & \in & A_\lambda(x(t)) + F(t, x(t), x'(t)) \text{ a.e. on } T \\ (\varphi(x'(0)), -\varphi(x'(b))) & \in & \xi(x(0), x(b)), \lambda > 0. \end{array} \right\}$$

Here for every $\lambda > 0, A_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ is the Yosida approximation of the maximal monotone map A. First we will establish the existence of solutions for problem (2), when F takes convex values ("convex problem"). For this purpose we introduce the following hypothesis on the data of (2).

$\underline{H(A)_1}$: $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $0 \in A(0)$.

Remark. In fact, it is enough to assume that $0 \in \text{dom}A = \{x \in \mathbb{R}^N : A(x) \neq \emptyset\}$ and then by translation we can have $0 \in A(0)$.

$\underline{H(F)_1}$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N,\ t \to F(t, x, y)$ is graph measurable;

(ii) for almost all $t \in T, (x, y) \to F(t, x, y)$ has a closed graph;

(iii) for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t, x, y)$, we have

$$(v, x)_{\mathbb{R}^N} \geq -a\|x\|^p - \gamma\|x\|^r\|y\|^{p-r} - c(t)\|x\|^s$$

with $a, \gamma \geq 0, 1 \leq r, s < p$ and $c \in L^1(T)$;

(iv) there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{\mathbb{R}^N} = 0$, then we can find $\delta > 0$ and $\xi > 0$ such that for almost all $t \in T$

$$\inf\left[(v, x)_{\mathbb{R}^N} + \|y\|^p : \|x - x_0\| + \|y - y_0\| < \delta, v \in F(t, x, y)\right] \geq \xi > 0;$$

(v) for almost all $t \in T$, all $x, y \in \mathbb{R}^N$and all $v \in F(t, x, y)$

$$\|v\| \leq \gamma_1(t, \|x\|) + \gamma_2(t, \|x\|)\|y\|^{p-1}$$

with $\sup_{0 \leq r \leq k} \gamma_1(t, r) \leq \eta_{1,k}(t)$ a.e. on T, $\eta_{1,k} \in L^q(T)$ ($\frac{1}{p} + \frac{1}{q} = 1$) and $\sup_{0 \leq r \leq k} \gamma_2(t, r) \leq \eta_{2,k}(t)$ a.e. on T, $\eta_{2,k} \in L^\infty(T)$.

Remark. Hypothesis $H(F)_1$ (iv) is an appropriate extension of the Nagumo-Hartman condition (see Hartman [12], p. 432–433).

$\underline{H(\xi)}$: $\xi : \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map such that $(0, 0) \in \xi(0, 0)$ and one of the following holds:

(i) for every $(a', d') \in \xi(a, d)$ we have $(a', a)_{\mathbb{R}^N} \geq 0$ and $(d', d)_{\mathbb{R}^N} \geq 0$;
or

(ii) $\mathrm{dom}\xi = \{(a, d) \in \mathbb{R}^N \times \mathbb{R}^N : a = d\}$.

Proposition 2. *If hypotheses $H(A)_1, H(F)_1$ and $H(\xi)$ hold, then problem* (2) *has a solution $x \in C^1(T, \mathbb{R}^N)$.*

Proof. Let

$$D = \{x \in C^1(T, \mathbb{R}^N) : \|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N),$$
$$(\varphi(x'(0)), -\varphi(x'(b))) \in \xi(x(0), x(b))\}$$

and let $V : D \subseteq L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be defined by
$V(x)(\cdot) = -\left(\|x'(\cdot)\|^{p-2}x'(\cdot)\right)'$, $x \in D$.

From Proposition 3 of Bader-Papageorgiou [2], we know that V is maximal monotone. Also let $\hat{A}_\lambda : L^p(T,\mathbb{R}^N) \to L^q(T,\mathbb{R}^N)$ be the Nemitsky operator corresponding to the Yosida approximation A_λ, i.e. $\hat{A}_\lambda(x)(\cdot) = A_\lambda(x(\cdot))$ and let $J : L^p(T,\mathbb{R}^N) \to L^q(T,\mathbb{R}^N)$ be defined by $J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$. Consider the map $K_\lambda = V + \hat{A}_\lambda + J$. Note that $\hat{A}_\lambda$ and J are both monotone, continuous, thus maximal monotone. So from Theorem III. 3.3, p. 334, of Hu-Papageorgiou [13], we have that K_λ is maximal monotone. Because $0 = A_\lambda(0)$, we have

$$(K_\lambda(x), x)_{pq} \geq (V(x), x)_{pq} + (J(x), x)_{pq},$$

where by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(T,\mathbb{R}^N), L^q(T,\mathbb{R}^N))$. Using Green's identity, the fact that if $x \in D$, then $(\varphi(x'(0)), -\varphi(x'(b))) \in \xi(x(0), x(b))$ and hypothesis $H(\xi)$, we obtain

$$(V(x), x)_{pq} = -\int_0^b \big((\|x'(t)\|^{p-2}x'(t))', x(t)\big)_{\mathbb{R}^N}\, dt$$

$$= \big(-\|x'(b)\|^{p-2}x'(b), x(b)\big)_{\mathbb{R}^N} + \big(\|x'(0)\|^{p-2}x'(0), x(0)\big)_{\mathbb{R}^N} + \|x'\|_p^p \geq \|x'\|_p^p.$$

Also $(J(x), x)_{pq} = \|x\|_p^p$. So we obtain

$$(K_\lambda(x), x)_{pq} \geq \|x'\|_p^p + \|x\|_p^p = \|x\|_{1,p}^p,$$

where $\|\cdot\|_{1,p}$ denotes the norm in the Sobolev space $W^{1,p}(T,\mathbb{R}^N)$. From this last inequality we infer that K_λ is coercive. But recall (see Section 2) that a maximal monotone and coercive operator is surjective. So $R(K_\lambda) = L^q(T,\mathbb{R}^N)$. Also it is clear that J is strictly monotone and so it follows that K_λ is injective. Hence we can define the operator $K_\lambda^{-1} : L^q(T,\mathbb{R}^N) \to D \subseteq W^{1,p}(T,\mathbb{R}^N)$.

Claim 1. $K_\lambda^{-1} : L^q(T,\mathbb{R}^N) \to D \subseteq W^{1,p}(T,\mathbb{R}^N)$ is completely continuous. Note that by virtue of the reflexivity of $L^q(T,\mathbb{R}^N)$, complete continuity implies compactness. Assume that $y_n \xrightarrow{w} y$ in $L^q(T,\mathbb{R}^N)$ and set $x_n = K_\lambda^{-1}(y_n), n \geq 1$, and $x = K_\lambda^{-1}(y)$. We have

$$\begin{aligned} & y_n = V(x_n) + \hat{A}_\lambda(x_n) + J(x_n). \\ \Rightarrow \quad & (y_n, x_n)_{pq} = (V(x_n), x_n)_{pq} + (\hat{A}_\lambda(x_n), x_n)_{pq} + (J(x_n), x_n)_{pq} \\ \Rightarrow \quad & \|x_n\|_{1,p}^p \leq \|y_n\|_q \|x_n\|_p. \end{aligned}$$

From the last inequality it follows that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(T, \mathbb{R}^N)$ is bounded. Hence by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} z$ in $W^{1,p}(T, \mathbb{R}^N)$ and from the compact embedding of $W^{1,p}(T, \mathbb{R}^N)$ into $L^p(T, \mathbb{R}^N)$, we also have $x_n \to z$ in $L^p(T, \mathbb{R}^N)$. Then

$$(y_n, x_n - z)_{pq}, (\hat{A}_\lambda(x_n), x_n - z)_{pq} \text{ and } (J(x_n), x_n - z)_{pq} \to 0 \text{ as } n \to \infty,$$

$$\Rightarrow \lim (V(x_n), x_n - z)_{pq} = 0.$$

But as we have already mentioned V is maximal monotone and so according to Remark III. 6.3, p. 365, of Hu-Papageorgiou [13], we have that V is also generalized pseudomonotone, which implies that $V(x_n) \xrightarrow{w} V(z)$ in $L^q(T, \mathbb{R}^N)$. So in the limit as $n \to \infty$ we obtain

$$y = V(z) + \hat{A}_\lambda(z) + J(z)$$

$$\Rightarrow \quad z = K_\lambda^{-1}(y), \text{ i.e. } z = x.$$

Moreover, note that $\{u_n = \|x_n'\|^{p-2} x_n'\}_{n\geq 1} \subseteq W^{1,q}(T, \mathbb{R}^N)$ is bounded and so we may assume that $u_n \xrightarrow{w} u$ in $W^{1,q}(T, \mathbb{R}^N)$. From the compact embedding of $W^{1,q}(T, \mathbb{R}^N)$ into $C(T, \mathbb{R}^N)$, we have that $u_n \xrightarrow{w} u$ in $C(T, \mathbb{R}^N)$. Since $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is a homeomorphism, we have that $\varphi^{-1}(u_n) = x_n' \to \varphi^{-1}(u)$ in $L^p(T, \mathbb{R}^N)$. Therefore $\varphi^{-1}(u) = x'$ and so $x_n \to x$ in $W^{1,p}(T, \mathbb{R}^N)$ which proves the claim. Next let $N_F : W^{1,p}(T, \mathbb{R}^N) \to 2^{L^q(T, \mathbb{R}^N)}$ be the multivalued Nemitsky operator corresponding to F, i.e. $N_F(x) = S^q_{F(\cdot, x(\cdot), x'(\cdot))}$.

Claim 2. N_F has values in $P_{wkc}(L^q(T, \mathbb{R}^N))$ and is usc from $W^{1,p}(T, \mathbb{R}^N)$ into $L^q(T, \mathbb{R}^N)_w$. Note that hypotheses $H(F)_1$ (i) and (ii), do not imply joint measurability of F and so it is not immediately clear that N_F has nonempty values. Let $x \in W^{1,p}(T, \mathbb{R}^N)$. We can find $\{s_n\}_{n\geq 1}, \{r_n\}_{n\geq 1}$ step functions such that $\|s_n(t)\| \leq \|x(t)\|, \|r_n(t)\| \leq \|x'(t)\|, s_n(t) \to x(t)$ and $r_n(t) \to x'(t)$ a.e. on T. By virtue of hypothesis $H(F)_1$ (i), for every $n \geq 1, t \mapsto F(t, s_n(t), r_n(t))$ is measurable. So we can apply the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [13], Theorem II. 2.14, p. 158) and obtain $f_n : T \to \mathbb{R}^N$ a measurable map such that $f_n(t) \in F(t, s_n(t), r_n(t))$ a.e. on T. Evidently $\{f_n\}_{n\geq 1} \subset L^q(T, \mathbb{R}^N)$ is bounded (see hypothesis $H(F)_1$ (v)) and so we may assume that $f_n \xrightarrow{w} f$ in $L^q(T, \mathbb{R}^N)$. Invoking Proposition VII. 3.9, p. 694, of Hu-Papageorgiou [13], we obtain

$$f(t) \in \overline{\text{conv}}\, \overline{\lim}\, F(t, s_n(t), r_n(t)) \subset F(t, x(t), x'(t)) \text{ a.e. on } T,$$

the last inclusion following from the fact that for almost all $t \in T, \mathrm{Gr}F(t,\cdot,\cdot)$ is closed (see hypothesis $H(F)_1$ (i)). So $f \in N_F(x)$ and we have established that N_F has nonempty values. It is clear that the values of N_F are bounded, closed, convex, hence they belong to $P_{wkc}(L^q(T,\mathbb{R}^N))$. Next we will show the upper semicontinuity of N_F from $W^{1,p}(T,\mathbb{R}^N)$ into $L^q(T,\mathbb{R}^N)_w$.

From hypothesis $H(F)_1$ (v), we see that N_F is locally compact into $L^q(T,\mathbb{R}^N)_w$ and so by virtue of Proposition I. 2.23, p. 43, of Hu-Papageorgiou [13] and the fact that on bounded sets in $L^q(T,\mathbb{R}^N)_w$, the weak topology is metrizable, it suffices to show that $\mathrm{Gr}N_F$ is sequentially closed in $W^{1,p}(T,\mathbb{R}^N) \times L^q(T,\mathbb{R}^N)_w$. So let $(x_n, f_n) \in \mathrm{Gr}N_F$, $n \geq 1$, and assume that $x_n \to x$ in $W^{1,p}(T,\mathbb{R}^N)$ and $f_n \xrightarrow{w} f$ in $L^q(T,\mathbb{R}^N)$. We may assume that $x_n(t) \to x(t)$ and $x_n'(t) \to x'(t)$ a.e. on T and so as above via Proposition VII. 3.9, p. 694, of Hu-Papageorgiou [13], we show that $(x,f) \in \mathrm{Gr}N_F$. This completes the proof of the claim. Let $N_1 = -N_F + J$. Evidently $N_1 : W^{1,p}(T,\mathbb{R}^N) \to P_{wkc}(L^q(T,\mathbb{R}^N))$ is usc from $W^{1,p}(T,\mathbb{R}^N)$ into $L^q(T,\mathbb{R}^N)_w$. We consider the multivalued operator

$$K_\lambda^{-1} \circ N_1 : W^{1,p}(T,\mathbb{R}^N) \to P_k(W^{1,p}(T,\mathbb{R}^N)).$$

This map is usc and maps bounded sets into relatively compact ones (i.e. $K_\lambda^{-1} \circ N_1$ is compact). So in order to be able to apply Proposition 1 and obtain a fixed point of $K_\lambda^{-1} \circ N_1$, we need to prove the following claim:

Claim 3. The set

$$S = \{x \in W^{1,p}(T,\mathbb{R}^N) : x \in \beta K_\lambda^{-1} \circ N_1(x), 0 < \beta < 1\}$$

is bounded.

Let $x \in S$. We have

$$\begin{aligned}
& K_\lambda\left(\frac{1}{\beta}x\right) \in N_1(x), \\
\Rightarrow\quad & V\left(\frac{1}{\beta}x\right) + \hat{A}_\lambda\left(\frac{1}{\beta}x\right) + J\left(\frac{1}{\beta}x\right) \in -N_F(x) + J(x), \\
\Rightarrow\quad & \left(V(\frac{1}{\beta}x), x\right)_{pq} + \left(\hat{A}_\lambda(\frac{1}{\beta}x), x\right)_{pq} + \left(J(\frac{1}{\beta}x), x\right)_{pq} \\
& = (-f, x)_{pq} + (J(x), x)_{pq},\ f \in N_F(x).
\end{aligned}$$

Since $\hat{A}_\lambda$ is monotone and $0 = \hat{A}_\lambda(0)$, it follows that $\left(\hat{A}_\lambda(\frac{1}{\beta}x), x\right)_{pq} \geq 0$. Also

$$\begin{aligned}\left(V(\frac{1}{\beta}x), x\right)_{pq} &= -\int_0^b \frac{1}{\beta^{p-1}} \left((\|x'(t)\|^{p-2}x'(t))', x(t)\right)_{\mathbb{R}^N} dt \\ &= -\left(\frac{1}{\beta^{p-1}}\|x'(b)\|^{p-2}x'(b), x(b)\right)_{\mathbb{R}^N} \\ &\quad + \left(\frac{1}{\beta^{p-1}}\|x'(0)\|^{p-2}x'(0), x(0)\right)_{\mathbb{R}^N} + \frac{1}{\beta^{p-1}}\|x'\|_p^p \\ &\geq \frac{1}{\beta^{p-1}}\|x'\|_p^p \quad \text{(from the boundary conditions and hypothesis } H(\xi)).\end{aligned}$$

So we obtain

$$\begin{aligned}&\frac{1}{\beta^{p-1}}\|x'\|_p^p + \frac{1}{\beta^{p-1}}\|x\|_p^p \leq -(f,x)_{pq} + \|x\|_p^p \\ \Rightarrow\quad &\|x'\|_p^p \leq -\beta^{p-1}(f,x)_{pq} + (\beta^{p-1}-1)\|x\|_p^p \\ &\leq -\beta^{p-1}(f,x)_{pq} \text{ (since } 0 < \beta < 1).\end{aligned} \tag{3}$$

From hypothesis $H(F)_1$ (iii), we have

$$\begin{aligned}-\beta^{p-1}(f,x)_{pq} &= \beta^{p-1}\int_0^b -\left(f(t), x(t)\right)_{\mathbb{R}^N} dt \\ &\leq \beta^{p-1}a\|x\|_p^p + \beta^{p-1}\gamma\int_0^b \|x(t)\|^r \|x'(t)\|^{p-r} dt + \beta^{p-1}\|c\|_1\|x\|_\infty^s.\end{aligned}$$

Let $\tau = p - r$, $\mu = \frac{p}{r}$ abd $\mu' = \frac{p}{\tau}$ ($\frac{1}{\mu} + \frac{1}{\mu'} = 1$). Applying Hölder's inequality with this pair of conjugate exponents, we obtain

$$\int_0^b \|x(t)\|^r \|x'(t)\|^{p-r} dt \leq \left(\int_0^b \|x(t)\|^{r\mu} dt\right)^{\frac{1}{\mu}} \left(\int_0^b (\|x'(t)\|^{\tau\mu'} dt\right)^{\frac{1}{\mu'}} \leq \|x\|_p^r \|x'\|_p^\tau.$$

It follows that

$$-\beta^{p-1}(f,x)_{pq} \leq \beta^{p-1}a\|x\|_p^p + \beta^{p-1}\gamma\|x\|_p^r\|x'\|_p^\tau + \beta^{p-1}\|c\|_1\|x\|_\infty^s. \tag{4}$$

We will show that for every $x \in S$, $\|x\|_\infty \leq M$ with M as in hypothesis $H(F)_1$ (iv). For this purpose we introduce the function $r(t) = \|x(t)\|^p$

and let $t_0 \in T$ be the point where r attains its maximum on T. Suppose that $M^p < r(t_0)$ and first suppose that $t_0 \in (0, b)$. We have $0 = r'(t_0) = p\|x(t_0)\|^{p-2}(x(t_0), x'(t_0))_{\mathbb{R}^N}$ and so $(x(t_0), x'(t_0))_{\mathbb{R}^N} = 0$ (unless $x \equiv 0$ in which case we trivially have $\|x\|_\infty \leq M$). Then from hypothesis $H(F)_1$ (iv), we know that there exists $\delta > 0$ and $\xi > 0$ such that

$$\inf \left[(v, x)_{\mathbb{R}^N} + \|y\|^p : \|x - x(t_0)\| + \|y - x'(t_0)\| < \delta, v \in F(t, x, y) \right] \geq \xi > 0.$$

Because $x \in S$, we have $x \in D$ and so $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N) \subseteq C(T, \mathbb{R}^N)$. Because φ is a homeomorphism, it follows that $\varphi^{-1}(\|x'(\cdot)\|^{p-2}x'(\cdot)) = x'(\cdot) \in C(T, \mathbb{R}^N)$. Also $x \in W^{1,p}(T, \mathbb{R}^N) \subseteq C(T, \mathbb{R}^N)$. Thus for $\delta > 0$ as above, we can find $\delta_1 > 0$ such that if $t \in (t_0, t_0 + \delta_1]$, we have

$$\|x(t) - x(t_0)\| + \|x'(t) - x'(t_0)\| < \delta.$$

Therefore for almost all $t \in (t_0, t_0 + \delta_1]$ and all $v \in F(t, x(t), x'(t))$, we have

$$\begin{aligned} &\beta^{p-1}(v, x(t))_{\mathbb{R}^N} + \beta^{p-1}\|x'(t)\|^p \geq \beta^{p-1}\xi \\ \Rightarrow\quad &\beta^{p-1}(f(t), x(t))_{\mathbb{R}^N} + \beta^{p-1}\|x'(t)\|^p \geq \beta^{p-1}\xi. \end{aligned} \tag{5}$$

Recall that

$$V\left(\frac{1}{\beta}x\right) + \hat{A}_\lambda\left(\frac{1}{\beta}x\right) + J\left(\frac{1}{\beta}x\right) = -f + J(x)$$

$$\begin{aligned} \Rightarrow\quad & -\left(\|x'(t)\|^{p-2}x'(t)\right)' + \beta^{p-1}A_\lambda\left(\frac{1}{\beta}x(t)\right) \\ & = -\beta^{p-1}f(t) + (\beta^{p-1} - 1)\|x(t)\|^{p-2}x(t) \text{ a.e. on } T, \\ \Rightarrow\quad & \beta^{p-1}f(t) = \left(\|x'(t)\|^{p-2}x'(t)\right)' - \beta^{p-1}A_\lambda\left(\frac{1}{\beta}x(t)\right) \\ & + (\beta^{p-1} - 1)\|x(t)\|^{p-2}x(t) \text{ a.e. on } T. \end{aligned}$$

Using this in (5), we obtain

$$\begin{aligned} & \left((\|x'(t)\|^{p-2}x'(t))', x(t)\right)_{\mathbb{R}^N} - \beta^{p-1}\left(A_\lambda(\frac{1}{\beta}x(t)), x(t)\right)_{\mathbb{R}^N} \\ & + (\beta^{p-1} - 1)\|x(t)\|^p + \beta^{p-1}\|x'(t)\|^p \\ & \geq \beta^{p-1}\xi \text{ a.e. on } (t_0, t_0 + \delta_1]. \end{aligned}$$

Note that $(\beta^{p-1}-1)\|x(t)\|^p \leq 0$ (since $0<\beta<1$) and $(A_\lambda(\frac{1}{\beta}x(t)), x(t))_{\mathbb{R}^N} \geq 0$ (since A_λ is monotone and $A_\lambda(0)=0$). Thus after integration over $[t_0,t]$, $t\in(t_0,t_0+\delta_1]$, we obtain

$$(6)\ \int_{t_0}^{t}\big((\|x'(s)\|^{p-2}x'(s))', x(s)\big)_{\mathbb{R}^N}\,ds + \beta^{p-1}\int_{t_0}^{t}\|x'(s)\|^p ds \geq \beta^{p-1}\xi(t-t_0).$$

From Green's identity, we have

$$\begin{aligned}
&\int_{t_0}^{t}\big((\|x'(s)\|^{p-2}x'(s))', x(s)\big)_{\mathbb{R}^N}\,ds = \|x'(t)\|^{p-2}(x'(t),x(t))_{\mathbb{R}^N}\\
&- \|x'(t_0)\|^{p-2}(x'(t_0),x(t_0))_{\mathbb{R}^N} - \int_{t_0}^{t}\|x'(s)\|^p ds\\
&= \|x'(t)\|^{p-2}(x'(t),x(t))_{\mathbb{R}^N} - \int_{t_0}^{t}\|x'(s)\|^p ds \ \ (\text{since } (x(t_0),x'(t_0))_{\mathbb{R}^N}=0).
\end{aligned}$$

Using this equality in (6), we have

$$\begin{aligned}
&\|x'(t)\|^{p-2}(x'(t),x(t))_{\mathbb{R}^N} + (\beta^{p-1}-1)\int_{t_0}^{t}\|x'(s)\|^p ds \geq \beta^{p-1}\xi(t-t_0)\\
\Rightarrow\ &\|x'(t)\|^{p-2}(x'(t),x(t))_{\mathbb{R}^N} \geq \beta^{p-1}\xi(t-t_0)\\
&(\text{since } 0<\beta<1), t\in(t_0,t_0+\delta_1]\\
\Rightarrow\ &(x(t),x'(t))_{\mathbb{R}^N} > 0\\
\Rightarrow\ &r'(t)>0 \text{ for all } t\in(t_0,t_0+\delta_1],
\end{aligned}$$

i.e. r is strictly increasing on $(t_0,t_0+\delta_1]$.

This contradicts the choice of $t_0\in T$. So $\|x(t_0)\|\leq M$ and this proves the desired bound when $t\in(0,b)$. Suppose $t_0=0$. Then $r'(0)\leq 0$ and so $(x(0),x'(0))_{\mathbb{R}^N}\leq 0$. If condition $H(\xi)$ (i) is satisfied we have $(x(0),x'(0))_{\mathbb{R}^N}\geq 0$ and so $(x(0),x'(0))_{\mathbb{R}^N}=0$, i.e. $r'(0)=0$. Thus we can proceed as before. If hypothesis $H(\xi)$ (ii) holds, we have $x(0)=x(b)$ and $r'(0)\leq 0\leq r'(b)$. So $(x'(0),x(0))_{\mathbb{R}^N}\leq 0\leq (x'(b),x(b))_{\mathbb{R}^N}$, while from the fact that $(\varphi(x'(0)),-\varphi(x'(b)))\in\xi(x(0),x(b))$, we have that $(x'(b),x(b))_{\mathbb{R}^N}\leq (x'(0),x(0))_{\mathbb{R}^N}$ (recall that $(0,0)\in\xi(0,0)$). Thus finally, we have $0=(x'(0),x(0))_{\mathbb{R}^N}=(x'(b),x(b))_{\mathbb{R}^N}$ and so $r'(0)=r'(b)=0$ and we can repeat the previous argument. Similarly, we can analyze the case $t_0=b$. Hence we have proved that for all $x\in S, \|x\|_\infty\leq M$.

Using (4) in (3) we obtain

$$\begin{aligned}\|x'\|_p^p &\leq \beta^{p-1}a\|x\|_p^p + \beta^{p-1}\gamma\|x\|_p^r\|x'\|_p^\tau + \beta^{p-1}\|c\|_1\|x\|_\infty^s \\ &\leq c_1 + c_2\|x'\|_p^\tau \ \ (\tau < p) \ \text{ for some } \ c_1, c_2 > 0.\end{aligned}$$

So $\{x'\}_{x\in S}$ is bounded in $L^p(T,\mathbb{R}^N)$. Since $\|x\|_\infty \leq M$ for all $x \in S$, we infer that $S \subseteq W^{1,p}(T,\mathbb{R}^N)$ is bounded. Applying Proposition 1, we obtain $x \in D$ such that $x \in K_\lambda^{-1} \circ N_1(x) \Rightarrow K_\lambda(x) \in N_1(x) \Rightarrow x$ solves the auxiliary problem (2). ■

In the above result F was convex-valued. We can still have an existence theorem for (2), even if F is not necessarily convex-valued. More precisely, we assume on F as follows:

$\underline{H(F)_2}$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is a multifunction such that

(i) $(t,x,y) \in \mathbb{R}^N \times \mathbb{R}^N,\ t \to F(t,x,y)$ is graph measurable;

(ii) for almost all $t \in T, (x,y) \to F(t,x,y)$ is lsc;

and conditions (iii), (iv) and (v) of hypothesis $H(F)_1$ hold.

Proposition 3. *If hypothesis $H(A)_1, H(F)_2$ and $H(\xi)$ hold then there exists a solution $x \in C^1(T,\mathbb{R}^N)$ to problem* (2).

Proof. As before let $N_F : W^{1,p}(T,\mathbb{R}^N) \to 2^{L^q(T,\mathbb{R}^N)}$ be the multivalued Nemitsky operator corresponding to F, i.e. $N_F(x) = S^p_{F(\cdot,x(\cdot),x'(\cdot))}$. By virtue of hypothesis $H(F)_2$ and the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [13], Theorem II. 2.14, p. 158), we have that N_F has values in $P_f(L^q(T,\mathbb{R}^N))$.

Claim. N_F is lsc from $W^{1,p}(T,\mathbb{R}^N)$ to $L^q(T,\mathbb{R}^N)$. Let $C \subseteq L^q(T,\mathbb{R}^N)$ closed. We will show that $N_F^+(C) = \{x \in W^{1,p}(T,\mathbb{R}^N) : N_F(x) \subseteq C\}$ is closed. To this end, let $x_n \in N_F^+(C)$ and assume that $x_n \to x$ in $W^{1,p}(T,\mathbb{R}^N)$. By passing to a subsequence if necessary, we may assume that $x_n(t) \to x(t)$ for all $t \in T$ and $x_n'(t) \to x'(t)$ a.e. on T. Let $f \in N_F(x)$ and let $f_n \in N_F(x_n)$ such that $\|f - f_n\|_q \leq d(f, N_F(x_n)) + \frac{1}{n}$.

From Hu-Papageorgiou [13], p. 237, we have

$$d(f, N_F(x_n)) = \left(\int_0^b d(f(t), F(t, x_n(t), x_n'(t)))^q dt\right)^{\frac{1}{q}}$$

Since $F(t,\cdot,\cdot)$ is lsc, from Proposition I. 2.26, p. 45, of Hu-Papageorgiou [13], we have that for all $v \in \mathbb{R}^N$, $(x,y) \mapsto d(v, F(t,x,y))$ is upper semicontinuous. Then by Fatou's lemma, we have

$$\overline{\lim}\, d(f, N_F(x_n)) \leq \left(\int_0^b d(f(t), F(t,x(t),x'(t)))^q dt\right)^{\frac{1}{q}} = 0$$

and therefore $\|f - f_n\|_q \to 0$ as $n \to \infty$. Because $f_n \in C$ and C is closed, we have $f \in C$. Since $f \in N_F(x)$ was arbitrary, we infer that $x \in N_F^+(C)$ and so N_F is lsc as claimed. Apply Theorem II. 8.7, p. 245, of Hu-Papageorgiou [13], to obtain $u : W^{1,p}(T,\mathbb{R}^N) \to L^q(T,\mathbb{R}^N)$ continuous such that $u(x) \in N_F(x)$ for every $x \in W^{1,p}(T,\mathbb{R}^N)$. Consider the following single-valued boundary value problem

$$(7) \qquad \left\{ \begin{array}{c} \left(\|x'(t)\|^{p-2}x'(t)\right)' \in A_\lambda(x(t)) + u(x)(t) \text{ a.e. on } T \\ (\varphi(x'(0)), -\varphi(x'(b))) \in \xi(x(0),\ x(b)),\ \lambda > 0. \end{array} \right\}$$

As in the proof of Proposition 2, using the Leray-Schauder alternative theorem (the single-valued version of Proposition 1), we obtain a solution $x \in C^1(T,\mathbb{R}^N)$ to problem (7). Evidently this solves problem (2). ∎

4. Existence Theorems

In this case using Propositions 2 and 3, we obtain existence theorems for the convex and nonconvex problems, for two different situations. In the first one we assume that $\mathrm{dom}A = \mathbb{R}^N$, while in the other we are able to assume that $\mathrm{dom}A \neq \mathbb{R}^N$ at the expense of slightly strengthening the growth hypothesis on $F(t,x,y)$. This case is of special interest since it incorporates variational inequalities. We start with the analysis of the problem in which A is defined everywhere, i.e. $\mathrm{dom}A = \{x \in \mathbb{R}^N :\ A(x) \neq \emptyset\} = \mathbb{R}^N$. More precisely, we assume:

$\underline{H(A)_2}$: $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map with $\mathrm{dom}A = \mathbb{R}^N$ and $0 \in A(0)$.

Theorem 4. *If hypotheses $H(A)_2, H(F)_1$ and $H(\xi)$ hold, then problem* (1) *has a solution $x \in C^1(T,\mathbb{R}^N)$.*

Proof. Let $\lambda_n \to 0, \lambda_n > 0$ and let $x_n \in C^1(T, \mathbb{R}^N)$ be solutions to the approximate problem (2) (Proposition 2). From the proof of Proposition 2, we know that $\|x_n\|_\infty \leq M$ for all $n \geq 1$. We have

$$V(x_n) + \hat{A}_{\lambda_n}(x_n) = -f_n, \ f_n \in N_F(x_n)$$
$$\Rightarrow \quad (V(x_n), x_n)_{pq} + (\hat{A}_{\lambda_n}(x_n), x_n)_{pq} = -(f_n, x_n)_{pq}.$$

Using Green's identity and the boundary conditions on the first term of the left hand side and exploiting the monotonicity of $\hat{A}_{\lambda_n}$ and the fact that $0 = \hat{A}_{\lambda_n}(0)$, we obtain

$$\|x_n'\|_p^p \leq \|f_n\|_q \|x_n\|_p \leq c_3 \|f_n\|_q \quad \text{for some } c_3 > 0 \text{ and all } n \geq 1$$
$$\Rightarrow \quad \|x_n'\|_p^p \leq c_3 \left(\|\eta_{1,M}\|_q + \|\eta_{2,M}\|_\infty \|x_n'\|_p^{p-1}\right), \ (\text{hypothesis } H(F)_1(v))$$
$$\Rightarrow \quad \{x_n'\}_{n\geq 1} \subseteq L^p(T, \mathbb{R}^N) \text{ is bounded,}$$
$$\Rightarrow \quad \{x_n\}_{n\geq 1} \subseteq W^{1,p}(T, \mathbb{R}^N) \text{ is bounded.}$$

So we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(T, \mathbb{R}^N)$ and $x_n \to x$ in $L^p(T, \mathbb{R}^N)$. From Proposition III. 2.29, p. 325, of Hu-Papageorgiou [13], we know that $\|A_{\lambda_n}(x_n(t))\| \leq \|A^0(x_n(t))\|$. Since $\mathrm{dom} A = \mathbb{R}^N$, from Theorem III. 1.21, p. 306 of Hu-Papageorgiou [13], we know that A^0 is bounded on compact sets. Note that since $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(T, \mathbb{R}^N)$ is bounded, from the compact embedding of $W^{1,p}(T, \mathbb{R}^N)$, we have that $\{x_n\}_{n\geq 1}$ is relatively compact in $C(T, \mathbb{R}^N)$ and so $\sup_{n\geq 1} \|A^0(x_n(t))\| \leq c_4$ for some $c_4 > 0$. Hence $\|A_{\lambda_n}(x_n(t))\| \leq c_4$ for all $t \in T$ and all $n \geq 1$. Thus we may assume that $\hat{A}_{\lambda_n}(x_n) \xrightarrow{w} u$ in $L^q(T, \mathbb{R}^N)$ as $n \to \infty$. Repeating the argument in Claim 1 in the proof of Proposition 2, we obtain $x_n \to x$ in $W^{1,p}(T, \mathbb{R}^N)$ and $\|x_n'\|^{p-2} x_n' \xrightarrow{w} \|x'\|^{p-2} x'$ in $W^{1,q}(T, \mathbb{R}^N)$. Since $f_n \in N_F(x_n), n \geq 1$, from Hypothesis $H(F)_1$ (v), we see that $\{f_n\}_{n\geq 1} \subseteq L^q(T, \mathbb{R}^N)$ is bounded and so we may assume that $f_n \xrightarrow{w} f$ in $L^q(T, \mathbb{R}^N)$. As in the proof of Proposition 2, using Proposition VII. 3.9, p. 694, of Hu-Papageorgiou [13], we obtain $f \in N_F(x)$. Taking $n \to \infty$, we obtain

$$(\|x'(t)\|^{p-2} x'(t))' = u(t) + f(t) \text{ a.e. on } T.$$

Also since $\|x_n'\|^{p-2} x_n' \xrightarrow{w} \|x'\|^{p-2} x'$ in $W^{1,q}(T, \mathbb{R}^N)$, we have $\|x_n'\|^{p-2} x_n' \to \|x'\|^{p-2} x'$ in $C(T, \mathbb{R}^N)$ and so $\varphi^{-1}(\|x_n'(t)\|^{p-2} x_n'(t)) = x_n'(t) \to x'(t) =$

$\varphi^{-1}(\|x'(t)\|^{p-2}x'(t))$ for all $t \in T$. Also, at least for a subsequence, we have $x_n \to x$ in $C(T, \mathbb{R}^N)$. Since $(\varphi(x'_n(0)), -\varphi(x'_n(b)) \in \xi(x_n(0), x_n(b))$ and because $\mathrm{Gr}\xi$ is closed we conclude that $(\varphi(x'(0)), -\varphi(x'(b)) \in \xi(x(0), x(b))$. So it remains to show that $u(t) \in A(x(t))$ a.e. on T. To this end, let $\hat{A} : \hat{D} \subseteq L^p(T, \mathbb{R}^N) \to 2^{L^q(T,\mathbb{R}^N)}$ be defined by

$$\hat{A}(x) = \{g \in L^q(T, \mathbb{R}^N) : g(t) \in A(x(t)) \text{ a.e. on } T\}$$

for all

$$x \in \hat{D} = \{x \in L^p(T, \mathbb{R}^N) : \text{ there is } g \in L^q(T, \mathbb{R}^N) \\ \text{satisfying } g(t) \in A(x(t)) \text{ a.e. on } T\}.$$

We show that $\hat{A}$ is maximal monotone. Let $J : L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be defined by $J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$. We will show that $R(\hat{A} + J) = L^q(T, \mathbb{R}^N)$. To this end, let $h \in L^q(T, \mathbb{R}^N)$ be given and let

$$\Gamma(t) = \{(x, a) \in \mathbb{R}^N \times \mathbb{R}^N : a + \varphi(x) = h(t),\ a \in A(x),\ \|x\| \le r(t)\}$$

with $r(t) = \|h(t)\|^{\frac{1}{p-1}} + 1$. Note that $A + \varphi$ is maximal monotone and so by Theorem III. 6.28, p. 371, of Hu-Papapgeorgiou [13], we have that $\Gamma(t) \neq \emptyset$ a.e. on T. Moreover,

$$\begin{aligned} &\mathrm{Gr}\Gamma \\ &= \{(t, x, a) \in T \times \mathbb{R}^N \times \mathbb{R}^N : a + \varphi(x) = h(t),\ d(a, A(x)) = 0,\ \|x\| \le r(t)\} \\ &\in \mathcal{L} \times B(\mathbb{R}^N) \times B(\mathbb{R}^N), \text{ with } \mathcal{L} \text{ being the Lebesgue } \sigma\text{-field of } T. \end{aligned}$$

Invoking the Yankov-von Neumann-Aumann selection theorem, we obtain measurable maps $x, a : T \to \mathbb{R}^N$ such that $(x(t), a(t)) \in \Gamma(t)$ a.e. on T. Evidently $x \in L^p(T, \mathbb{R}^N)$ and $a \in L^q(T, \mathbb{R}^N)$. So $R(\hat{A} + J) = L^q(T, \mathbb{R}^N)$. Now we will show that this surjectivity implies the maximality of the monotone operator $\hat{A}$. Indeed suppose $y \in L^p(T, \mathbb{R}^N), v \in L^q(T, \mathbb{R}^N)$ and assume that

$$(a - v, x - y)_{pq} \ge 0 \text{ for all } x \in \hat{D}, a \in \hat{A}(x).$$

Since $R(\hat{A} + J) = L^q(T, \mathbb{R}^N)$, we can find $x_1 \in \hat{D}$ such that $v + J(y) = a_1 + J(x_1),\ a_1 \in \hat{A}(x_1)$. Then $(a_1 - a_1 + J(y) - J(x_1), x_1 - y)_{pq} \ge 0 \Rightarrow (J(y) - J(x_1), x_1 - y)_{pq} \ge 0 \Rightarrow x_1 = y \in \hat{D}$ (since J is strictly monotone) and $v = a_1 \in \hat{A}(x_1)$ i.e. $(y, v) \in \mathrm{Gr}\hat{A}$ and this proves the maximality of $\hat{A}$. Let $J_{\lambda_n} : \mathbb{R}^N \to \mathbb{R}^N$ be the resolvent map. Then for all $t \in T$

$$\begin{aligned} \|J_{\lambda_n}(x_n(t)) - x(t)\| &\leq \|J_{\lambda_n}(x_n(t)) - J_{\lambda_n}(x(t))\| + \|J_{\lambda_n}(x(t)) - x(t)\| \\ &\leq \|x_n(t) - x(t)\| + \|J_{\lambda_n}(x(t)) - x(t)\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Recall that $A_{\lambda_n}(x_n(t)) \in A(J_{\lambda_n}(x_n(t)))$ for all $t \in T$. We have

$$(J_{\lambda_n}(x_n(\cdot)), A_{\lambda_n}(x_n(\cdot))) \in \mathrm{Gr}\hat{A}$$

and $J_{\lambda_n}(x_n) \to x$ in $L^p(T, \mathbb{R}^N)$, $\hat{A}_{\lambda_n}(x_n) \xrightarrow{w} u$ in $L^q(T, \mathbb{R}^N)$. Since $\hat{A}$ is maximal monotone, in the limit we have $(x, u) \in \mathrm{Gr}\hat{A}$, i.e. $u(t) \in A(x(t))$ a.e. on T. This proves that $x \in C^1(T, \mathbb{R}^N)$ is a solution of (1). ∎

In the same way, using this time Proposition 3, we can have a"nonconvex" existence theorem.

Theorem 5. *If the hypotheses $H(A)_2, H(F)_2$ and $H(\xi)$, then problem* (1) *has a solution $x \in C^1(T, \mathbb{R}^N)$.*

Now we will prove existence theorems for the case when $\mathrm{dom}A \neq \mathbb{R}^N$. As we have already mentioned, this requires a slightly stronger growth condition. More precisely, the new hypotheses on F are the following:

$\underline{H(F)_3}$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that hypothesis $\overline{H(F)_1}$ (i) – (iv) hold and

(v) for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t, x, y)$

$$\|v\| \leq \gamma_1(t, \|x\|) + \gamma_2(t, \|x\|)\|y\|$$

with $\sup_{0 \leq r \leq k} \gamma_1(t, r) \leq \eta_{1,k}(t)$ a.e. on T, $\eta_{1,k} \in L^2(T)$ and $\sup_{0 \leq r \leq k} \gamma_2(t, r) \leq \eta_{2,k}(t)$ a.e. on T, $\eta_{2,k} \in L^{\frac{2p}{p-2}}(T)$ (as usual $\frac{r}{0} = \infty$ for $r > 0$).

Similarily, we can have the hypothesis of the nonconvex problem.

$\underline{H(F)_4}$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is a multifunction such that hypothesis $\overline{H(F)_2}$ (i) – (iv) and $H(F)_3(v)$ hold.

Also we will need a compatibility condition between the maps A and ξ.

$\underline{H_0}$: for all $(a,d) \in \mathrm{dom}\xi$ and all $(a',d') \in \xi(a,d)$, we have $(A_\lambda(a), a')_{\mathbb{R}^N} + (A_\lambda(d), d')_{\mathbb{R}^N} \geq 0$ for all $\lambda > 0$.

Remark. Suppose $\xi = \partial\psi$ with $\psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ convex (hence locally Lipschitz). Let $\partial_k\psi, k = 1,2$, denote the partial subdifferential of $\psi(\cdot,\cdot)$ with respect to the first variable ($k = 1$) or to the second variable ($k = 2$). We know $\partial\psi \subseteq \partial_1\psi \times \partial_2\psi$. In the case $(A_\lambda(a), a')_{\mathbb{R}^N} \geq 0$ and $(A_\lambda(d), d')_{\mathbb{R}^N} \geq 0$ for all $(a',d') \in \xi(a,d), (a,d) \in \mathrm{dom}\xi$, is equivalent to saying that $\psi(J_\lambda(a), d) \leq \psi(a,d), \psi(a, J_\lambda(d)) \leq \psi(a,d)$ respectively (see Hu-Papageorgiou [13]).

Theorem 6. *If hypotheses $H(A)_1, H(F)_3, H(\xi)$ and H_0 hold, then problem* (1) *has a solution $x \in C^1(T, \mathbb{R}^N)$.*

Proof. As in the proof of Theorem 4, we take $\lambda_n \to 0, \lambda_n > 0$ and $x_n \in C^1(T, \mathbb{R}^N)$ solutions of the auxiliary problems (2). We know that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(T, \mathbb{R}^N)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(T, \mathbb{R})^N$. Keeping the notation introduced in the previous proofs, we have

$$\begin{aligned} & V(x_n) + \hat{A}_{\lambda_n}(x_n) = -f_n,\ f_n \in N_F(x_n), n \geq 1 \\ \Rightarrow\quad & (V(x_n), \hat{A}_{\lambda_n}(x_n))_{pq} + \|\hat{A}_{\lambda_n}(x_n)\|_2^2 = -(f_n, \hat{A}_{\lambda_n}(x_n)_{pq} \end{aligned} \tag{8}$$

(recall that $\hat{A}_{\lambda_n}(x_n) \in C(T, \mathbb{R}^N)$). From the definition of V and Green's identity, we obtain

$$\begin{aligned} & (V(x_n), \hat{A}_{\lambda_n}(x_n))_{pq} \\ & = -\int_0^b \left(\left(\left\|x_n'(t)\right\|^{p-2} x_n'(t)\right)', A_{\lambda_n}(x_n(t))\right)_{\mathbb{R}^N} dt \\ & = -\|x_n'(b)\|^{p-2}(x_n'(b), A_{\lambda_n}(x_n(b)))_{\mathbb{R}^N} + \|x_n'(0)\|^{p-2}(x_n'(0), A_{\lambda_n}(x_n(0)))_{\mathbb{R}^N} \\ & + \int_0^b \left\|x_n'(t)\right\|^{p-2} \left(x_n'(t), \frac{d}{dt} A_{\lambda_n}(x_n(t))\right)_{\mathbb{R}^N} dt. \end{aligned}$$

We know that $A_{\lambda_n} : \mathbb{R}^N \to \mathbb{R}^N$ is $\frac{1}{\lambda_n}$-Lipschitz and so it is differentiable almost everywhere (Rademacher's theorem). Also A_{λ_n} is monotone. Exploiting the monotonicity, we have for every $x \in \mathbb{R}^N$ being point of differentiability of A

$$\left(y, \frac{A_{\lambda_n}(x+ty) - A_{\lambda_n}(x)}{t}\right)_{\mathbb{R}^N} \geq 0$$

for all $t \geq 0$ and all $y \in \mathbb{R}^N$ and hence $(y, A'_{\lambda_n}(x)y)_{\mathbb{R}^N} \geq 0$.

From the chain rule of Marcus-Mizel [16] (Theorem 4.1), we know that

$$\frac{d}{dt} A_{\lambda_n}(x_n(t)) = A'_{\lambda_n}(x_n(t))x'_n(t) \text{ a.e. on } T.$$

Using hypothesis H_0, we have

$$(V(x_n), \hat{A}_{\lambda_n}(x_n))_{pq} \geq \int_0^b \|x'_n(t)\|^{p-2} \left(x'_n(t), A'_{\lambda_n}(x_n(t))x'_n(t)\right)_{\mathbb{R}^N} dt \geq 0.$$

Using this in (8), we obtain

$$\|\hat{A}_{\lambda_n}(x_n)\|_2^2 \leq \|f_n\|_2 \|\hat{A}_{\lambda_n}(x_n)\|_2$$

and thus $\{\hat{A}_{\lambda_n}(x_n)\}_{n\geq 1} \subseteq L^2(T, \mathbb{R}^N)$ is bounded. So we may assume that $\hat{A}_{\lambda_n}(x_n) \xrightarrow{w} u$ in $L^2(T, \mathbb{R}^N)$. Moreover, as in the proof of Theorem 4, we can have $x_n \to x$ in $W^{1,p}(T, \mathbb{R}^N)$ and $\|x'_n(\cdot)\|^{p-2}x'_n(\cdot) \xrightarrow{w} \|x'(\cdot)\|^{p-2}x'(\cdot)$ in $W^{1,q}(T, \mathbb{R}^N)$, while $f_n \xrightarrow{w} f$ in $L^2(T, \mathbb{R}^N)$ (see hypothesis $H(F)_3$ (v)). In the limit we have $f \in N_F(x)$ and

$$\left\{ \begin{array}{rl} \left(\|x'(t)\|^{p-2}x'(t)\right)' \in & u(t) + f(t, x(t), x'(t)) \text{ a.e. on } T \\ (\varphi(x'(0)), -\varphi(x'(b))) \in & \xi(x(0), x(b)). \end{array} \right\}$$

As before to finish the proof, we need to show that $u(t) \in A(u(t))$ a.e. on T. For this purpose, let $J_\lambda : \mathbb{R}^N \to \mathbb{R}^N$, $\lambda > 0$, be there solvent of A and $\hat{J}_\lambda : L^p(T, \mathbb{R}^N) \to L^p(T, \mathbb{R}^N)$ be defined by $J_\lambda(x)(\cdot) = J_\lambda(x(\cdot))$. Since J_λ is nonexpansive, from Marcus-Mizel [16] we have that $\hat{J}_\lambda(x_n) \in W^{1,p}(T, \mathbb{R}^N)$ and $\frac{d}{dt} J_{\lambda_n}(x_n(t)) = J'_{\lambda_n}(x_n(t))x'_n(t)$ (chain rule) and $\|J'_{\lambda_n}(x_n(t))\| \leq 1$. So $\|J'_{\lambda_n}(x_n(t))x'_n(t)\| \leq \|x'_n(t)\|$ a.e. on T, from which it follows that $\{\hat{J}_{\lambda_n}(x_n)\}_{n\geq 1} \subseteq W^{1,p}(T, \mathbb{R}^N)$ is bounded. Thus we may assume that $\hat{J}_{\lambda_n}(x_n) \xrightarrow{w} v$ in $W^{1,p}(T, \mathbb{R}^N)$, $\hat{J}_{\lambda_n}(x_n) \to v$ in $L^p(T, \mathbb{R}^N)$. From the definition of the Yosida approximation $A_{\lambda_n} = \frac{1}{\lambda_n}(I - J_{\lambda_n})$, we have

$$\begin{aligned} & J_{\lambda_n}(x_n(t)) + \lambda_n A_{\lambda_n}(x_n(t)) = x_n(t) \\ \Rightarrow \quad & \hat{J}_{\lambda_n}(x_n) + \lambda_n \hat{A}_{\lambda_n}(x_n) = x_n. \end{aligned}$$

Recall that $\{\hat{A}_{\lambda_n}(x_n)\}_{n\geq 1} \subseteq L^2(T, \mathbb{R}^N)$ is bounded and $\lambda_n \to 0$. Moreover, $\hat{J}_{\lambda_n}(x_n) \to v$ and $x_n \to x$ in $L^p(T, \mathbb{R}^N)$, hence in $L^2(T, \mathbb{R}^N)$, too (since $2 \leq p$). So passing to the limit in the last inequality, we obtain $v = x$. Hence $\hat{J}_{\lambda_n}(x_n) \xrightarrow{w} x$ in $W^{1,p}(T, \mathbb{R}^N)$ and $\hat{J}_{\lambda_n}(x_n) \to x$ in $C(T, \mathbb{R}^N)$. Let

$$S = \{t \in T : \text{ there exists } (y, w) \in \mathrm{Gr}A \text{ such that } (u(t)-w, x(t)-y)_{\mathbb{R}^N} < 0\}.$$

If we can show that $|S| = 0$ (i.e. S is a Lebesgue-null), then by virtue of the maximal monotonicity of $A(\cdot)$, we will have that $u(t) \in A(u(t))$ a.e. on T. Let $\Gamma(t) = \{(y, w) \in \mathrm{Gr}A : \ (u(t) - w, x(t) - y)_{\mathbb{R}^N} < 0\}$. Evidently $S = \{t \in T : \Gamma(t) \neq \emptyset\}$. Also $\mathrm{Gr}\Gamma = (T \times \mathrm{Gr}A) \cap \{(t, y, w) \in T \times \mathbb{R}^N \times \mathbb{R}^N : \xi(t, y, w) < 0\}$, where $\xi(t, y, w) = (u(t) - w, x(t) - y)_{\mathbb{R}^N}$. Clearly, $t \mapsto \xi(t, y, w)$ is measurable and $(y, w) \mapsto \xi(t, y, w)$ is continuous. Hence ξ is jointly measurable and so $\mathrm{Gr}\Gamma \in \mathcal{L} \times B(\mathbb{R}^N) \times B(\mathbb{R}^N)$, with $\mathcal{L}$ being the Lebesgue σ-field of T. Invoking the Yankov-von Neumann- Aumann projection theorem (see Hu-Papageorgiou [13], Theorem II. 1.33, p. 149), we have $\mathrm{proj}_T\, \mathrm{Gr}\Gamma = \{t \in T : \Gamma(t) \neq \emptyset\} = S \in \mathcal{L}$. If $|S| > 0$, by the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [13], Theorem II. 2.14, p. 158), we obtain $y : S \to \mathbb{R}^N, \ w : S \to \mathbb{R}^N$ measurable functions such that $(y(t), w(t)) \in \Gamma(t)$ for all $t \in T$. From Lusin's theorem, we know that we can find $S_1 \subseteq S$ closed, with $|S_1| > 0$ such that $y|_{S_1}, \ w|_{S_1}$ are both continuous, hence bounded. Since $A_{\lambda_n}(x_n(t)) \in A(J_{\lambda_n}(x_n(t)))$, we have

$$\begin{aligned}
& (A_{\lambda_n}(x_n(t)) - w(t), J_{\lambda_n}(x_n(t)) - y(t))_{\mathbb{R}^N} \geq 0 \\
\Rightarrow \ & \int_{S_1} (A_{\lambda_n}(x_n(t)) - w(t), J_{\lambda_n}(x_n(t)) - y(t))_{\mathbb{R}^N}\, dt \geq 0 \\
\Rightarrow \ & \int_{S_1} (u(t) - w(t), x(t) - y(t))_{\mathbb{R}^N}\, dt \geq 0.
\end{aligned}$$

On the other hand, since $(y(t), w(t)) \in \Gamma(t)$ for all $t \in S$ and $|S_1| > 0$, we have

$$\int_{S_1} (u(t) - w(t), x(t) - y(t))_{\mathbb{R}^N}\, dt < 0,$$

a contradiction. Therefore $|S| = 0$ and so we conclude that $u(t) \in A(x(t))$ a.e. on T. This shows that $x \in C^1(T, \mathbb{R}^N)$ is a solution of (1). ■

As for Theorem 4, we can have the "nonconvex" counterpart of Theorem 6.

Theorem 7. *If the hypothesis $H(A)_1, H(F)_4, H(\xi)$ and H_0 hold, then problem* (1) *has a solution $x \in C^1(T, \mathbb{R}^N)$.*

5. Special cases

(a) Let $K_1, K_2 \subseteq \mathbb{R}^N$ be nonempty, closed, convex and $0 \in K_1 \cap K_2$. Let $\delta_{K_1 \times K_2}$ be the indicator function of $K_1 \times K_2$, i.e.

$$\delta_{K_1\times K_2}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in K_1 \times K_2 \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\partial\delta_{K_1\times K_2} = N_{K_1\times K_2} = N_{K_1} \times N_{K_2}$ (here if $C \subseteq \mathbb{R}^N$ by N_C we denote the normal cone to C). Set $\xi = \partial\delta_{K_1\times K_2}$. Problem (1) takes the form

$$(9) \qquad \left\{ \begin{array}{rcl} \left(\|x'(t)\|^{p-2}x'(t)\right)' & \in & A(x(t)) + F(t, x(t), x'(t)) \\ x(0) \in K_1, \ x(b) & \in & K_2 \\ (x'(0), x(0))_{\mathbb{R}^N} & = & \sigma(x'(0), K_1) \\ (-x'(b), x(b))_{\mathbb{R}^N} & = & \sigma(-x'(b), K_2), \end{array} \right\}$$

where for $C \subseteq \mathbb{R}^N, \sigma(x^*, C) = \sup[(x^*, c)_{\mathbb{R}^N} : c \in C] =$ the support function of C. Note that ξ is maximal monotone and because $0 \in K_1 \cap K_2, (0,0) \in \xi(0,0)$ and for $(a', d') \in N_{K_1}(a) \times N_{K_2}(d)$ we have $(a', a)_{\mathbb{R}^N} \geq 0, \ (d', d)_{\mathbb{R}^N} \geq 0$. So $H(\xi)$ is valid and the results of this paper apply to problem (9). If $K_1, K_2 \subseteq \mathbb{R}^N_+, \ \psi = \delta_{\mathbb{R}^N_+}$ and

$$A(x) = \partial\psi(x) = N_{\mathbb{R}^N_+}(x)$$
$$= \begin{cases} \{0\} & \text{if } x_k > 0 \text{ for all } k \in 1, \dots, N \\ -\mathbb{R}^N_+ \cap \{x\}^\perp & \text{if } x_k = 0 \text{ for at least one } k \in 1, \dots, N \end{cases}$$

$(x = (x_k)_{k=1}^N)$. We can check that $A_\lambda(x) = \frac{1}{\lambda}(x - p(x; \mathbb{R}^N_+))$, with $p(\cdot; \mathbb{R}^N_+)$ being the metric projection on $\mathbb{R}^N_+$. For $x \in K_1$ or $x \in K_2, \ p(x; \mathbb{R}^N_+) = x$ and so $A_\lambda(x) = 0$, which means that H_0 holds. Problem (1) becomes

$$(10)\quad \left\{\begin{array}{l} \left(\|x'(t)\|^{p-2}x'(t)\right)' \in F(t,x(t),x'(t)) \text{ a.e. on} \\ \qquad\qquad \{t \in T : x_k(t) > 0 \text{ for all } k \in 1,\dots,N\} \\ \left(\|x'(t)\|^{p-2}x'(t)\right)' \in F(t,x(t),x'(t)) - u(t) \text{ a.e. on} \\ \qquad\qquad \{t \in T : x_k(t) = 0 \text{ for some } k \in 1,\dots,N\} \\ \qquad x(t) \in \mathbb{R}^N_+ \text{ for all } t \in T,\ x(0) \in K_1,\ x(b) \in K_2, \\ \qquad\qquad (x(t),u(t))_{\mathbb{R}^N} = 0, u(t) \in \mathbb{R}^N_+ \\ (x'(0),x(0))_{\mathbb{R}^N} = \sigma(x'(0),K_1), (-x'(b),x(b))_{\mathbb{R}^N} = \sigma(-x'(b),K_2). \end{array}\right\}$$

The results of the paper apply to the multivalued variational inclusion (10).

(b) If $K_1 = K_2 = \{0\}$, then $N_{K_1} = N_{K_2} = \mathbb{R}^N$ and so there are no constraints on $x'(0)$ and $x'(b)$. Therefore problem (1) becomes the classical Dirichlet problem. Since $A_\lambda(0) = 0$, hypothesis H_0 holds and the results of this work apply.

(c) If $K_1 = K_2 = \mathbb{R}^N$, then $N_{K_1} = N_{K_2} = \{0\}$ and so there are no constraints on $x(0)$ and $x(b)$, while $x'(0) = x'(b) = 0$. Thus problem (1) becomes the classical Neumann problem. In this case (H_0) is trivially true and our results apply.

(d) If $K = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x = y\}$ and

$$\xi = \partial\delta_K = K^\perp = \{(v,w) \in \mathbb{R}^N \times \mathbb{R}^N : v = -w\},$$

then problem (1) becomes the periodic problem. In this case we satisfy $H(\xi)$ (ii) and $(A_\lambda(a),a')_{\mathbb{R}^N} + (A_\lambda(d),d')_{\mathbb{R}^N} = 0$, i.e. H_0 is true. So our results incorporate the periodic problem.

(e) Let $\xi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ be defined by $\xi(x,y) = (\frac{1}{\vartheta^{p-1}}\varphi(x), \frac{1}{\eta^{p-1}}\varphi(y)), \vartheta, \eta > 0$, then from (1) we obtain a Sturm-Louville type problem with $x(0) - \vartheta x'(0) = 0$ and $x(b) + \eta x'(b) = 0$. It is easy to see that $H(\xi)$ (i) and H_0 hold and so our results apply.

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