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OPTIMAL CONTROL OF ∞-DIMENSIONAL STOCHASTIC SYSTEMS VIA GENERALIZED SOLUTIONS OF HJB EQUATIONS

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Abstract

In this paper, we consider optimal feedback control for stochastc infinite dimensional systems. We present some new results on the solution of associated HJB equations in infinite dimensional Hilbert spaces. In the process, we have also developed some new mathematical tools involving distributions on Hilbert spaces which may have many other interesting applications in other fields. We conclude with an application to optimal stationary feedback control.

Keywords: optimal control, stochastic systems, infinite dimension, HJB equation, stationary feedback control.

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1. Introduction

We consider the following system governed by a controlled stochastic differential equation as described below:

$$
\begin{aligned}
dz &= Azdt + F(z)dt + B(z)udt + \sqrt{Q}dW, \ t \geq 0, \\
z(0) &= x.
\end{aligned}
\tag{1.1}
$$

Let H and U be any two Hilbert spaces, the first denoting the state space and the other the space of controls. Generally, A is an unbounded operator with domain and range in H, F is a nonlinear operator in H and B is a nonlinear map from H to the space of bounded linear operators from U

to H. The map Q is a bounded symmetric positive operator in H and W is a cylindrical Brownian motion in H. Let $\mathcal{U}_{ad}$ denote the class of admissible control policies. Precise hypothesis will be introduced shortly. This paper is motivated by the following problem. Find a control $u^o \in \mathcal{U}_{ad}$ that minimizes over the time interval $I \equiv [0,T]$, the expected (average) cost functional

$$J(u) = E\Big\{\int_0^T [g(t,z(t)) + h(u(t))]dt + \varphi_0(z(T))\Big\}, \tag{1.2}$$

where g, h and φ_0 are suitable real valued functions defined on $I \times H, U,$ and H respectively. Two basic questions arising in control theory are existence of optimal control policies and necessary conditions that optimal policies must satisfy. Formally for a fully observed problem as stated above, the solution to both the problems are provided by the solution of the so called Hamilton-Jacobi-Bellman (HJB) equation. To use the dynamic programming principle, one introduces the value function

$$V(t,x) \equiv Inf\{J(t,x,u), u \in \mathcal{U}_{ad}\}, \tag{1.3}$$

where

$$J(t,x,u) \equiv E\Big\{\int_t^T [g(\theta,\xi_{t,x}(\theta)) + h(u(\theta))]d\theta + \varphi_0(\xi_{t,x}(T))\Big\}$$

and $\xi_{t,x}$ is the solution of equation (1.1) starting from time $t \in I$ and $x \in H$. In other words $\xi_{t,x}$ is the solution of

$$\begin{aligned} dy(s) &= (Ay(s) + F(y(s)) + B(y(s))u(s))ds + \sqrt{Q}dW(s),\ s \ge t,\\ y(t) &= x. \end{aligned}$$

Let $B_r \subset U$ denote the closed ball in U of radius $r > 0$, centered at the origin. For admissible open loop controls, denoted by $\mathcal{U}_{ad}^o$, we take the class of all progressively measurable random processes $\{u(t), t \ge 0\}$, taking values from B_r. For admissible feedback controls, denoted by $\mathcal{U}_{ad}^c$, we take all Borel measurable maps from H to B_r. For arbitrary $x \in H$ and $q \in B^*(x)H \subset U$ and $u \in B_r$ define $L(x,q,u) \equiv (u,q) + h(u)$ and the Hamiltonian

$$H(x,q) = \inf\{L(x,q,u), u \in B_r\}. \tag{1.4}$$

Then by use of dynamic programming principle one can formally derive the following partial differential equation for the value function V,

$$\begin{gathered} \frac{\partial}{\partial t}V(t,x) + (\mathcal{A}_0 V)(t,x) + \mathcal{F}_1(V)(t,x) + \mathcal{F}_2(V)(t,x) + g(t,x) = 0,\\ t \in [0,T),\ x \in H,\ V(T,x) = \varphi_0(x),\ x \in H, \end{gathered} \tag{1.5}$$

where the operators $\mathcal{A}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by

$$\begin{aligned} \mathcal{A}_0\psi(x) &\equiv (1/2)Tr(QD^2\psi(x)) + (A^*D\psi(x), x), \\ \mathcal{F}_1(\psi)(x) &\equiv (F(x), D\psi(x))_H, \\ \mathcal{F}_2(\psi)(x) &\equiv H(x, B^*(x)D\psi(x)), \end{aligned} \tag{1.6}$$

with $D^k\psi, k \in N$, denoting the Frechet differential of ψ of order k. Solving this equation one expects to obtain the value (the optimal cost)

$$J^o \equiv V(0, x) \tag{1.7}$$

of the control problem as stated. If the initial state is an H valued random variable with associated measure, say, ν_0 then it is given by

$$J^o \equiv \int_H V(0, x)\nu_0(dx). \tag{1.8}$$

The optimal feedback control law is given by

$$u^*(t, x) \equiv H_q(x, B^*(x)DV(t, x)), \tag{1.9}$$

where H_q denotes the Frechet derivative of $H(x, q)$ with respect to the variable $q \in U$. In general, HJB equation does not have a classical solution even when $\dim H < \infty$. In any case, if a solution exists which is C^1 in $t \in I$ and C^2 in $x \in H$, then the optimal cost is given by (1.7) or (1.8) and the optimal control is given by $u^o(t) = u^*(t, z^o(t))$ where z^o is the solution of equation (1.1) corresponding to the control u^o.

For infinite horizon problem or stationary problem, the objective functional is given by the discounted cost

$$J(u, x) \equiv E\left\{\int_0^\infty e^{-\delta t}[g(z(t)) + h(u(t))]dt\right\}, \tag{1.10}$$

for a suitable positive number δ known as the discount factor. Here $z = z(t, x)$ is the solution of (1.1) corresponding to the control u and initial state $z(0) = x$. The value function is defined by

$$V(x) \equiv \inf\{J(u, x), u \in \mathcal{U}_{ad}\}.$$

In this case one has the stationary HJB equation

$$\delta\Phi = \mathcal{A}_0\Phi + \mathcal{F}_1(\Phi) + \mathcal{F}_2(\Phi) + g(x), x \in H. \tag{1.11}$$

If this equation has a solution, formally the optimal cost is given by $V(x) = \Phi(x)$ and the optimal feedback control law is given by

$$u^*(x) = H_q(x, B^*(x)D\Phi(x))$$

and the optimal control $u^o(t) = u^*(z(t))$ where again $z(t), t \geq 0$, is the solution of equation (1.1) corresponding to the control u^o and the initial state $z(0) = x$.

By reversing the flow of time, that is, setting $t \longrightarrow T-t$ one can rewrite the equation (1.5) as

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t,x) &= \mathcal{A}_0\varphi + \mathcal{F}_1(\varphi) + \mathcal{F}_2(\varphi) + \tilde{g}(t,x) = 0, t \in (0,T], x \in H, \\ \varphi(0,x) &= \varphi_0(x),\ x \in H, \end{aligned} \tag{1.12}$$

where $\tilde{g}(t,x) = g(T-t,x)$. Our primary concern is to prove the existence of solution of the HJB equation (1.12) and the stationary problem (1.11) in some generalized sense. In recent years this problem has been considered intensively by many notable workers in the field, specially Da Prato [2], Gozzi and Rouy [1], Goldys and Maslowski [3]. These are fully observed control problems treated using purely analytic techniques. Partially observed control problems based on filter theory [12] have been treated in [8, 9, 10] using stochastic and analytic techniques. Gozzi and Rouy treated the stationary HJB equation (1.11) in [1]. They used differentiability property of the Ornstein-Uhlenbeck semigroup $R_t, t \geq 0$, and its weak continuity in the sense of Cerrai [7] and formulated a fixed point problem in the form

$$V(x) = \int_0^\infty e^{-\delta t} R_t\{\mathcal{F}_1(V)(x) + \mathcal{F}_2(V)(x) + g(x)\}dt, \tag{1.13}$$

in the Banach space $BUC^1(H)$, the space of bounded uniformly continuous and once Frechet differentiable functions. This is equivalent to the question of existence of a fixed point in $BUC^1(H)$ of the operator $R(\delta, \mathcal{A}_0)G_g$ or the functional equation,

$$V = R(\delta, \mathcal{A}_0)G_g(V), \tag{1.14}$$

where

$$G_g(V) \equiv \mathcal{F}_1(V) + \mathcal{F}_2(V) + g$$

and $R(\delta, \mathcal{A}_0)$ denotes the resolvent of the semigroup $R_t, t \geq 0$, with $\mathcal{A}_0$ being its weak infinitesimal generator. They used Banach fixed point theorem to prove the existence and uniqueness of solution to this problem. This was proved under the assumptions that F is Lipschitz continuous and bounded uniformly on H. Goldys and Maslowski [3] used similar techniques to extend these results to the case where F is merely Lipschitz and proved the existence of solutions in weighted BUC_m^1 spaces whose elements are uniformly continuous with atmost polynomial growth of order $2m$. Da Prato [2] used perturbation theory of semigroups and logarithmic transform to prove the existence of solution of the HJB equation for a linear system with quadratic cost in control and subquadratic cost in state. By use of the well known logarithmic transform, the nonlinear problem is transformed into a linear (parabolic) problem with a potential. The perturbed operator is shown to be selfadjoint and dissipative generating a positivity preserving C_0 semigroup of contractions on $\mathcal{H}$ (see Section 4, Proposition 4.1) giving an explicit representation of the value function. Our approach is based on bilinear forms and coercivity of Ornstein-Uhlenbeck operator involving the Gelfand triple ($\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$) (see Section 4, Proposition 4.1). We use standard fixed point theorems to prove existence and regularity properties of generalized solutions of HJB equations and finally use a theorem on measurable selections for existence of optimal feedback controls.

The rest of the paper is organized as follows: In Section 2, basic notations are introduced; in Section 3, basic assumptions and some fundamental results of independent interest are presented following Da Prato [2]. In Section 4, we study Ornstein-Uhlenbeck operator in Sobolev spaces and their duals introduced in this paper. In Section 5, we present existence, uniqueness and regularity properties of generalized solutions for an abstract version of the HJB equation which are then applied to the HJB equation (1.12). In the final section, we conclude the paper with a result on the stationary control problem.

2. Basic notations

Let H be a separable Hilbert space with the norm and inner product denoted by $|\cdot|, (\cdot,\cdot)$. Let $C_b(H)(B_b(H))$ be the vector space of bounded continuous (bounded Borel measurable) functions on H with the topology induced by the norm

$$\| f \|_0 \equiv \sup\{|f(x)|, x \in H\}.$$

With respect to this norm topology these are Banach spaces. We let $BUC(H)$ denote the space of bounded uniformly continuous functions on

H to R with the same supnorm topology as above. With this topology, $BUC(H)$ is also a Banach space. For any Banach space X with dual X^*, the duality pairing of $y \in X^*$ and $x \in X$ is denoted by $< y, x >_{X^*,X} = < x, y >_{X,X^*}$. For any pair of Banach spaces X, Y, $\mathcal{L}(X, Y)$ will denote the space of bounded linear operators from X to Y. For $Y = X$, $\mathcal{L}(X, X) \equiv \mathcal{L}(X)$. For any Banach space X, $\mathcal{B}_X$ denotes the Borel sigma algebra of subsets of the set X and $(X, \mathcal{B}_X)$ the measurable space.

3. Basic assumptions and some preperatory results

For the study of the HJB equation (1.12) or its stationary version (1.11) we shall make use of the Ornstein-Uhlenbeck Semigroup associated with the Markov transition operator corresponding to the stochastic evolution equation:

$$\begin{aligned} dz &= Azdt + \sqrt{Q}dW, \\ z(0) &= x, \end{aligned} \tag{3.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, in a separable Hilbert space H, W is an H-cylindrical Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$ and Q is a bounded positive selfadjoint operator in H. For any bounded Borel measurable function ϕ, that is, $\phi \in B_b(H)$, one defines

$$R_t\phi(x) \equiv E\{\phi(z(t, x))) = \int_H \phi(z)\mathcal{N}(S(t)x, Q_t)(dz), t \geq 0, \tag{3.2}$$

where E denotes the expectation and $\mathcal{N}(S(t)x, Q_t)(dz)$ is the Gaussian measure induced by $z(t, x)$ with mean $S(t)x$ and covariance Q_t given by

$$Q_t \equiv \int_0^t S(\theta)QS^*(\theta)d\theta, t \geq 0. \tag{3.3}$$

One can verify that $R_0 = I$, $R_{t+s} = R_tR_s, t, s \geq 0$. Clearly, it is a contraction on $B_b(H)$ and that it is positivity preserving in the sense that $\phi(x) \geq 0$ implies that $(R_t\phi)(x) \geq 0$, however it is not a strongly continuous semigroup on $B_b(H)$ or $C_b(H)$. It is, however, a weakly continuous semi group on $BUC(H)$ as illustrated in an interesting paper by Cerrai [7]. This semigroup is well known as the Ornstein-Uhlenbeck semigroup; see for details [5, 6].

In a recent paper by Da Prato and Zabczyk [4], see also [5], it was shown that by weakening the topology, or more precisely, using a larger space with a weaker topology one can obtain a strongly continuous semigroup by extension while preserving contraction and positivity properties. Our study of the HJB equation is based on this strongly continuous semigroup. For this purpose we must introduce some basic assumptions:

(H1): A is the infinitesimal generator of a C_0-semigroup, $S(t)$, $t \geq 0$, in H satisfying

$$\| S(t) \|_{\mathcal{L}(H)} \leq Me^{-\omega t}, t \geq 0, \omega > 0, M \geq 1.$$

(H2): Q is a positive, symmetric, bounded operator in H so that the operator Q_t given by (3.3) is nuclear for all $t \geq 0$ and $Sup_{t \geq 0} TrQ_t < \infty$.

(H3): W is a cylindrical Wiener process in H with $CovW(1) = I$.

(H4): $ImS(t) \subset ImQ_t^{1/2}$ and the operator valued function $\Gamma(t) \equiv (Q_t^{-1/2}S(t))$, $t \geq 0$, is Laplace transformable.

Remarks. It may be interesting to make some comments on the hypotheses (H1) – (H4). A sufficient condition for the assumption (H2) to hold is that Q has finite trace. This follows from the estimate

$$TrQ_\infty \leq (M^2/2\omega)TrQ$$

which is easily verified by direct computation. Alternatively, (H2) holds if Q is merely a bounded positive operator in H and the semigroup $S(t)$, $t > 0$, is Hilbert-Schmidt. This assumption guarantees the existence of an invariant measure for the Markov semigroup $R_t, t \geq 0$, corresponding to the linear system (3.1). Hypothesis (H4) is equivalent to the null controllability [6] of the deterministic system

$$\dot{x} = Ax + \sqrt{Q}\ u, t \geq 0,$$

where u denotes the control. This condition guarantees strong Feller property for the Markov semigroup $R_t, t \geq 0$, [5, Theorem 7.2.1] and this, in turn, guarantees uniqueness of invariant measure whenever it exists. The existence and uniqueness of an invariant measure are crucial in our study of the HJB equation.

Under the hypothesis (H1 – H4), the process z given by the (mild) solution of equation (3.1) has a unique invariant measure μ which is Gaussian, that is,

$\mu(\cdot) = \mathcal{N}(0, Q_\infty)(\cdot)$, with mean zero and covariance Q_∞. Using this invariant measure one can construct the following Hilbert and Sobolev spaces.
Let $D\phi$ and $D^2\phi$ denote the first and the second Frechet derivatives of the function $\phi : H \longrightarrow R$, whenever they exist as elements of H and $\mathcal{L}(H)$ respectively. Let $L_2(H, \mu)$ denote the equivalence classes of real valued functions on H which are square integrable with respect to the invariant measure μ. Furnished with the natural scalar product and the associated norm, it is a Hilbert space. Let $W^{1,2}(H, \mu)$ and $W^{2,2}(H, \mu)$ denote the Sobolev spaces furnished with the norm topologies

$$\| \phi \|_{W^{1,2}(H,\mu)} \equiv \Big(\| \phi \|^2_{L_2(H,\mu)} + \| \sqrt{Q} D\phi \|^2_{L_2(H,\mu)} \Big)^{1/2}$$
$$\| \phi \|_{W^{2,2}(H,\mu)} \equiv \left(\| \phi \|^2_{W^{1,2}(H,\mu)} + \int_H \| QD^2\phi \|^2_{H.S}\, \mu(dx) \right)^{(1/2)}.$$

Using Schwartz inequality, it follows from (3.2) that

$$|(R_t\phi)(x)|^2 \leq (R_t\phi^2)(x) \ \forall \phi \in B_b(H).$$

Integrating this with respect to the invariant measure μ we have

$$\int_H |R_t\phi|^2 \mu(dx)$$
$$\leq \int_H R_t\phi^2(x)\mu(dx) =< \phi^2, R_t^*\mu >=< \phi^2, \mu >= \int_H \phi^2(x)\mu(dx).$$

Since $B_b(H)$ is dense in $L_2(H, \mu)$ it follows from this inequality that the semigroup R_t can be extended from $B_b(H)$ to $L_2(H, \mu)$ while preserving the contraction property. We shall denote this extension by the same symbol $R_t, t \geq 0$. Clearly, on this space $\{R_t, t \geq 0\}$ is a strongly continuous contraction semigroup. Its infinitesimal generator is given by $\mathcal{C} \equiv \bar{\mathcal{A}}_0$, the closure of $\mathcal{A}_0$ in $L_2(H, \mu)$. The following result is well known [see 4, 5].

Proposition 3.1. *Suppose the assumptions* (H1) – (H4) *hold. Then the operator* $\mathcal{C}$ *generates a* C_0*-semigroup of contractions,* $R_t, t \geq 0$*, in* $L_2(H, \mu)$ *and it is the extension of the original Markov transition operator from* $B_b(H)$ *to* $L_2(H, \mu)$*. Further* $D(\mathcal{C}) \subset W^{1,2}(H, \mu)$ *and for* $t > 0$*,* R_t *is a family of compact operators in* $L_2(H, \mu)$*.*

Proof. See Da Prato-Zabczyk [4, 5].
Unless otherwise stated, throughout the rest of the paper the assumptions (H1) – (H4) remain in force even though they are not stated explictly.

We start with the following integration by parts formula. See also Da Prato [2].

Proposition 3.2. *For each $\varphi, \psi \in W^{1,2}(H,\mu)$ and $e \in H$ the following identity holds.*

$$\int_H \{(D\varphi, Q_\infty e)\psi + (D\psi, Q_\infty e)\varphi\}\mu(dx) = \int_H \varphi(x)\psi(x)(x,e)\mu(dx). \tag{3.4}$$

Proof. The proof essentially follows from Cameron-Martin-Girsanov formula. For $h \in H$, let $\mu_h(dz)$ denote the shift of the measure μ. Since μ is a centered Gaussian measure we have $\mu_h(dz) = \mathcal{N}(h, Q_\infty)(dz) = \rho_h(z)\mu(dz)$ where

$$\rho_h(z) = exp\{(Q_\infty^{-1}h, z) - (1/2)(Q_\infty^{-1}h, h)\}. \tag{3.5}$$

It is clear from this expression that all shifts in the direction $Im(Q_\infty) \subset H$ are admissible. Thus for admissible shifts, μ_h is absolutely continuous with respect to the measure μ. Hence for any $e \in H$ we have

$$\begin{aligned}
&\int_H (D\varphi, Q_\infty e)\psi\mu(dx) \\
&= \lim_{\epsilon\to 0}(1/\epsilon)\int_H \{\varphi(x+\epsilon Q_\infty e) - \phi(x)\}\psi(x)\mu(dx) \\
&= \lim_{\epsilon\to 0}(1/\epsilon)\left\{\int_H \varphi(z)\psi(z-\epsilon Q_\infty e)\mu_{\epsilon Q_\infty e}(dz) - \int_H \varphi(z)\psi(z)\mu(dz)\right\} \\
&= \lim_{\epsilon\to 0}(1/\epsilon)\left\{\int_H \varphi(z)\Big(\psi(z-\epsilon Q_\infty e)\rho_{\epsilon Q_\infty e}(z) - \psi(z)\Big)\mu(dz)\right\} \\
&= -\int_H (D\psi, Q_\infty e)\varphi\ \mu(dz) + \int_H \varphi(z)\psi(z)(e,z)\ \mu(dz).
\end{aligned} \tag{3.6}$$

This is (3.4). ■

Let $\{\lambda_i, e_i\}$ and $\{q_i, e_i\}$ be the set of eigen values and eigen vectors for the covariance operators Q_∞ and Q respectively and suppose that $\{e_i\}$ is orthonormal in H forming a basis for H.

Proposition 3.3. *Suppose $\gamma \equiv \sum(\lambda_i/q_i)^2 < \infty$. Then for any $\psi \in W^{1,2}(H,\mu)$, $x \longrightarrow |x|\psi(x)$ belongs to $L_2(H,\mu)$ and there exists a constant*

$C > 0$ such that

$$\int_H |\psi|^2 |x|^2 \mu(dx) \tag{3.7}$$
$$\leq C \left\{ \int_H |\psi|^2 \mu(dx) + \int_H |\sqrt{Q} D\psi|^2 \mu(dx) \right\} \equiv C \parallel \psi \parallel^2_{W^{1,2}(H,\mu)}$$

for all $\psi \in W^{1,2}(H,\mu)$, where $C = 2Tr(Q_\infty) + 4\gamma \parallel \sqrt{Q} \parallel^2$.

Proof. Taking $\varphi(x) = \psi(x)(x,e)$ and substituting in the expression (3.4) we have

$$\int \psi^2(x)(x,e)^2 \mu(dx)$$
$$= \int_H \psi^2(x)(e, Q_\infty e)\mu(dx) + 2\int_H (D\psi, Q_\infty e)\psi(x)(x,e)\mu(dx).$$

Using Cauchy inequality applied to the second term on the right hand side we obtain

$$\int \psi^2(x)(x,e)^2 \mu(dx) \leq 2(e, Q_\infty e) \int_H \psi^2(x)\mu(dx) + 4\int_H (D\psi, Q_\infty e)^2 \mu(dx).$$

Then taking $e = e_i$ and summing over all $i \geq 1$, it follows from the preceding inequality that

$$\int \psi^2(x)|x|^2 \mu(dx)$$
$$\leq 2Tr(Q_\infty) \int_H \psi^2(x)\mu(dx) + 4\sum_{i\geq 1} \int_H (D\psi, (\lambda_i/q_i)Qe_i)^2 \mu(dx)$$
$$\leq 2Tr(Q_\infty) \int_H \psi^2(x)\mu(dx) + 4\sum_{i\geq 1} (\lambda_i/q_i)^2 \int_H (D\psi, Qe_i)^2 \mu(dx).$$

The estimate (3.7) follows immediately from this inequality. ∎

Remark. Note that if $\{q_i = 1, i \geq 1\}$ then Q is the identity operator and in this case the assumption, $\gamma < \infty$, is trivially satisfied. This follows from the simple fact that Q_∞ is a bounded positive nuclear operator (assumption (H2)) and hence Hilbert-Schmidt. In fact, we have $\parallel Q_\infty \parallel^2_{H.S} \leq \parallel Q_\infty \parallel TrQ_\infty$. Further the assumption, $\gamma < \infty$, implies that the injection $W^{1,2}(H,\mu) \hookrightarrow L_2(H,\mu)$ is compact.

Corollary 3.4. *For every $\phi \in W^{2,2}(H,\mu)$, $x \longrightarrow |x|^2\phi(x)$ belongs to $L_2(H,\mu)$.*

Proof. Since $\phi \in W^{2,2}(H,\mu) \subset W^{1,2}(H,\mu)$ it follows from the Proposition 3.3 that

$$x \longrightarrow |x|\phi(x)$$

belongs to $L_2(H,\mu)$. Thus it suffices to verify that $x \longrightarrow |x|\phi(x)$ actually belongs to $W^{1,2}(H,\mu)$. But this follows immediately from the fact that each of the terms in the following expression

$$D(|x|\phi(x)) = |x|D\phi(x) + (x/|x|)\phi$$

belongs to $L_2(H,\mu)$. ■

Proposition 3.5. *For each $\Phi \in W^{2,2}(H,\mu)$ there exists a positive constant C_1 such that*

$$\int_H |x|^4\Phi^2(x)\mu(dx) \leq C_1 \parallel \Phi \parallel^2_{W^{2,2}(H,\mu)} . \tag{3.8}$$

Proof. Here we give only an outline of the proof. Using the identity (3.4) with $\varphi \equiv (x,e_i)\Phi(x)$ and $\psi \equiv (x,e_j)^2\Phi(x)$ and then summing over the indices $\{i,j\}$, we obtain

$$\int_H |x|^4\Phi^2(x)\mu(dx) \leq 4\int_H (Q_\infty D\Phi, x)^2\mu(dx) + \beta\int_H \Phi^2|x|^2\mu(dx), \tag{3.9}$$

where $\beta = 2 \parallel Q_\infty \parallel + TrQ_\infty$. Dealing with the first term of (3.9) we have

$$\begin{aligned}
&\int_H (Q_\infty D\Phi, x)^2\mu(dx)\\
&= \sum_{i\geq 1}\int_H (\sqrt{Q}_\infty D\Phi, e_i)^2(\sqrt{Q}_\infty x, e_i)^2\mu(dx)\\
&= \sum_{i\geq 1}(\lambda_i/q_i)^2\int_H (\sqrt{Q}D\Phi, e_i)^2(\sqrt{Q}x, e_i)^2\mu(dx)\\
&\leq \parallel \sqrt{Q} \parallel^2 \gamma\int_H |\sqrt{Q}D\Phi|^2|x|^2\mu(dx)\\
&\leq \parallel \sqrt{Q} \parallel^2 \gamma\sum_{i\geq 1}\int_H (\sqrt{Q}D\Phi, e_i)^2|x|^2\mu(dx).
\end{aligned} \tag{3.10}$$

Using the estimate (3.7) for $\psi \equiv (\sqrt{Q}D\Phi, e_i)$ we have

$$\begin{aligned}
&\sum_{i\geq 1}\int_H (\sqrt{Q}D\Phi, e_i)^2|x|^2\mu(dx)\\
&\leq C\Big(\int_H |\sqrt{Q}D\Phi|^2\mu(dx) + \int_H \parallel \sqrt{Q}D^2\Phi(x)\sqrt{Q} \parallel^2_{H.S}\mu(dx)\Big).
\end{aligned} \tag{3.11}$$

Similarly, using the estimate (3.7) for $\psi \equiv \Phi$ the second term of (3.9) gives

$$\beta \int_H \Phi^2(x)|x|^2\mu(dx) \le \beta C\Big\{\int_H \Phi^2\mu(dx) + \int_H |\sqrt{Q}D\Phi|^2\mu(dx)\Big\}. \tag{3.12}$$

Using (3.10) – (3.12) into (3.9), the inequality (3.8) follows, where the constant C_1 can be taken as

$$C_1 = (2 \parallel Q_\infty \parallel + TrQ_\infty + 4 \parallel \sqrt{Q} \parallel^2 \gamma)C.$$

■

4. Distributions and Ornstein-Uhlenbeck operator

Here, first we consider the Ornstein-Uhlenbeck operator $\mathcal{A}_0$ given by

$$(\mathcal{A}_0\varphi)(x) = (1/2)Tr((QD^2\varphi)(x)) + (A^*D\varphi(x), x). \tag{4.1}$$

We show that, associated to this formal differential operator, there exists a bilinear form on the Sobolev space $W^{1,2}(H,\mu)$ and hence a bounded linear operator from this space to its dual which is characterized here. Let $\mathcal{E}_A(H)$ denote the class of exponential functions of the form

$$\mathcal{E}_A(H) \equiv \{\phi : \phi(x) = Re(\sum_{k=1}^m a_k\ e^{i(z_k,x)}), z_k \in D(A^*), a_k \in C, m \in N\}.$$

It is clear that $\mathcal{E}_A(H)$ is dense in all of the spaces $W^{2,2}(H,\mu)$, $W^{1,2}(H,\mu)$, $L_2(H,\mu)$ and that $\mathcal{E}_A(H) \subset D(\mathcal{A}_0)$. A non unique characterization of the dual of the Hilbert space $W^{1,2}(H,\mu)$ is given as follows. Let $\varphi_0, \varphi_1 \in L_2(H,\mu)$ and define

$$\theta(x) \equiv a_0\varphi_0(x) + a_1(D\varphi_1(x), Q_\infty e), x \in H, \tag{4.2}$$

for any $e \in H$ and $a_0, a_1 \in R$, where D denotes the Frechet derivative in some generalized sense to be clarified shortly. Define

$$\ell_\theta(\psi) \equiv \int_H a_0\varphi_0\psi\mu(dx) + \int_H a_1(D\varphi_1, Q_\infty e)\psi\mu(dx). \tag{4.3}$$

We verify that ℓ_θ defines a continuous linear functional on $W^{1,2}(H,\mu)$. Since $\varphi_0 \in L_2(H,\mu)$ and $\psi \in W^{1,2}(H,\mu)$, the first integral is well defined.

For the second term, using Cameron-Martin formula we have

$$
\begin{aligned}
&\int_H a_1(D\varphi_1, Q_\infty e)\psi\mu(dx) \\
(4.4)\qquad &= \int_H \varphi_1(x)\{-a_1(D\psi(x), Q_\infty e) + a_1\psi(x)(x,e)\}\mu(dx) \\
&= \int_H \varphi_1(x)\{-a_1(\sqrt{Q}D\psi(x), Q^{-1/2}Q_\infty e) + a_1\psi(x)(x,e)\}\mu(dx).
\end{aligned}
$$

One can easily verify that

$$\| Q^{-1/2}Q_\infty \| \le \sqrt{\gamma \| Q \|}.$$

Since $\psi \in W^{1,2}(H,\mu)$, it follows from this that the first term within the braces belongs to $L_2(H,\mu)$ and by proposition 3.3 the second term also belongs to $L_2(H,\mu)$. This implies that the last integral in (4.3) is well defined for any $\varphi_1 \in L_2(H,\mu)$ and hence every θ of the form (4.2) induces a continuous linear functional ℓ_θ on $W^{1,2}(H,\mu)$. Thus $\theta \in (W^{1,2}(H,\mu))^*$. Similarly, we can define the dual of $W^{2,2}(H,\mu)$. An element of the form

$$
\begin{aligned}
(4.5)\qquad \theta(x) &\equiv a_0\phi_0(x) + a_1(D\phi_1(x), Q_\infty e) + a_2(D^2\phi_2(x)Q_\infty f, Q_\infty g) \\
&\equiv \theta_0 + \theta_1 + \theta_2.
\end{aligned}
$$

for arbitrary $\{\phi_0,\phi_1,\phi_2\} \in L_2(H,\mu)$, $\{e,f,g\} \in H$ and $\{a_0,a_1,a_2\} \in R$ defines a continuous linear functional on $W^{2,2}(H,\mu)$. It suffices to justify this for the last component θ_2. Define

$$\ell_{\theta_2}(\psi) = a_2\int_H (D^2\phi_2 Q_\infty f, Q_\infty g)\psi\mu(dx).$$

By similar computation one can easily verify that

$$
\begin{aligned}
\ell_{\theta_2}(\psi) &= a_2\int_H (D^2\phi_2 Q_\infty f, Q_\infty g)\psi\mu(dx) \\
&= a_2\int_H \{-(D\phi_2, Q_\infty f)(D\psi, Q_\infty g) + (D\phi_2, Q_\infty f)(g,x)\psi\}\mu(dx) \\
(4.6)\qquad &= a_2\int_H \phi_2\{(D^2\psi Q_\infty f, Q_\infty g) - (D\psi, Q_\infty g)(x,f)\}\mu(dx) \\
&\quad - a_2\int \phi_2\{(D\psi, Q_\infty f)(x,g) + \psi(Q_\infty g, f)\}\mu(dx) \\
&\quad + a_2\int \phi_2\psi\ (x,f)(x,g)\mu(dx).
\end{aligned}
$$

Since $\psi \in W^{2,2}(H,\mu)$ the first term of the last expression within the braces belongs to $L_2(H,\mu)$; by Proposition 3.3, the second and the third terms within the braces also belong to $L_2(H,\mu)$. Clearly, the fourth term within the brace belongs to $L_2(H,\mu)$ and by virtue of the Proposition 3.5, the map $x \longrightarrow \psi(x)(x,f)(x,g)$ belongs to $L_2(H,\mu)$. Thus ℓ_{θ_2} is a continuous linear functional on $W^{2,2}(H,\mu)$ and consequently corresponding to each θ given by (4.5), ℓ_θ induces a continuous linear functional on $W^{2,2}(H,\mu)$ and hence $\theta \in (W^{2,2}(H,\mu))^*$. In line with the finite dimensional cases we may denote this dual by $W^{-2,2}(H,\mu)$. With respect to the topology induced by the norm defined by

$$\| \theta \| \equiv \sup\{\ell_\theta(\psi), \| \psi \|_{W^{2,2}(H,\mu)} = 1\}$$

it is a Banach space. In general, using the Banach spaces $L_p(H,\mu)$, $1 \leq p \leq \infty$, we can prove the following general result.

Proposition 4.1. *For $1 < p, q < \infty$, satisfying $1/p + 1/q = 1$, and for any $m \in N$, the dual of $W^{m,p}(H,\mu)$ denoted by $W^{-m,q}(H,\mu)$ is a Banach space. The "negative" norm of any element $\theta \in W^{-m,q}(H,\mu)$ is given by*

$$\| \theta \|_{W^{-m,q}(H,\mu)} = \sup\{\ell_\theta(\psi), \psi \in W^{m,p}(H,\mu), \| \psi \|_{W^{m,p}(H,\mu)} = 1\}.$$

As in the L_2 case, an element θ of the form

$$\theta(x) \equiv \sum_{k=0}^{m} a_k (D^k \phi_k)(x)(Q_\infty g_1, Q_\infty g_2, \cdots, Q_\infty g_k)$$

for $\{a_k \in R, \phi_k \in L_q(H,\mu), 0 \leq k \leq m\}$ induces a continuous linear functional on $W^{m,p}(H,\mu)$ where $(D^k\phi_k)(x)(h_1, h_2, \cdots, h_k)$ is a multilinear (k-linear) form on H for μ almost all $x \in H$.

For convenience of notation, we set $L_2(H,\mu) \equiv \mathcal{H}$ and $W^{1,2}(H,\mu) \equiv \mathcal{V}$ and let $\mathcal{V}^* = W^{-1,2}(H,\mu)$ denote the dual of $\mathcal{V}$. Identifying $\mathcal{H}$ with its own dual we have the so called Gelfand triple

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*,$$

with continuous and dense embeddings. Recall that we used $< \xi, \eta >_{\mathcal{V}^*,\mathcal{V}}$ to denote the duality pairing of $\xi \in \mathcal{V}^*$ and $\eta \in \mathcal{V}$. In case $\xi \in \mathcal{H}$ and $\eta \in \mathcal{V}$, it is clear that

$$< \xi, \eta >_{\mathcal{V}^*,\mathcal{V}} = (\xi, \eta)_{\mathcal{H}}.$$

Now define the bilinear form

$$a(\phi,\psi) \equiv (1/2)\int_H (\sqrt{Q}D\varphi, \sqrt{Q}D\psi)\mu(dx). \tag{4.7}$$

Lemma 4.2. *The Ornstein-Uhlenbeck operator $\mathcal{A}_0$ satisfies the following identity*

$$\begin{aligned}\int_H (\mathcal{A}_0\phi)\ \psi\ \mu(dx) &= -(1/2)\int_H (\sqrt{Q}D\phi, \sqrt{Q}D\psi)\mu(dx)\\ &= \int_H \phi\ (\mathcal{A}_0\psi)\ \mu(dx),\end{aligned} \tag{4.8}$$

for all $\phi,\psi \in D(\mathcal{A}_0)$. Further, there exists a unique (linear) operator $\mathcal{A} \in \mathcal{L}(\mathcal{V},\mathcal{V}^*)$ such that the bilinear form has the representation

$$a(\phi,\psi) = -< \mathcal{A}\phi,\psi >_{\mathcal{V}^*,\mathcal{V}} = -< \phi, \mathcal{A}\psi >_{\mathcal{V},\mathcal{V}^*} = a(\psi,\phi) \tag{4.9}$$

for all $\phi,\psi \in \mathcal{V}$, and $-\mathcal{A}$ is coercive. The part of $\mathcal{A}$ in $\mathcal{H}$ denoted by $\mathcal{A}_{\mathcal{H}}$ coincides with the infinitesimal generator $\mathcal{C}$ of the Ornstein-Uhlenbeck semigroup $R_t, t \geq 0$, in $\mathcal{H}$.

Proof. The first statement concerning the identity (4.8) follows from the integration by parts formula (3.4) and the Lyapunov equation $AQ_\infty + Q_\infty A^* = -Q$. For the second statement we consider the bilinear form

$$a(\phi,\psi) \equiv (1/2)\int_H (\sqrt{Q}D\varphi, \sqrt{Q}D\psi)\mu(dx). \tag{4.10}$$

Clearly, it follows from the following inequality

$$\begin{aligned}|a(\phi,\psi)| &\leq (1/2)\Big(\int_H |\sqrt{Q}D\phi|^2\mu(dx)\Big)^{1/2}\Big(\int_H |\sqrt{Q}D\psi|^2\mu(dx)\Big)^{1/2}\\ &\leq (1/2)\parallel \phi \parallel_{\mathcal{V}} \parallel \psi \parallel_{\mathcal{V}},\end{aligned} \tag{4.11}$$

that it is a continuous map from $\mathcal{V}\times\mathcal{V}$ to R. It is evident from this that for any fixed $\phi \in \mathcal{V}$, the map

$$\psi \longrightarrow a_\phi(\psi) = a(\phi,\psi)$$

is a continuous linear functional on $\mathcal{V}$. Hence by Riesz theorem, there exists a unique $\eta = \eta_\phi \in \mathcal{V}^*$ such that

$$a_\phi(\psi) = < \eta, \psi >_{\mathcal{V}^*,\mathcal{V}} \ \text{ for all } \psi \in \mathcal{V}. \tag{4.12}$$

Consequently, there exists a unique continuous linear map $\mathcal{A}$ from $\mathcal{V}$ to $\mathcal{V}^*$ such that $\eta_\phi = -\mathcal{A}\phi$. The same statement is valid if the roles of ϕ and ψ are interchanged. Thus the bilinear form a has the characterization

$$a(\phi,\psi) =< -\mathcal{A}\phi, \psi >= - < \phi, \mathcal{A}\psi >= a(\psi,\phi). \tag{4.13}$$

Further, the coercivity follows from the following inequality,

$$a(\phi,\phi) + (1/2) \parallel \phi \parallel_{\mathcal{H}}^2 \geq (1/2) \parallel \phi \parallel_{\mathcal{V}}^2 . \tag{4.14}$$

The part of $\mathcal{A}$ in $\mathcal{H}$, denoted by $\mathcal{A}_{\mathcal{H}}$, is a closed densely defined linear operator in $\mathcal{H}$. By use of Lax-Milgram theorem one can show that $\mathcal{A}_{\mathcal{H}}$ is m-dissipative, hence it follows from Lumer-Phillips theorem that it generates a unique C_0-semigroup in $\mathcal{H}$. The coincidence of $\mathcal{A}_{\mathcal{H}}$ with $\mathcal{C}$ follows from uniqueness of the semigroup $R_t, t \geq 0$, they generate. This completes the proof. ∎

5. Generalized solution on finite time horizon

Throughout the remainder of this paper we assume, without further notice, that the basic hypotheses of Section 3 remain in force. Before we consider the HJB equation (1.5), or equivalently, (1.12), we treat an abstract version of this with reference to the Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$. First we consider the linear problem:

$$\begin{aligned} dy/dt &= \mathcal{A}y + f(t), t \geq 0 \\ y(0) &= y_0. \end{aligned} \tag{5.1}$$

The following result is important in our study of the HJB equation. For each $t > 0$, let $I_t \equiv [0,t]$ denote the closed interval and set $I \equiv I_T$ for T finite.

Introduce the vector spaces $\mathcal{W}_t \equiv \{\varphi : \varphi \in L_2(I_t, \mathcal{V}),\ \dot{\varphi} \in L_2(I_t, \mathcal{V}^*)\}$. Furnished with the norm topology,

$$\parallel \varphi \parallel_{\mathcal{W}_t} \equiv \sqrt{\parallel \varphi \parallel_{L_2(I_t,\mathcal{V})}^2 + \parallel \dot{\varphi} \parallel_{L_2(I_t,\mathcal{V}^*)}^2}, \tag{5.2}$$

this is a Hilbert space. Further, it is well known that the embedding $\mathcal{W}_t \hookrightarrow C(I_t, \mathcal{H})$ is continuous. Strictly speaking, the inclusion is understood in the sense that every element $\varphi \in \mathcal{W}_t$ has a continuous modification with values in $\mathcal{H}$. We write $\mathcal{W}_T \equiv \mathcal{W}$.

Lemma 5.1. *For every $y_0 \in \mathcal{H}$ and $f \in L_2(I, \mathcal{V}^*)$, the evolution equation* (5.1) *has a unique solution $y \in L_\infty(I, \mathcal{H}) \cap L_2(I, \mathcal{V})$. The solution y is differentiable in the sense of $\mathcal{V}^*$-valued distributions and $\dot{y} \in L_2(I, \mathcal{V}^*)$, and hence $y \in \mathcal{W}$. Further $\{y_0, f\} \longrightarrow y$ is a continuous linear map from $\mathcal{H} \times L_2(I, \mathcal{V}^*)$ to $\mathcal{W}$. In fact, it is Lipschitz and there exists a* (*Lipschitz*) *constant $C > 0$ such that*

$$
\begin{aligned}
&\| y_1 - y_2 \|_{L_2(I,\mathcal{V})} \leq C\{|y_{01} - y_{02}|_{\mathcal{H}} + \| f_1 - f_2 \|_{L_2(I,\mathcal{V}^*)}\} \\
&\| y_1 - y_2 \|_{C(I,\mathcal{H})} \leq C\{|y_{01} - y_{02}|_{\mathcal{H}} + \| f_1 - f_2 \|_{L_2(I,\mathcal{V}^*)}\} \\
&\| \dot{y}_1 - \dot{y}_2 \|_{L_2(I,\mathcal{V}^*)} \leq C\{|y_{01} - y_{02}|_{\mathcal{H}} + \| f_1 - f_2 \|_{L_2(I,\mathcal{V}^*)}\},
\end{aligned}
\tag{5.3}
$$

where $\{y_i, i = 1, 2\}$ denote the solutions of (5.1) *corresponding to the pairs $\{y_{0i}, f_i\}, i = 1, 2$.*

Proof. We have set up the problem in the framework of J.L. Lions using the Gelfand triple. Thus the proof is classical which is based on Fadeo-Galerkin approximation. ■

Remark. Under the assumption of Proposition 3.3, $\gamma < \infty$, the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$ is compact. Let $\Lambda : \mathcal{V} \longrightarrow \mathcal{V}^*$ denote the canonical isomorphism (duality map) of $\mathcal{V}$ onto $\mathcal{V}^*$. Then it is clear that Λ^{-1} is a compact operator in $\mathcal{V}^*$ and hence there exists a countable sequence of eigen values and eigen vectors of Λ^{-1} each with finite multiplicity. The eigen vectors of Λ^{-1} are also the eigen vectors of Λ and they are complete in all the Gelfand triple. In fact, they are orthogonal in $\mathcal{V}$ and $\mathcal{V}^*$ and orthonormal in $\mathcal{H}$. For proof of Lemma 5.1 one can use this basis or any other complete orthonormal basis. A more concrete basis (the Hermite basis) was introduced by Da Prato-Zabczyk in [4, 5], which may be used not only for the proof of existence but also for computation of solutions.

Now we are prepared to consider the following semilinear evolution equation in $\mathcal{H}$

$$
\begin{aligned}
d\phi/dt &= \mathcal{A}\phi + G(\phi) + f, t \in I, \\
\phi(0) &= \phi_0.
\end{aligned}
\tag{5.4}
$$

We present here two general existence results for equation (5.4) and two corollaries as application to the HJB equation (1.12).

Consider the operator $\mathcal{A}$ of Lemma 4.2, and set $\alpha = (1/2)$ and $\beta \geq (1/2)$. In view of the identity (4.13), we may write the ellipticity condition (4.14) as

$$< -\mathcal{A}\phi, \phi > +\beta \parallel \phi \parallel_{\mathcal{H}}^2 \geq \alpha \parallel \phi \parallel_{\mathcal{V}}^2 .$$

Theorem 5.2. *Let $\mathcal{A}$ denote the generalized Ornstein-Uhlenbeck generator as introduced in Lemma* 4.2 *and suppose G maps $\mathcal{V}$ to $\mathcal{V}^*$ and there exists a positive constant $K < \alpha$ such that*

(G1): $\parallel G(\xi) \parallel_{\mathcal{V}^*} \leq K(1+ \parallel \xi \parallel_{\mathcal{V}})$ *for all* $\xi \in \mathcal{V}$

(G2): $\parallel G(\xi) - G(\eta) \parallel_{\mathcal{V}^*} \leq K \parallel \xi - \eta \parallel_{\mathcal{V}}$ *for all* $\xi, \eta \in \mathcal{V}$.

Then, for every $\phi_0 \in \mathcal{H}$ and $f \in L_2(I, \mathcal{V}^*)$, the evolution equation (5.4) *has a unique solution in $\mathcal{W}$.*

Proof. First we give an a priori bound for the solution. Let ϕ denote a solution of equation (5.4). Scalar multiplying equation (5.4) by ϕ, in the sense of $\mathcal{V}^*, \mathcal{V}$ pairing, and using Cauchy-Schwartz inequalities one obtains, for any $\epsilon > 0$, the following expression

$$\begin{aligned} &|\phi(t)|_{\mathcal{H}}^2 + 2(\alpha - (K + \epsilon)) \int_0^t \parallel \phi(s) \parallel_{\mathcal{V}}^2 ds \\ &\leq \left\{ |\phi_0|^2 + (K^2/\epsilon)T + 2\beta \int_0^t |\phi(s)|^2 ds + (1/\epsilon) \int_0^t \parallel f(s) \parallel_{\mathcal{V}^*}^2 ds \right\} \end{aligned} \tag{5.5}$$

for all $t \in I$. Since by our assumption $0 < K < \alpha$, we can choose ϵ sufficiently small so that $K_\epsilon \equiv K + \epsilon < \alpha$. Fixing ϵ at this value, and using Gronwall inequality one can easily verify that

$$|\phi(t)|_{\mathcal{H}}^2 \leq e^{2\beta t} \left\{ |\phi_0|_{\mathcal{H}}^2 + (K^2/\epsilon)T + (1/\epsilon) \int_0^T \parallel f(s) \parallel_{\mathcal{V}^*} ds \right\} \tag{5.6}$$

for all $t \in I$. Using this estimate in (5.5) we also obtain

$$\begin{aligned} &\int_0^T \parallel \phi(t) \parallel_{\mathcal{V}}^2 dt \\ &\leq [e^{2\beta T}/2(\alpha - K_\epsilon)] \left\{ |\phi_0|_{\mathcal{H}}^2 + (K^2/\epsilon)T + (1/\epsilon) \int_0^T \parallel f(s) \parallel_{\mathcal{V}^*} ds \right\}. \end{aligned} \tag{5.7}$$

Clearly it follows from (5.6) and (5.7) that $\phi \in L_\infty(I, \mathcal{H}) \cap L_2(I, \mathcal{V})$. Using the estimate (5.7) and the fact that $\mathcal{A}$ is a bounded linear operator from $\mathcal{V}$ to $\mathcal{V}^*$, it follows from (5.4) that $\dot{\phi} \in L_2(I, \mathcal{V}^*)$. In fact, there exist positive constants $C_1 = C_1(\alpha, \beta, K, T), C_2 = C_2(\alpha, \beta, K, T)$ such that

$$\parallel \dot{\phi} \parallel_{L_2(I,\mathcal{V}^*)}^2 \leq C_1 + C_2(|\phi_0|_{\mathcal{H}}^2 + \parallel f \parallel_{L_2(I,\mathcal{V}^*)}^2). \tag{5.8}$$

Hence if (5.4) has any solution it must belong to $\mathcal{W}$. Now we prove the existence. Let $y, z \in L_2(I, \mathcal{V}) \cap C(I, \mathcal{H})$ with $y(0) = z(0) = \phi_0$. Consider the system of equations

$$
\begin{aligned}
&d\phi/dt = \mathcal{A}\phi + G(y) + f, \phi(0) = \phi_0, \\
&d\psi/dt = \mathcal{A}\psi + G(z) + f, \psi(0) = \phi_0.
\end{aligned} \tag{5.9}
$$

For the given y, z, it follows from the linear growth that both $G(y), G(z) \in L_2(I, \mathcal{V}^*)$, and hence by Lemma 5.1 each of these equations has unique solutions $\phi, \psi \in \mathcal{W}$. Let $\mathcal{S} \equiv \mathcal{S}_T$ denote the solution map so that $\phi = \mathcal{S}(y)$ and $\psi = \mathcal{S}(z)$. First, we show that $\mathcal{S}$ has a unique fixed point in $L_2(I, \mathcal{V})$. Subtracting the second equation from the first it follows from (5.9) that

$$
d(\phi - \psi)/dt = \mathcal{A}(\phi - \psi) + G(y) - G(z), \phi(0) - \psi(0) = 0. \tag{5.10}
$$

Following the procedure used in deriving the a-priori estimates (5.6) and (5.7) one can easily show that for all $t \in I$,

$$
\begin{aligned}
&|\phi(t) - \psi(t)|^2 + \alpha \int_0^t \parallel \phi(s) - \psi(s) \parallel_{\mathcal{V}}^2 ds \\
&\leq 2\beta \int_0^t |\phi(s) - \psi(s)|_{\mathcal{H}}^2 ds + (K^2/\alpha) \int_0^t \parallel y(s) - z(s) \parallel_{\mathcal{V}}^2 ds.
\end{aligned} \tag{5.11}
$$

Using Gronwall lemma the following estimates follow from (5.11)

$$
\begin{aligned}
&(E1) : \sup_{t \in I} |\phi(t) - \psi(t)|^2 \leq (K^2/\alpha) e^{2\beta T} \int_0^T \parallel y(s) - z(s) \parallel_{\mathcal{V}}^2 ds, \\
&(E2) : \int_0^T \parallel \phi(s) - \psi(s) \parallel_{\mathcal{V}}^2 ds \leq (K/\alpha)^2 e^{2\beta T} \int_0^T \parallel y(s) - z(s) \parallel_{\mathcal{V}}^2 ds.
\end{aligned} \tag{5.12}
$$

Since by assumption $\alpha > K$ it follows from the estimate $(E2)$ that there exists a τ, $0 < \tau < T_0 \equiv (1/\beta)\ell n(\alpha/K)$ such that for $T = \tau$ we have $(K/\alpha)e^{\beta\tau} \equiv \gamma_\tau < 1$, and consequently

$$
\parallel \phi - \psi \parallel_{L_2(I_\tau, \mathcal{V})} \leq \gamma_\tau \parallel y - z \parallel_{L_2(I_\tau, \mathcal{V})} . \tag{5.13}
$$

Thus the map $\mathcal{S}_\tau \equiv \mathcal{S}|_{[0,\tau]}$, the restriction of $\mathcal{S}$ to $L_2(I_\tau, \mathcal{V})$, is a contraction on $L_2(I_\tau, \mathcal{V})$ and hence equation (5.4), considered over the interval I_τ, has a unique solution $\varphi \in L_2(I_\tau, \mathcal{V})$. By virtue of the arguments leading to the a-priori estimates, we have $\varphi \in \mathcal{W}_\tau$ and hence $\varphi(\tau) \in \mathcal{H}$. Since $I = I_T$ is a compact interval it can be covered by a finite union of closed intervals of length equal to τ. Obviously, on each of these intervals the solution of (5.4) is uniquely determined starting from the states $\{\varphi(k\tau),\ k = 0, 1, 2\cdots, n\}$ where n is the largest integer satisfying $n\tau < T$. Hence, for each $\phi_0 \in \mathcal{H}$ and $f \in L_2(I, \mathcal{V}^*)$, the evolution equation (5.4) has a unique solution $\varphi \in \mathcal{W} = \mathcal{W}_T$. This completes the proof. ■

Next we remove the restriction, $\alpha > K$, and replace this with the condition that the nonlinear operator G is more regular and now maps $\mathcal{V}$ to $\mathcal{H}$ instead of $\mathcal{V}$ to $\mathcal{V}^*$.

Theorem 5.3. *Let $\mathcal{A}$ denote the generalized Ornstein-Uhlenbeck generator as introduced in Lemma* 4.2 *and suppose G maps $\mathcal{V}$ to $\mathcal{H}$ and there exists a positive constant K such that*

(G1): $\| G(\xi) \|_{\mathcal{H}} \leq K(1+ \| \xi \|_{\mathcal{V}})$ *for all* $\xi \in \mathcal{V}$

(G2): $\| G(\xi) - G(\eta) \|_{\mathcal{H}} \leq K \| \xi - \eta \|_{\mathcal{V}}$ *for all* $\xi, \eta \in \mathcal{V}$.

Then, for every $\phi_0 \in \mathcal{H}$ and $f \in L_2(I, \mathcal{V}^*)$, the evolution equation (5.4) *has a unique solution in $\mathcal{W}$. Further, equation* 5.4 *holds in the sense of $\mathcal{V}^*$-valued distributions on I or more precisely as elements of $L_2(I, \mathcal{V}^*)$.*

Proof. The proof of this result requires a slight modification of that of Theorem 5.2. Again starting from (5.9), instead of (5.11) we have

$$\begin{aligned}
&|\phi(t) - \psi(t)|^2 + 2\alpha \int_0^t \| \phi(s) - \psi(s) \|_{\mathcal{V}}^2 \, ds \\
&\leq ((1 + 2\beta\epsilon)/\epsilon) \int_0^t |\phi(s) - \psi(s)|_{\mathcal{H}}^2 ds \\
&+ (\epsilon K^2) \int_0^t \| y(s) - z(s) \|_{\mathcal{V}}^2 \, ds,
\end{aligned} \tag{5.14}$$

for arbitrary $\epsilon > 0$, and for all $t \in I$. By virtue of Gronwall lemma, it follows from this that, for any $\tau \in I$,

(5.15)

$$(E1) : \sup_{t \in I_\tau} |\phi(t) - \psi(t)|^2 \leq (\epsilon K^2) e^{((1+2\beta\epsilon)/\epsilon)\tau} \int_0^\tau \| y(s) - z(s) \|_{\mathcal{V}}^2 \, ds,$$

$$(E2) : \int_0^\tau \| \phi(s) - \psi(s) \|_{\mathcal{V}}^2 \, ds \leq \{(\epsilon K^2/2\alpha) e^{((1+2\beta\epsilon)/\epsilon)\tau}\} \int_0^\tau \| y(s) - z(s) \|_{\mathcal{V}}^2 \, ds.$$

Clearly for contraction, it suffices to choose τ so that

$$0 < \tau < -(\epsilon/(1 + 2\beta\epsilon)) \ell n(\epsilon K^2/2\alpha). \tag{5.16}$$

For (5.16) to hold we must choose ϵ so that

$$0 < \epsilon < (2\alpha/K^2). \tag{5.17}$$

Defining

$$\tau(\epsilon) \equiv -(\epsilon/(1+2\beta\epsilon))\ell n(\epsilon K^2/2\alpha)$$

one can verify that there is an optimum ϵ in the open interval specified by (5.17) that maximizes this expression. In fact, this is given by the solution of the following equation

$$\epsilon = \nu(\epsilon) \equiv (2\alpha/K^2)exp\{-(1+2\beta\epsilon)/(1+4\beta\epsilon)\}. \tag{5.18}$$

Note that $\nu(\epsilon), \epsilon \geq 0$, is a bounded, positive and nondecreasing function of ϵ. Hence this equation has a unique fixed point ϵ_m in the interval specified by (5.17). Thus it suffices to choose $\tau < \tau(\epsilon_m)$ for which the solution map $\mathcal{S}_\tau$ is a contraction. The proof is now completed using the arguments of the preceding theorem. ∎

Remark. Though the operator G is very general in theorem 5.2, we require the coercivity of the linear operator $\mathcal{A}$ to dominate over the Lipschitz coefficient of G. In Theorem 5.3, the operator G is more regular and no such dominance is necessary.

Now we consider the HJB equation (1.12) and prove the existence, uniqueness and regularity properties of its solutions as corollaries of Theorems 5.2 and 5.3 under two different assumptions on the operators $\mathcal{F}_1$ and $\mathcal{F}_2$. To this end, we identify the nonlinear operator G of equation (5.4) as follows:

$$G(\phi) \equiv \mathcal{F}_1(\phi) + \mathcal{F}_2(\phi). \tag{5.19}$$

First we present an existence result for the HJB equation (1.12) as a corollary to Theorem 5.2.

Corollary 5.4. *Let $\mathcal{A}$ denote the generalized Ornstein-Uhlenbeck generator, associated with the formal differential operator $\mathcal{A}_0$, as introduced in Lemma 4.2 and suppose both $F(x)$ and $\{B(x)u, u \in B_r \subset U\}$ are Borel measurable maps in H and are in the image of $\sqrt{Q}$ and there exist constants $k_1, k_2 \geq 0$ such that*

$$\begin{aligned} |Q^{-1/2}F(x)|_H &\leq k_1(1+|x|), \text{ for all } x \in H, \\ \| B^*(x)Q^{-1/2} \|_{\mathcal{L}(H,U)} &\leq k_2(1+|x|), \text{ for all } x \in H, \end{aligned} \tag{5.20}$$

and

$$\sup\{|h(u)|, u \in B_r\} \equiv \bar{h} < \infty.$$

Then, for every $\varphi_0 \in \mathcal{H}$ *and* $\tilde{g} \in L_2(I, \mathcal{V}^*)$, *the HJB equation* (1.12) *has a unique solution* $\varphi \in \mathcal{W}$ *provided* $\{k_1, k_2\}$ *are sufficiently small.*

Proof. It suffices to verify that the operator G given by (5.19) satisfies the assumptions (G1) and (G2) of Theorem 5.2. First, we verify the Lipschitz condition. Let $\psi \in \mathcal{V}$ and $\varphi_1, \varphi_2 \in \mathcal{V}$. Then using assumption (5.20) we have

$$\begin{aligned}
&|\int_H (\mathcal{F}_1(\varphi_1) - \mathcal{F}_1(\varphi_2))\psi\mu(dx)| \\
&= |\int_H (F(x), D(\varphi_1 - \varphi_2))\psi\mu(dx)| \\
&= |\int_H (Q^{-1/2}F(x), \sqrt{Q}D(\varphi_1 - \varphi_2))\psi\mu(dx)| \\
&\leq k_1 \int_H |\sqrt{Q}D(\varphi_1 - \varphi_2)|_H(1 + |x|_H)|\psi(x)|\mu(dx)
\end{aligned}$$

where $Q^{-1/2}$ is the pseudoinverse of $\sqrt{Q}$. Using the estimate (3.7) of Proposition 3.3, it follows that

$$\begin{aligned}
&|\int_H (\mathcal{F}_1(\varphi_1) - \mathcal{F}_1(\varphi_2))\psi\mu(dx)| \leq\ k_1 \parallel \varphi_1 - \varphi_2 \parallel_{\mathcal{V}} (|\psi|_{\mathcal{H}} + \sqrt{C} \parallel \psi \parallel_{\mathcal{V}}) \\
&\leq\ k_1(1 + \sqrt{C}) \parallel \varphi_1 - \varphi_2 \parallel_{\mathcal{V}} \quad \parallel \psi \parallel_{\mathcal{V}} .
\end{aligned}$$

Since this is true for arbitrary $\psi \in \mathcal{V}$, it follows that

$$\parallel \mathcal{F}_1(\varphi_1) - \mathcal{F}_1(\varphi_2) \parallel_{\mathcal{V}^*} \leq k_1(1 + \sqrt{C}) \parallel \varphi_1 - \varphi_2 \parallel_{\mathcal{V}} . \tag{5.21}$$

This is simply based on the arguments of distribution theory of Section 4. For the operator $\mathcal{F}_2$, note that the Hamiltonian H is Lipschitz (see equation 1.4),

$$|H(x,q) - H(x,p)| \leq r|q - p|_U, \quad \text{for all} \quad x \in H. \tag{5.22}$$

Then using assumption (5.20), it is easy to see that

$$\begin{aligned}
&| \int_H (\mathcal{F}_2(\varphi_1) - \mathcal{F}_2(\varphi_2))\psi\mu(dx)| \\
&\leq | \int_H r \parallel B^*(x)D(\varphi_1 - \varphi_2) \parallel_U |\psi|\mu(dx)| \\
&\leq r \int_H \parallel B^*(x)Q^{-1/2} \parallel \parallel \sqrt{Q}D(\varphi_1 - \varphi_2) \parallel_H |\psi|\mu(dx) \\
&\leq rk_2 \int_H |\sqrt{Q}D(\varphi_1 - \varphi_2)|_H(1 + |x|_H)|\psi(x)|\mu(dx).
\end{aligned}$$

Again using the estimate (3.7) of Proposition 3.3, it follows from this that

$$\parallel \mathcal{F}_2(\varphi_1) - \mathcal{F}_2(\varphi_2) \parallel_{\mathcal{V}^*} \leq k_2 r(1 + \sqrt{C}) \parallel \varphi_1 - \varphi_2 \parallel_{\mathcal{V}} . \tag{5.23}$$

Now it follows from (5.19), (5.21) and (5.23) that

$$\parallel G(\varphi_1) - G(\varphi_2) \parallel_{\mathcal{V}^*} \leq (k_1 + k_2 r)(1 + \sqrt{C}) \parallel \varphi_1 - \varphi_2 \parallel_{\mathcal{V}} . \tag{5.24}$$

Following similar procedure one can verify the growth condition

$$\parallel G(\varphi) \parallel_{\mathcal{V}^*} \leq c_2(1 + \parallel \varphi \parallel_{\mathcal{V}}). \tag{5.25}$$

for a suitable constant c_2 dependent only on $\{k_1, k_2, R, C, \bar{h}\}$, where $\bar{h} \equiv \{|h(u)|, u \in B_r\}$. Let K denote the smallest positive number equal or less than $(k_1 + k_2 r)(1 + \sqrt{C})$ such that

$$\sup_{\|\psi\|_{\mathcal{V}}=1} \{| < G(\varphi_1) - G(\varphi_2), \psi >_{\mathcal{V}^*,\mathcal{V}} |\} \leq K \parallel \varphi_1 - \varphi_2 \parallel_{\mathcal{V}} \tag{5.26}$$

for all $\varphi_1, \varphi_2 \in \mathcal{V}$. Then it follows from Theorem 5.2 that for $K < \alpha$, the HJB equation (1.12) has a unique solution $\varphi \in \mathcal{W}$. ■

The following result is a corollary of Theorem 5.3.

Corollary 5.5. *Let $\mathcal{A}$ denote the generalized Ornstein-Uhlenbeck generator as introduced in Lemma 4.2 and suppose both $F(x)$ and $\{B(x)u,\ u \in B_r \subset U\}$ are Borel measurable maps in H and are in the image of $\sqrt{Q}$ and there exist constants $k_1, k_2 \geq 0$ such that*

$$\sup_{x \in H} |Q^{-1/2)} F(x)|_H = k_1 < \infty, \ \sup_{x \in H} \parallel B^*(x)Q^{-1/2} \parallel_{\mathcal{L}(H,U)} = k_2 < \infty.$$

Then, for every $\varphi_0 \in \mathcal{H}$ and $\tilde{g} \in L_2(I, \mathcal{V}^*)$, the HJB equation (1.12) *has a unique solution $\varphi \in \mathcal{W}$.*

Proof. In view of Theorem 5.3, it suffices to verify that, under the given assumptions, the operator $G \equiv \mathcal{F}_1 + \mathcal{F}_2$ satisfies the hypotheses of Theorem 5.3. For the growth condition it is clear that for each $\phi \in \mathcal{V}$,

$$\begin{aligned} |\mathcal{F}_1(\phi)|^2_{\mathcal{H}} &\equiv \int_H (F(x), D\phi)^2 \mu(dx) \\ &= \int_H (Q^{-1/2}F(x), \sqrt{Q}D\phi)^2 \mu(dx) \leq k_1^2 \parallel \phi \parallel^2_{\mathcal{V}} . \end{aligned} \tag{5.27}$$

Similarly, we have

$$\begin{aligned} |\mathcal{F}_2(\phi)|^2_{\mathcal{H}} &\equiv \int_H |H(x, B^*(x)D\phi)|^2 \mu(dx) \\ &\leq \int_H (\bar{h} + r|B^*(x)Q^{-1/2}Q^{1/2}D\phi|_U)^2 \mu(dx) \\ &\leq \int_H (\bar{h} + rk_2|\sqrt{Q}D\phi(x)|_H)^2 \mu(dx), \end{aligned} \tag{5.28}$$

where $\bar{h} \equiv \sup\{|h(u)|, u \in B_r \subset U\}$. Thus

$$|\mathcal{F}_2(\phi)|_{\mathcal{H}} \leq (\bar{h} + rk_2 \parallel \phi \parallel_{\mathcal{V}}). \tag{5.29}$$

Clearly, it follows from this that both $\mathcal{F}_1, \mathcal{F}_2$ map $\mathcal{V}$ to $\mathcal{H}$ and further, by virtue of the estimates (5.27) and (5.29), there exists a constant k_3 such that

$$\parallel G(\phi) \parallel_{\mathcal{H}} \leq k_3(1 + \parallel \phi \parallel_{\mathcal{V}}). \tag{5.30}$$

Similarly, for the Lipschitz property, it is easy to see that

$$\begin{aligned} &\parallel \mathcal{F}_1(\phi_1) - \mathcal{F}_1(\phi_2) \parallel_{\mathcal{H}} \leq k_1 \parallel \phi_1 - \phi_2 \parallel_{\mathcal{V}} \\ &\parallel \mathcal{F}_2(\phi_1) - \mathcal{F}_2(\phi_2) \parallel_{\mathcal{H}} \leq rk_2 \parallel \phi_1 - \phi_2 \parallel_{\mathcal{V}} . \end{aligned}$$

From these estimates we have

$$\parallel G(\phi_1) - G(\phi_2) \parallel_{\mathcal{H}} \leq (k_1 + rk_2) \parallel \phi_1 - \phi_2 \parallel_{\mathcal{V}} . \tag{5.31}$$

Taking $K = \max\{k_3, (k_1 + rk_2)\}$ the nonlinear operator G satisfies the assumptions of Theorem 5.3. This completes the proof. ∎

Remark. In view of Corollaries 5.4 and 5.5, it is very interesting to see that the HJB equation (1.12) has generalized solutions under very general assumptions, like Borel measurability and linear growth conditions, for the maps F and B. By imposing further regularity conditions on the maps F and B, such as Lipschitz continuity and linear growth, one can prove that these generalized solutions are truly the value function of nonstationary control problems (1.1 – 1.3).

In the following section, we demonstrate how the preceding results are applied to control problems. Here we are interested only in the stationary control problem.

6. Stationary HJB equation and feedback control

We consider the stationary HJB equation (1.11) or equivalently the functional equation (1.14). We prove that for sufficiently large $\delta > 0$, it has a unique solution in $\mathcal{V}$. We replace the operator $\mathcal{A}_0$ by its extension $\mathcal{A}$. In other words we solve the equation

$$\Psi = R(\delta, \mathcal{A}) G_g(\Psi) \equiv R(\delta, \mathcal{A}) G(\Psi) + R(\delta, \mathcal{A}) g, \tag{6.1}$$

in the Sobolev space $\mathcal{V}$. We add to the hypothesis (H4) the following assumption.

$(H4)'$: The Laplace transform $\gamma(\lambda)$ given by $\gamma(\lambda) \equiv \int_0^\infty e^{-\lambda t} \parallel \Gamma(t) \parallel dt$, is a nonicreasing function of $\lambda \in [0, \infty)$ and $\lim_{\lambda \to \infty} \gamma(\lambda) = 0$.

A necessary and sufficient condition under which this hypothesis holds is given by Gozzi and Rouy [1]. For example, if

$$\parallel \Gamma(t) \parallel \leq C_1 / t^\theta + C_2 e^{-\nu t}$$

for some constants $C_1, C_2, \nu \geq 0$, and $\theta \in [0, 1)$, then $(H4)'$ holds.

Theorem 6.1. *Suppose the assumptions of Corollary* 5.5 *and the hypothesis* $(H4)'$ *hold. Then for sufficiently large discount factor* $\delta > 0$, *and for every* $g \in \mathcal{H}$, *the stationary HJB equation* (1.11) *or equivalently the functional equation* (6.1) *has a unique solution* $\Phi \in \mathcal{V}$.

Proof. We prove that, for sufficiently large $\delta > 0$, the functional equation (6.1) has a unique solution in $\mathcal{V}$. This is equivalent to showing that, for sufficiently large $\delta > 0$, the composition map $R(\delta, \mathcal{A}) G_g$ has a unique fixed point in $\mathcal{V}$. First we verify that this operator maps $\mathcal{V}$ into itself. For this we must show that, for any $\delta > 0$, $R(\delta, \mathcal{A})$ maps $\mathcal{H}$ to $\mathcal{V}$. It follows from Cameron-Martin formula that, for any $h \in H$,

$$(DR_t\phi(x), h) = \int_H \phi(y)(\Gamma(t)h, Q_t^{-1/2}y)\mathcal{N}(S(t)x, Q_t)(dy), \tag{6.2}$$

for all $\phi \in \mathcal{H}$. For details see Da Prato-Zabczyk [5, 6]. Using Schwartz inequality it follows from (6.2) that

$$(DR_t\phi(x), h)^2 \leq R_t\phi^2(x)|\Gamma(t)h|_H^2, x \in H. \tag{6.3}$$

Since μ is an invariant measure with respect to the Ornstein-Uhlenbeck semigroup $R_t, t \geq 0$, it follows from this that

$$|DR_t\phi|_{\mathcal{H}} \leq \parallel \Gamma(t) \parallel |\phi|_{\mathcal{H}}, \tag{6.4}$$

where $\Gamma(t) \equiv Q_t^{-1/2}S(t)$ as given in hypothesis (H4). Hence

$$|DR(\delta, \mathcal{A})\phi|_{\mathcal{H}} \leq \gamma(\delta)|\phi|_{\mathcal{H}}, \tag{6.5}$$

where $\gamma(\delta) \equiv \int_0^\infty e^{-\delta t} \parallel \Gamma(t) \parallel dt$. Since $R_t, t \geq 0$, is a C_0-semigroup of contractions in $\mathcal{H}$, it follows from this that

$$\parallel R(\delta, \mathcal{A})\phi \parallel_{\mathcal{V}} \leq ((1/\delta) + \parallel \sqrt{Q} \parallel \gamma(\delta))|\phi|_{\mathcal{H}}. \tag{6.6}$$

This shows that for each $\delta > 0$, $R(\delta, \mathcal{A})$ maps $\mathcal{H}$ to $\mathcal{V}$. We have already seen in Corollary 5.5 (see equation (5.30)) that G maps $\mathcal{V}$ into $\mathcal{H}$. Since $g \in \mathcal{H}$, it follows from these results that the composition map $R(\delta, \mathcal{A})G_g$ maps $\mathcal{V}$ into itself. Now we prove that, for sufficiently large $\delta > 0$, it has a fixed point. By Corollary (5.5), G satisfies the Lipschitz condition (see equation (5.31)) with Lipschitz constant K. Hence

$$\begin{aligned}
&\parallel R(\delta, \mathcal{A})G_g(\phi_1) - R(\delta, \mathcal{A})G_g(\phi_2) \parallel_{\mathcal{V}} \\
&\leq ((1/\delta) + \parallel \sqrt{Q} \parallel \gamma(\delta))|G_g(\phi_1) - G_g(\phi_2)|_{\mathcal{H}} \\
&\leq K((1/\delta) + \parallel \sqrt{Q} \parallel \gamma(\delta)) \parallel \phi_1 - \phi_2 \parallel_{\mathcal{V}} \\
&\leq \tilde{\gamma}(\delta) \parallel \phi_1 - \phi_2 \parallel_{\mathcal{V}} .
\end{aligned} \tag{6.7}$$

By virtue of the hypothesis $(H4)'$, $\tilde{\gamma}$ is also a positive nonincreasing function on $[0, \infty)$ and $\tilde{\gamma}(\delta) \to 0$ as $\delta \to \infty$ and hence there exists a $\delta_0 > 0$ such that the operator $R(\delta, \mathcal{A})G_g$ is a contraction in $\mathcal{V}$ for all $\delta \geq \delta_0$. Thus for any $\delta \geq \delta_0$, it follows from Banach fixed point theorem that the operator $R(\delta, \mathcal{A})G_g$ has a unique fixed point Φ in $\mathcal{V}$. Hence $\Phi \in \mathcal{V}$ is the unique solution of the stationary HJB equation (1.11) provided $\delta \geq \delta_0$. ■

Remark. Note that since $\Phi \in \mathcal{V}$ and $G_g(\Phi) \in \mathcal{H}$ and Φ satisfies equation (6.1), we have $\mathcal{A}\Phi \in \mathcal{H}$.

Here we present an application of theorem 6.1 to the infinite horizon control problem. Let $\delta \geq \delta_0$ and consider the stationary control problem (1.10) and suppose Φ is the unique solution of equation (1.11) for the given δ. For the next result we need separability of the control space U. In fact, we can take any separable metric space and not necessarily Hilbert space. We prove the following result.

Theorem 6.2. *Suppose h is weakly lower semicontinuous on U and the assumptions of Theorem* 6.1 *hold with $\delta \geq \delta_0$. Let Φ be the unique solution of the stationary HJB equation* (1.11). *Then the stationary control problem has a solution in the sense that there exists a Borel measurable map* (*control law*) *$u^* : H \longrightarrow B_r \subset U$ which is optimal and for each initial state $x \in H$, $\Phi(x)$ coincides with the value function $V(x)$.*

Proof. Since Φ is the solution of (1.11) and $B^*(x)Q^{-1/2} \in \mathcal{L}(H, U)$ we have $|\sqrt{Q}D\Phi| \in \mathcal{H}$ and

$$\| B^* Q^{-1/2}\sqrt{Q}D\Phi \|_U = \| B^* D\Phi \|_U \in \mathcal{H}.$$

Hence $B^*D\Phi$ is a μ-measurable U-valued function on H. Recall that the function $q \longrightarrow H(x, q)$ is continuous (even Lipschitz) on U μ-a.e. $x \in H$. Since a continuous function of a measurable function is measurable, the function ψ given by

$$\psi(\cdot) \equiv H(\cdot, B^*(\cdot)D\Phi(\cdot))$$

is measurable and belongs to the class $\mathcal{H} = L_2(H, \mu)$. Define the multi-function $M : (H, \mathcal{B}_H) \longrightarrow 2^{B_r} \subset 2^U$ by

$$M(x) \equiv \{u \in B_r : \psi(x) = L(x, B^*(x)D\Phi(x), u) \equiv (u, B^*(x)D\Phi(x)) + h(u)\}.$$

Since h is weakly lower semicontinuous and B_r is a closed ball in U, and ψ is Measurable, M is a nonempty graph measurable multifunction from $(H, \mathcal{B}_H)$ to $(U, \mathcal{B}_U)$. Then it follows from a well known selection theorem due to Yankov-Von Neumann-Aumann [12, Theorems 2.14, 2.25], that M has a Borel measurable selection $u^* : H \longrightarrow B_r$ so that $\psi(x) = (u^*(x), B^*(x)D\Phi(x)) + h(u^*(x)), \mu$-a.e. $x \in H$. In fact, this is true if h is only bounded Borel measurable. For details on measurable multifunctions and their measurable selections we refer the reader to the excellent book by Hu and Papageorgiou [13]. Take any admissible control $u \in \mathcal{U}_{ad}^c$ and let $X(t,x)$ denote the solution of the SDE,

$$\begin{aligned} dX &= (AX + F(X))dt + B(X)u(X)dt + \sqrt{Q}d\hat{W}, \\ X(0,x) &= x, \end{aligned} \tag{6.9}$$

corresponding to this control law. Since under the given assumptions $B(x)u(x)$ is a bounded Borel measurable function with values in H, the solution exists in the martingale sense on an extended probability space (Skorohod extension) where a cylindrical Brownian motion $\hat{W}$ is defined. This is justified by the technique of substitution of drifts via Girsanov formula. For details see [6, 3]. By virtue of the remark following Theorem 6.1, $\mathcal{A}\Phi \in \mathcal{H}$, and hence we can apply Ito formula to the process

$$\Psi(t, X(t,x)) \equiv e^{-\delta t}\Phi(X(t,x)), t \geq 0,$$

for a fixed $\delta \geq \delta_0$ and obtain

$$\begin{aligned} &e^{-\delta t}E\Phi(X(t,x)) \\ &= \Phi(x) + E\int_0^t e^{-\delta s}\Big\{-\delta\Phi + \mathcal{A}\Phi + \mathcal{F}_1(\Phi) + (D\Phi, B(X)u(X))\Big\}ds \\ &= \Phi(x) + E\int_0^t e^{-\delta s}\Big\{-\delta\Phi + \mathcal{A}\Phi + \mathcal{F}_1(\Phi) + (D\Phi, B(X)u^*(X)) \\ &\quad + h(u^*(X)) + g(X)\Big\}ds \\ &\quad + E\int_0^t e^{-\delta s}\Big\{(D\Phi, B(X)(u - u^*)) - g(X) - h(u^*(X))\Big\}ds. \end{aligned}$$

Clearly, it follows from the definition of the operator $\mathcal{F}_2$ as given in (1.6), that

$$(D\varphi(x), B(x)u^*(x)) + h(u^*(x)) = \mathcal{F}_2(\varphi)(x).$$

Thus letting $\mathcal{L}$ denote the operator

$$\mathcal{L}\varphi \equiv -\delta\varphi + \mathcal{A}\varphi + \mathcal{F}_1(\varphi) + \mathcal{F}_2(\varphi) + g,$$

we can rewrite the above expression as

$$
\begin{aligned}
&e^{-\delta t}E\Phi(X(t,x)) = \Phi(x) + E\int_0^t e^{-\delta s}\mathcal{L}\Phi(X(s))ds \\
&\quad + E\int_0^t e^{-\delta s}\Big\{(D\Phi, B(X)(u-u^*)) \\
&\quad + h(u(X)) - h(u^*(X)) - h(u(X)) - g(X)\Big\}ds \\
&= \Phi(x) + E\int_0^t e^{-\delta s}\Big\{L(X, B^*(X)D\Phi(X), u(X)) \\
&\quad - L(X, B^*(X)D\Phi(X), u^*(X))\Big\}ds - E\int_0^t e^{\delta s}\{g(X) + h(u(X))\}ds.
\end{aligned}
$$

The last line follows from the fact Φ satisfies the stationary HJB equation equivalent to (6.1) and hence $\mathcal{L}\Phi(x) = 0$ μ-a.e. Now letting $t \to \infty$ it follows from the above expression that

$$
\begin{aligned}
J(u,x) &= \Phi(x) + E\int_0^\infty e^{-\delta t}\Big\{L(X, B^*(X)D\Phi(X), u(X)) \\
&\quad - L(X, B^*(X)D\Phi(X), u^*(X))\Big\}dt \\
&= \Phi(x) + E\int_0^\infty e^{-\delta t}\Big\{L(X, B^*(X)D\Phi(X), u(X)) \\
&\quad - H(X, B^*(X)D\Phi(X))\Big\}dt.
\end{aligned}
$$

Taking infimum over the class $\mathcal{U}_{ad}^c$ and recalling that $u^* \in \mathcal{U}_{ad}^c$, it follows from this that

$$
V(x) \equiv \inf\{J(u,x), u \in \mathcal{U}_{ad}^c\} = \Phi(x).
$$

Hence the solution Φ of the stationary HJB equation is the optimal cost. ■

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