OPTIMIZATION PROBLEMS FOR STRUCTURAL ACOUSTIC MODELS WITH THERMOELASTICITY AND SMART MATERIALS

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Abstract

Optimization problem for a structural acoustic model with controls governed by *unbounded* operators on the state space is considered. This type of controls arises naturally in the context of "smart material technology". The main result of the paper provides an optimal synthesis and solvability of associated nonstandard Riccati equations. It is shown that in spite of the unboundedness of control operators, the resulting gain operators (feedbacks) are *bounded* on the state space. This allows to provide full solvability of the associated Riccati equations. The proof of the main result is based on exploiting propagation of analyticity from the structural component of the model into an acoustic medium.

Keywords: structural acoustic model with thermal effects, optimal control problem, smart controls, nonstandard Riccati equations, analyticity of semigroups.

1991 Mathematics Subject Classification: 35L70, 93D15, 35B40.

1. Introduction

1..1 PDE control system

We consider structural acoustic model consisting of an acoustic chamber with a combination of flexible (vibrating) and hard walls. Thermal effects

are accounted for in modeling flexible walls. The main goal of this paper is to formulate and to solve an optimization problem aiming at the reduction of a noise and acoustic pressure in an acoustic chamber. This will be accomplished by an appropriate use of smart materials technology in the form of smart actuators and smart sensors.

The PDE model describing the structure is given by a system of equations consisting of wave equation coupled with thermal plate equation [36, 9]. At the interface of the two regions the coupling between the wave and the plate occurs. The control action will be implemented via smart materials technology involving piezo-ceramic patches bounded to the flexible wall and responding to the change of the voltage actuating the process.

Let $\Omega \in R^n$, $n = 2, 3$, be an open bounded domain with boundary $\Gamma \subset R^{n-1}$ which is assumed either sufficiently smooth or convex. The boundary $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where Γ_0, Γ_1 are simply connected regions.

The pressure in the chamber (acoustic medium) is defined on a spatial domain Ω while the displacement of the flexible walls is defined on a flat segment ($n = 2$) or flat surface ($n = 3$) Γ_0. Γ_1 represents a "hard" wall. We assume that some frictional forces are active on the flexible wall Γ_0. These frictional forces will be described by "damping" operator $D : \mathcal{D}(D) \subseteq L_2(\Gamma_0) \to L_2(\Gamma_0)$; which is assumed closed, densely defined and positive, ie: $(Dz, z)_{L_2(\Gamma_0)} \geq 0$; $z \in \mathcal{D}(D)$.

In addition to classical notation used for Sobolev's spaces we shall use the following $H^1_\gamma(\Gamma_0) = H^1(\Gamma_0), \gamma > 0$; $L_2(\Gamma_0)$; $\gamma = 0$ with the inner product

(1.1)
$$(\omega_1, \omega_2)_{H^1_\gamma(\Gamma_0)} \equiv (\omega_1, \omega_2)_{L^2(\Gamma_0)} + \gamma(\nabla \omega_1, \nabla \omega_2)_{L^2(\Gamma_0)}$$

$$\forall \omega_1, \omega_2 \in H^1_\gamma(\Gamma_0).$$

The PDE model considered consists of the wave equation in the variable z (where the quantity ρz_t is the acoustic pressure, and ρ is the density of the fluid)

(1.2)
$$z_{tt} = c^2 \Delta z + f \quad \text{in} \quad \Omega \times (0, \infty),$$

$$\alpha \frac{\partial}{\partial \nu} z + z = 0; \ \alpha \geq 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty);$$

$$\frac{\partial}{\partial \nu} z + Dz_t = w_t \quad \text{on} \quad \Gamma_0 \times (0, \infty),$$

$$z(0) = z_0 \in H^1_{\Gamma_1}(\Omega), \qquad z_t(0) = z_1 \in L^2(\Omega);$$

where $H^1_{\Gamma_1}(\Omega) \equiv \{z \in H^1(\Omega); z = 0 \text{ on } \Gamma_1 \text{ if } \alpha = 0\}$ and the elastic equation representing the displacement of the wall w subject to thermal effects (see, e.g., [20]):

$$(1.3) \quad \left. \begin{array}{l} w_{tt} - \gamma \Delta w_{tt} + \Delta^2 w = -\Delta \theta - \rho z_t + \mathcal{B}u \\ \theta_t - \Delta \theta = \Delta w_t \end{array} \right\} \quad \text{on} \quad \Gamma_0 \times (0, \infty)$$

$$w(0) = w_0 \in H^2(\Gamma_0), \ w_t(0) = w_1 \in H^1_\gamma(\Gamma_0); \quad \theta(0) = \theta_0 \in L_2(\Gamma_0).$$

Here θ is the temperature, c^2 is the speed of sound as usual and ρz_t represents back pressure acting upon the wall. The operator $\mathcal{B}$ is a control operator acting upon a control function u and will be described below. The function f represents an unwanted noise entering the acoustic medium.

The vector ν (resp $\tilde{\nu}$) denotes the unit normal vector to the boundary Γ (resp. $\partial \Gamma_0$) and τ (resp, $\tilde{\tau}$) will denote the unit tangential direction to Γ (resp. $\partial \Gamma_0$). The constant γ accounts for rotational forces and here is taken to be small (proportional to the thickness of the plate) and non-negative. With the plate equation (1.3) we associate typical boundary conditions associated with plate models: clamped, hinged or free. These are defined below.

Clamped boundary conditions:

$$w = \frac{\partial}{\partial \nu} w = \theta = 0; \quad \text{on} \quad \partial \Gamma_0 \times (0, \infty).$$

Hinged boundary conditions:

$$w = \Delta w = \theta = 0; \quad \text{on} \quad \partial \Gamma_0 \times (0, \infty).$$

Free boundary conditions:

$$(1.4) \quad \left. \begin{array}{l} \Delta w + B_1 w + \theta = 0 \\ \dfrac{\partial}{\partial \tilde{\nu}} \Delta w + B_2 w - \gamma \dfrac{\partial}{\partial \tilde{\nu}} w_{tt} + \dfrac{\partial}{\partial \tilde{\nu}} \theta = 0 \\ \dfrac{\partial}{\partial \tilde{\nu}} \theta + \lambda \theta = 0; \ \lambda > 0 \end{array} \right\} \quad \text{on} \ \partial \Gamma_0 \times (0, \infty).$$

The constant λ is positive. The boundary operators B_1, B_2 are given by:

$$B_1 \equiv (1 - \mu) \left(2\nu_1 \nu_2 D^2_{x,y} - \nu_1^2 D^2_{y,y} - \nu_2^2 D^2_{x,x} \right); \quad [\nu_1, \nu_2] = \tilde{\nu},$$

$$B_2 \equiv (1 - \mu) \left\{ \frac{\partial}{\partial \tilde{\tau}} \left[(\nu_1^2 - \nu_2^2) D^2_{x,y} + \nu_1 \nu_2 (D^2_{y,y} - D^2_{x,x}) \right] + lw \right\}, \quad l > 0.$$

The constant μ denotes Poisson's modulus with the values between 0 and $1/2$ (depending on the characteristics of the material used).

It should be noted that the presence of the parameter γ in equation (1.3) changes the character of dynamics. Indeed, the "uncoupled" thermoelastic plate is of hyperbolic type, when $\gamma > 0$ and of analytic type, when $\gamma = 0$ [35, 29, 28, 30].

Choice for the control operator $\mathcal{B}$

A common choice for the control operator in structural acoustic models is the following

$$(1.5) \qquad \mathcal{B}u = \sum_{i=1}^{k} \alpha_i u_i \delta'_{\xi_i},$$

where ξ_i are either points (dim $\Gamma_0 = 1$), or closed, regular curves (dim $\Gamma_0 = 2$) in Γ_0 and the corresponding α_i are either constants (dim $\Gamma_0 = 1$), or smooth functions (dim $\Gamma_0 = 2$). The symbol δ'_{ξ_i} denotes the derivative of the delta distribution supported on ξ_i (when dim $\Gamma_0 = 2$, we take normal derivatives to the curves ξ_i). This choice of the boundary operator is typical in smart materials technology, where control action is implemented via piezo-ceramic patches bonded to the wall (see [7, 4]). The voltage applied to the patch creates a bending moment which, in turn, causes the bending of the wall. This has an effect on reducing vibrations.

We let $\mathcal{U}$ denote the space of controls u. The control space $\mathcal{U}$ is taken to be

$$(1.6) \qquad \mathcal{U} \equiv R^k.$$

We note that the control operator $\mathcal{B}$ is not defined on $L_2(\Gamma_0)$. The values of $\mathcal{B}u$ are in a larger (dual) space to be defined later.

Optimal control problem studied in this paper is formulated below:

Optimal Control Problem: Given $f \in L_2(0, T; L_2(\Omega))$, minimize

$$(1.7) \qquad \begin{aligned} J(u, z) &\equiv \int_0^T \left[\int_\Omega \left[|\nabla z(t)|^2 + |z_t(t)|^2 \right] dx \right. \\ &\left. + \int_{\Gamma_0} \left[|\Delta w(t)|^2 + |w_t(t)|^2 + \gamma |\nabla w_t(t)|^2 \right] dx + |u(t)|_{\mathcal{U}}^2 \right] dt, \end{aligned}$$

where z, w satisfy (1.2) and (1.3) subject to either clamped, hinged or free boundary conditions.

Problems related to optimization of structural acoustic models have attracted considerable attention in the literature [36, 13, 33, 14, 5, 6, 15]. However, the majority of the related work is in the context of engineering

or computational applications. Very little has been done from the point of view of the mathematical theory underlying the problem. In fact, a very first work providing a rigorous mathematical analysis of optimal synthesis and associated Riccati equations is [1]. However, the model treated in [1] (see also review paper [21]) involves structural damping (Kelvin-Voight type) on the wall. On the other hand, it is well known and acknowledged in the engineering literature, that structural damping is rare and its modeling is poorly understood. Moreover, structural dampers placed on the active walls may produce local "overdamping" effect undermining the effectiveness of active controllers. For these reasons, it is desirable to consider other models which do not exhibit structural damping. The contribution of this paper is to address the problem by using a model that includes thermal effects in the wall and dispenses all-together with the need for any type of damping affecting the walls. The interest in studying this problem, beside strong physical motivations, is that mathematical analysis of these models is much more subtle. Indeed, the lack of strongly regularizing effect due to structural damping, when combined with the intrinsic unboundedness of control operator $\mathcal{B}$, makes the arguments existing in the present literature non applicable. New mathematical tools/ideas need to be brought into play.

1..2 Semigroup model

In order to formulate our main results, we find convenient to introduce semigroup framework describing the dynamics considered in (1.2), (1.3). To accomplish this, we introduce the following spaces and operators:

$A_N : L_2(\Omega) \to L_2(\Omega),\ A_N \equiv -c^2 \Delta z,$
$\mathcal{D}(A_N) = \{z \in H^1(\Omega) :\ \Delta z \in L_2(\Omega);\ \alpha \frac{\partial z}{\partial \nu} + z = 0 \text{ on } \Gamma_1,\ \frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma_0\},$

and the operator $N : L_2(\Gamma_0) \to L_2(\Omega)$ defined by

$$\Delta N g = 0 \ in \ \Omega,\ \ \alpha \frac{\partial}{\partial \nu} N g + N g = 0 \text{ on } \Gamma_1;\ \ \frac{\partial}{\partial \nu} N g = g \text{ on } \Gamma_0.$$

It is well known ([34]) that $N \in \mathcal{L}(L_2(\Gamma_0);\ H^{3/2}(\Omega))$ and, moreover, by Green's Formula the operator $N^* A_N$ coincides with the trace operator ie:

(1.8) $$N^* A_N z \equiv c^2 z|_{\Gamma_0},\ z \in H^1_{\Gamma_1}(\Omega).$$

In order to provide an abstract framework for the plate equation we introduce the following elliptic operators:

$$\mathcal{A}_D;\ \mathcal{A}_N;\ \mathcal{A}_R\ : L_2(\Gamma_0) \to L_2(\Gamma_0)$$

given by

$$\mathcal{A}_D w \equiv -\Delta w, \quad \mathcal{D}(\mathcal{A}_D) = H^2(\Gamma_0) \cap H_0^2(\Gamma_0),$$

$$(1.9) \quad \mathcal{A}_N w \equiv -\Delta w, \quad \mathcal{D}(\mathcal{A}_N) \equiv \left\{ w \in H^2(\Gamma_0), \frac{\partial}{\partial \nu} w = 0 \text{ on } \partial \Gamma_0 \right\},$$

$$\mathcal{A}_R w \equiv -\Delta w, \quad \mathcal{D}(\mathcal{A}_R) \equiv \left\{ w \in H^2(\Gamma_0), \frac{\partial}{\partial \nu} w + \lambda w = 0 \text{ on } \partial \Gamma_0 \right\}.$$

Depending on the type of boundary conditions associated with each plate model, we introduce various elliptic operators describing the static part of the plate equation.

Clamped Plates/Beams

$$(1.10) \quad \begin{aligned} &\mathcal{A} : L_2(\Gamma_0) \to L_2(\Gamma_0); \quad \mathcal{A} \equiv \Delta^2; D(\mathcal{A}) = H_0^2(\Gamma_0); \\ &\mathcal{A}_0 \equiv \mathcal{A}_D; \quad \mathcal{M}_\gamma \equiv I + \gamma \mathcal{A}_D. \end{aligned}$$

Hinged Plates/Beams

$$(1.11) \quad \begin{aligned} &\mathcal{A} : L_2(\Gamma_0) \to L_2(\Gamma_0); \quad \mathcal{A} \equiv \mathcal{A}_D^2; \\ &\mathcal{A}_0 \equiv \mathcal{A}_D; \quad \mathcal{M}_\gamma \equiv I + \gamma \mathcal{A}_D. \end{aligned}$$

Free Plates/Beams

$$(1.12) \quad \begin{aligned} &\mathcal{A} : L_2(\Gamma_0) \to L_2(\Gamma_0); \quad \mathcal{A} \equiv \Delta^2; \\ &\mathcal{D}(\mathcal{A}) = \{ v \in H^4(\Gamma_0); \Delta v + B_1 v = 0, \frac{\partial}{\partial \nu} \Delta v + B_2 v = 0 \text{ on } \partial \Gamma_0 \}; \\ &\mathcal{G}_i : L_2(\partial \Gamma_0) \to L_2(\Gamma_0); \, i = 1, 2. \ \mathcal{G}_i g \equiv v_i, \text{ where } \Delta^2 v_i = 0 \text{ in } \Gamma_0; \\ &\text{and } \ \Delta v_1 + B_1 v_1 = g; \ \frac{\partial}{\partial \tilde{\nu}} \Delta v_1 + B_1 v_1 = 0 \text{ on } \partial \Gamma_0 \\ &\Delta v_2 + B_1 v_2 = 0; \ \frac{\partial}{\partial \tilde{\nu}} \Delta v_2 + B_2 v_2 = g \text{ on } \partial \Gamma_0 \\ &\mathcal{A}_0 \equiv \mathcal{A}_R; \quad \mathcal{M}_\gamma \equiv I + \gamma \mathcal{A}_N. \end{aligned}$$

With the above notation, we can rewrite equations (1.2), (1.3) in the following abstract form:

$$(1.13) \quad \begin{cases} z_{tt} + A_N z - A_N N w_t + A_N N D N^* A_N z_t = f \text{ on } \mathcal{D}(A_N)' \\ \mathcal{M}_\gamma w_{tt} + \mathcal{A}w - \mathcal{A}_0 \theta + \rho N^* A_N z_t = \mathcal{B}u \text{ on } \mathcal{D}(\mathcal{A})' \\ \theta_t + \mathcal{A}_0 \theta = -\mathcal{A}_D w_t \text{ on } \mathcal{D}(\mathcal{A}_0)' \end{cases}$$

for the *hinged* or *clamped* boundary conditions and

$$\begin{cases} z_{tt} + A_N z - A_N N w_t + A_N N D N^* A_N z_t = f \text{ on } \mathcal{D}(A_N)' \\ \mathcal{M}_\gamma w_{tt} + \mathcal{A}[w + \mathcal{G}_1 \theta|_{\partial \Gamma_0} + \lambda \mathcal{G}_2 \theta|_{\partial \Gamma_0}] - \mathcal{A}_0 \theta + \rho N^* A_N z_t = \mathcal{B}u \text{ on } \mathcal{D}(\mathcal{A})' \\ \theta_t + \mathcal{A}_0 \theta = \Delta w_t \text{ on } \mathcal{D}(\mathcal{A}_0)' \end{cases}$$

(1.14)

for the *free* boundary conditions [23]. We note, that in the case of free boundary conditions, the coupling between thermal and mechanical variables occurs also on the boundary.

We define next the following spaces

(1.15)
$$H_z \equiv H^1_{\Gamma_1}(\Omega) \times L_2(\Omega); \quad H_v \equiv D(\mathcal{A}^{1/2}) \times D(\mathcal{M}_\gamma^{1/2}) \times L_2(\Gamma_0);$$
$$H \equiv H_z \times H_v$$

and (unbounded) operator: $C : H_z \to H_v$:

$$C \begin{pmatrix} z \\ z_t \end{pmatrix} \equiv \begin{pmatrix} 0 \\ c^{-2} \rho \mathcal{M}_\gamma^{-1} N^* A_N z_t \\ 0 \end{pmatrix}.$$

The domain of the operator C is defined maximally ie

$$D(C) = \{(z, z_t) \in H_z; N^* A_N z_t \in [D(\mathcal{M}_\gamma^{1/2})]'\}.$$

Clearly, $H^1(\Omega) \times H^1(\Omega) \subset D(C)$. Thus, C is densely defined. Note however that the operator C, being a trace operator, is both unbounded and unclosable on the state space H_z. Define next:

$$A_z : H_z \to H_z; \quad A_v : H_v \to H_v; \quad A_z \equiv \begin{pmatrix} 0 & -I \\ A_N & -c^{-2} A_N N D N^* A_N \end{pmatrix};$$

$$A_v \equiv \begin{pmatrix} 0 & -I & 0 \\ \mathcal{M}_\gamma^{-1} \mathcal{A} & 0 & -\mathcal{M}_\gamma^{-1} \mathcal{A}_0 \\ 0 & -\mathcal{A}_0 & -\mathcal{A}_0 \end{pmatrix},$$

for the *clamped* or *hinged* boundary conditions, and

$$A_v \equiv \begin{pmatrix} 0 & -I & 0 \\ \mathcal{M}_\gamma^{-1} \mathcal{A} & 0 & -\mathcal{M}_\gamma^{-1}[\mathcal{A}_0 + \mathcal{A}\mathcal{G}_1 \theta|_{\partial \Gamma_0} + \lambda \mathcal{A}\mathcal{G}_2 \theta|_{\partial \Gamma_0}] \\ 0 & -\mathcal{A}_0 & \Delta \end{pmatrix}$$

for the *free* case.

We note that $-A_z$ corresponds to the generator of the wave equation with absorbing boundary conditions, while $-A_v$ corresponds to the generator of thermoelastic plate/beam or shell model accounting for moments of inertia [23]. It can be easily verified [22] that both generators are disspative on the respective topologies of H_z and H_v (we topologize $H^1(\Omega)$ by the graph norm of $A_N^{1/2}$). Moreover, if $\gamma = 0, -A_v$ generates generates an analytic semigroup on H_v [35, 29, 28].

With the above notations, the "generator" $A : H \to H$, describing the entire structure is given by:

$$(1.16) \qquad\qquad -A \equiv \begin{pmatrix} A_z & -c^2\rho^{-1}C^* \\ C & A_v \end{pmatrix}$$

where the adjoint of C is defined via duality:

$$\left(C\begin{pmatrix} z \\ z_t \end{pmatrix}, \begin{pmatrix} w \\ w_t \\ \theta \end{pmatrix}\right)_{H_v} = \left(\begin{pmatrix} z \\ z_t \end{pmatrix}, C^*\begin{pmatrix} w \\ w_t \\ \theta \end{pmatrix}\right)_{H_z}$$

for $\begin{pmatrix} z \\ z_t \end{pmatrix} \in D(C)$, $\mathbf{v} \equiv \begin{pmatrix} w \\ w_t \\ \theta \end{pmatrix} \in H_v$. Specifically:

$$C^*\begin{pmatrix} w \\ w_t \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ c^{-2}\rho A_N N w_t \end{pmatrix}.$$

The domain of A is defined maximally, i.e.

$$(1.17) \qquad D(A) = \Big\{ (z_1, z_2, v_1, v_2, v_3) \in H; \ z_2 \in H^1(\Omega); \ v_2 \in D(\mathcal{A}^{1/2})$$

$$z_1 - c^{-2}NDN^*A_N z_2 - Nv_2 \in D(A_N); \quad \mathcal{A}v_1 \in D(\mathcal{M}_\gamma^{1/2})', v_3 \in \mathcal{D}(\mathcal{A}_0)\Big\}$$

for the clamped or hinged boundary conditions and with analogous definition for the free case. Now we can rewrite the original dynamics as a first order system:

$$(1.18) \quad \frac{d}{dt}\begin{pmatrix} z \\ z_t \\ w \\ w_t \\ \theta \end{pmatrix} - A\begin{pmatrix} z \\ z_t \\ w \\ w_t \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathcal{M}_\gamma^{-1}\mathcal{B}u \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ f \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv Bu + F.$$

It can be easily verified that both A and A^* are dissipative (after rescaling the inner product on H_v to account for the factor $c^2\rho$), hence by standard semigroup theory [37, 3], A is the generator of a contraction semigroup on H. In what follows, we shall use the adjoint of B defined via duality as:

$$(B^*x, u)_{\mathcal{U}} = (x, Bu)_H \text{ for all } x \in D(B^*)$$
$$\equiv \{x \in H : (x, Bu)_H \leq \infty \text{ for all } u \in \mathcal{U}\},$$

which formula when applied with $x = (z_1, z_2, v_1, v_2, v_3)$ is equivalent to:

$$(B^*x, u)_{\mathcal{U}} = (v_2, \mathcal{B}u)_{L_2(\Gamma_0)}; \ x \in D(B^*)$$
$$\equiv \{x \in H : (v_2, \mathcal{B}u)_H \leq \infty \text{ for all } u \in \mathcal{U}\}.$$

For the control operator given by (1.5) we obtain

$$B^*(z, z_t, w, w_t, \theta) = \alpha_i w_t(\xi_i), \ i = 1, \ldots, k.$$

The dynamics described by (1.18) is an abstract semigroup formulation of the structural acoustic model given by (1.2), (1.3) with appropriate boundary conditions. We note, that while for $\gamma = 0$, $-A_v$ generates an analytic semigroup, this is not the case for the "big" operator A. The operator A has a substantial hyperbolic component. In addition, the operator B is intrinsically unbounded. These two features contribute to main difficulties in the analysis of the problem.

1..3   Formulation of the results

We recall $H \equiv H_z \times H_v$ where H_z, H_v are given by (1.15) We introduce the state variable: $\mathbf{y} \equiv [z, z_t, v] = [z, z_t, w, w_t, \theta]$.

Let R be a bounded operator from $H \to Z$, where Z is another Hilbert space (space of observations).

The following control problem will be studied in the context of the dynamics given by (1.18).

CONTROL PROBLEM: Minimize the functional

$$(1.19) \qquad J(u, \mathbf{y}) \equiv \int_0^T \left[|R\mathbf{y}|_Z^2 + |u|_U^2\right] dt$$

for all $u \in L_2(0, T; U)$ and $\mathbf{y} \in L_2(0, T; H)$ which satisfy (1.18) or equivalently equations (1.2), (1.3) which can be written as

$$(1.20) \qquad \mathbf{y}_t - A\mathbf{y} = Bu + F; \ \ \mathbf{y}(0) \in H.$$

The operator $A : H \to H$, given by (1.16) was shown to generate a C_0 semi-group on H, and the operator $B : U \to [D(A^*)]'$, $Bu = (0, 0, 0, \mathcal{M}_\gamma{}^{-1}\mathcal{B}u, 0)$ represents the *unbounded* control operator. The forcing term $F \in L_2(0, T; H)$ describes the effect of a noise. In what follows we shall identify $\mathcal{U} = U$.

The control problem stated above is a more general version of the control problem (1.7) formulated in Section 1.1.

If time T is finite ("finite horizon control problem"), the existence and uniqueness of optimal pair $u^0, \mathbf{y}^0$ follows from standard arguments in calculus of variations [10, 3]. Also, in the case when $T = \infty$, the same conclusion follows provided that the Finite Cost Condition is satisfied [26]. This, in turn, requires that Γ_1 satisfies certain geometric conditions to be specified later.

The main goal of this paper is to provide feedback representation of the optimal control which, as is well known, involves a solution of an appropriate Riccati equation.

At this point we notice that **if the control operator B were bounded from U to H**, then the problem of the optimal synthesis would be a straightforward consequence of the existing optimal control theory established for C_0 semigroups with bounded control actions (see [3, 10]). This is not the case for **unbounded** control operators. Indeed, the issue of solvability of Riccati equations and the meaning of the feedback **gain operator**, B^*P which is expressed as a product of **unbounded** B^* and the Riccati operator P are more delicate. The difficulty stems from the fact that the control operator B is highly unbounded on the state space and, therefore, the gain operator may not be even properly defined on a dense subset of the state space (see counterexamples for hyperbolic systems given in [47, 44] and references therein). This, however, can not happen in the case of analytic semigroups where the theory is much richer and it provides not only for the meaning to the gain operator but it also shows that the gain operator B^*P is in fact **bounded** (see [31, 10, 26]). This last phenomenon is due to regularizing effect of analyticity, which is then inherited by the optimal solution which is more regular (in fact analytic) than optimization predicts. Unfortunately, in our case, the system is *not analytic* and only some components of the structure may be analytic.

Thus, the novelty and interest of the present problem lies in the fact that the control operator is *intrinsicly unbounded* (not even "admissible"-in the terminology of [39]) and the overall dynamics *is not analytic*. As we shall see, our results will depend on the fact whether rotational moments are accounted for in the plate model (ie $\gamma > 0$), or not. In fact, in the case

$\gamma = 0$, the thermoelastic plate is connected with an analytic semigroup, therefore more regularity of the optimal synthesis is expected. On the other hand, if $\gamma > 0$, the entire system has strongly hyperbolic effect which, in turn, has ome beneficial effects on propagations of singularities caused by the unbounded control operator. Thus, the reasons and the arguments leading to solvability of optimal synthesis are very different in these two cases: $\gamma = 0$ and $\gamma > 0$.

In order to state our results, we formulate the following Assumptions.

Assumption 1.
- $\mathcal{A}^{-r}\mathcal{B} \in \mathcal{L}(\mathcal{U}; L_2(\Gamma_0))$, for some $r < 1/2$.
- There exist positive constants δ, M such that
  $\delta|z|^2_{\mathcal{D}(\mathcal{A}^{r_0})} \leq (Dz, z)_{L_2(\Gamma_0)} \leq M|z|^2_{\mathcal{D}(\mathcal{A}^{r_0})}$; where for $r \geq 1/4$, $r - 1/4 < r_0 \leq 1/4$, and $r_0 = 0$ for $r < 1/4$.

Assumption 2. $(x - x_0) \cdot \nu \leq 0$, $x_0 \in R^n$, $x \in \Gamma_1$. Moreover Γ_1 is assumed convex if $\alpha > 0$ in model (1.2).

Remark 1.1.
- It is easy to verify that the operator $\mathcal{B}$ in (1.5) satisfies the first part of Assumption 1. Thus, the first part of Assumption 1 is empty in the case of control operator represented by derivatives of delta functions. Indeed, by Sobolev's embeddings $H^{1/2+\epsilon}(\Gamma_0) \subset C(\Gamma_0)$ if dim $\Gamma_0 = 1$, and $H^{1/2+\epsilon}(\Gamma_0) \subset C(\xi_i)$ if dim $\Gamma_0 = 2$. Thus $\delta'_{\xi_i} \in [H^{3/2+\epsilon}(\Gamma_0)]'$ in both cases. Since, by [18], $D(\mathcal{A}^\theta) \subset H^{4\theta}(\Gamma_0)$, we have $[H^{3/2+\epsilon}(\Gamma_0)]' \subset [D(\mathcal{A}^{3/8+\frac{1}{4}\epsilon})]'$, we thus conclude via (1.5) that Assumption 1 is satisfied with $r = 3/8 + \frac{1}{4}\epsilon < 1/2$, for some small ϵ.
- Regarding the second part of the Assumption 1, we note that $\mathcal{D}(\mathcal{A}^{r_0})$ is topologically equivalent subject to appropriate boundary conditions to $H^{4r_0}(\Gamma_0) \subseteq H^1(\Gamma_0)$. Thus, for $r < 1/4$, the damping operator D may ba taken as the identity operator. For $r \geq 1/4$, one needs a stronger damping, whith D unbounded. Typical realization of such operator is via Laplace Beltrami operator defined on Γ_0. In order to comply with the boundary conditions imposed by $\mathcal{A}^{r_0}$ (relevant only if $r > 3/8$), we may take $\alpha \geq 0$, for the case of free boundary conditions imposed on the plate and $\alpha = 0$, for the case of clamped or hinged boundary conditions.

Theorem 1.1 ($\gamma = 0$, $T < \infty$)**.** *Consider the control problem governed by the dynamics described in* (1.18)*, with $\gamma = 0$, $T < \infty$, and the functional cost*

given in (1.19). The control operator $\mathcal{B}$ is subject to Assumption 1 and we assume that $F \in L_2(0, T; H)$. Then, for any initial condition $\mathbf{y}_0 \in H$, there exists an unique optimal pair $(u^0, \mathbf{y}^0) \in L_2(0, T; U \times H)$ with the following properties:

(i) [**regularity of the optimal pair**]

$$u^0 \in C(0, T; U); \ \mathbf{y}^0 \in C(0, T; H).$$

(ii) [**regularity of the gains and optimal synthesis**] *There exist a self-adjoint positive operator $P(t) \in \mathcal{L}(H)$ with the property*

$$B^* P(.) \in \mathcal{L}(H \to C(0, T; U))$$

and an element $r \in C(0, T; H)$ (depending on F) with the property

$$B^* r \in L_2(0, T; U)$$

such that:
$$u^0(t) = -B^* P(t) \mathbf{y}^0(t) - B^* r(t); \ t \geq 0.$$

If, in addition, $F \in C([0, T]; H)$, then

$$B^* r \in C([0, T]; U).$$

(iii) [**feedback semigroup**] *The operator $A_{P(t)} \equiv A - BB^* P(t) : H \to D(A^*)'$ generates a strongly continuous semigroup on H.*

(iv) [**Riccati equation**] *The operator $P(t)$ is a unique (within the class of selfadjoint positive operators subject to the regularity in part (ii)) solution of the following operator Differential Riccati Equation:*

$$
(1.21) \quad
\begin{aligned}
(P_t(t)x, y)_H = \ & (A^* P(t)x, y)_H + (P(t)Ax, y)_H + (R^* Rx, y)_H \\
& - (B^* P(t)x, B^* P(t)y)_U \ for \ x, y \in D(A).
\end{aligned}
$$

(v) [**equation for "r"**] *The operator $A_P(t) \equiv A - BB^* P(t)$ generates a strongly continuous evolution on H and the element $r(t)$ satisfies the differential equation*

$$
(1.22) \quad
\begin{aligned}
& r_t(t) = -A_P^*(t)r(t) - P(t)F; \ on \ [D(A)]' \\
& with \ the \ terminal \ condition \ \ r(T) = 0.
\end{aligned}
$$

Remark 1.2. We note that the results of parts (i), (ii) Theorem 1.1 provide more regularity properties of the optimal solution than optimization alone predicts. Indeed, the synthesis is pointwisely defined with all the gains represented by bounded operators. This is not the usual property of solutions to control problems with unbounded control actions (see [26]). On the other hand, these regularity results are necessary in order to give the meaning to the quadratic term in Riccati equation (1.21).

Remark 1.3. It is interesting to note the role played by the damping operator D. As seen in Assumption 1, the strength of this damping on the wave component offsets some unboundedness of the control operator $\mathcal{B}$ acting on the plate component (note taht for $r \geq 1/4$, B is not admissible). In contrast, for structurally damped plates, there is no need for the additional coercivity of the damping operator D affecting the wave [1, 21].

Remark 1.4. The statement of Theorem 1.1 provides a full pointwise feedback synthesis for the optimal control. It also gives an important regularity result for the Riccati operator. Indeed, typically, one can not expect to obtain bounded gains with unbounded control operator B [17, 10, 26, 27]. In our case, this additional regularity properties of the gains results from the presence of thermal damping on the wall. The regularizing effect of parabolicity present in the wall model is partially propagated onto the entire structure. An interesting aspect of the problem is that the overall coupled model is not *analytic* (in which case the boundedness of gains is well understood: [26, 10, 16, 27]. The "hidden"regularity of B^*P allows us to show that the external noise gain operator is also pointwisely defined. This, in turn, provides a meaningful optimal synthesis.

Our next result deals with the infinite horizon problem in the purely hyperbolic case which results from the presence of rotational inertia in the model.

Theorem 1.2 $(\gamma > 0, T = \infty)$**.** *Consider the control problem governed by the dynamics described in (1.18), with $\gamma > 0$, along with the functional cost given in (1.19) and $T = \infty$. We assume that the Assumption 2 is in force, $F \in L_2(0, \infty; H)$, the control operator $\mathcal{B}$ is of the form as in (1.5), and the damping operator D is bounded and coercive on $L_2(\Gamma_0)$.*

Then, for any initial condition $\mathbf{y}_0 \in H$, there exists an unique optimal pair $(u^0, \mathbf{y}^0) \in L_2(0, \infty; U \times H)$ with the following properties:

(i) [**regularity of the optimal pair**]

$$u^0 \in L_2(0, \infty; U); \mathbf{y}^0 \in C(0, \infty; H).$$

(ii) [**feedback semigroup**] *The operator $A_P \equiv A - BB^*P$ is a generator of a strongly continuous semigroup $e^{A_P t}$ on H, which is, moreover, exponentially stable.*

(iii) [**regularity of the gain and optimal synthesis**] *There exist a self-adjoint non-negative operator $P \in \mathcal{L}(H)$ with the property that*

$$B^*P : D(B^*P) \subset H \to U$$

is densely defined on H (its domain contains $D(A_P)$) and an element $r \in C(0, \infty; H)$ with the property such that $Py^0(t; y_0) + r(t) \in D(B^*)$ a.e in $t > 0$ and:

$$u^0(t; y_0) = -B^*[Py^0(t; y_0) + r(t)]; a.e \text{ in } t > 0; y_0 \in H,$$

(1.23)
$$u^0(t; y_0) = -B^*Py^0(t; y_0) - B^*r(t)$$

$$for \text{ all } t > 0; \ y_0 \in D(A_P), F \in H^1(0, \infty, H)$$

(iv) [**Riccati equation**] *The operator P is a unique (within the class of selfadjoint non-negative operators subject to the regularity in part (iii)) solution of the following operator Algebraic Riccati Equation:*

$$(A^*Px, y)_Y + (PAx, y)_Y + (R^*Rx, y)_Y = (B^*Px, B^*Py,)_U$$

$$for \ x, y \in D(A_P);$$

(1.24)
$$(A^*Px, y)_Y + (PAx, y)_Y + (R^*Rx, y)_Y = (B^*_{ext}Px, B^*_{ext}Py,)_U$$

$$for \ x, y \in D(A);$$

where B^*_{ext} is a suitable extension of B^* ie: $B^*_{ext}x = B^*x; \ x \in \mathcal{D}(B^*)$ [44, 8].

(v) [**equation for r**] *The element $r(t)$ is the unique solution to the differential equation*

(1.25)
$$r_t(t) = -A^*_P r(t) - P(t)F; \ on \ [D(A)]'; \quad \lim_{T \to \infty} r(T) = 0.$$

Remark 1.5.
- The main difference between the results in the parabolic-hyperbolic case given by Theorem 1.1 and these presented in Theorem 1.2 is the nature of the gain operator. Indeed, in the parabolic-hyperbolic case we have that the gain operator is actually a *bounded operator*, while in the hyperbolic-hyperbolic setup this is not the case. In fact, one can show by means of an example that the gain operator is *intrinsically* unbounded. (see also [17, 26, 31] in the context of stabilization). Also, the formulation of the Riccati equation on $D(A)$ requires a special extension of B^*. That this is a necessity follows from a counterexample given in [47].
- A similar comment applies to the regularity of the element $B^*r(t)$. Indeed, while in the case of Theorem 1.1 the optimal synthesis holds pointwisely with each entity B^*P and B^*r well defined on its own, this is not the case in the present, purely hyperbolic, context.

Remark 1.6. Note that the formulation of Theorem 1.2 requires that the control operator be of specific structure, rather than more general form in Assumption 1. This is due to the hyperbolic nature of the problem, and the fact that in order to take advantage of propagation of singularities, more precise information about the "rough" source is necessary.

Remark 1.7. The role of the damping operator D in the "hyperbolic" framework of Theorem 1.2 is very different from that in the "analytic" case of Theorem 1.1. Indeed, while in the "analytic" framework the damping operator had an effect on offsetting some of the unboundedness of control operator $\mathcal{B}$, this is not the case here. In fact, the operator D has no effect whatsoever on the "admissibility "properties of B. For the latter, propagation of singularities is of the essence. Instead, strong coercivity of the operator D is critical in showing Finite Cost Condition - a necessary condition for solvability of the infinite horizon problem. In fact, the boundary damping imposed on the wave part implies the uniform stability for the entire structure. This fact was proved in [24, 32] with the damping D which is assumed bounded and coercive. However, the same conclusion fails for the stronger (unbounded) damping D (ie with $r_0 > 0$). This "overdamping" phenomenon was observed earlier in the context of structurally damped plate equations, where "too unbounded" damping destroys uniform decays of the energy function. For this reason, the infinite horizon version of Theorem 1.1, in the case of "nonadmissible" control operator B (ie; $r \geq 1/4$), is an open problem. This issue is particularly difficult since "point" control

operators, of the type as in (1.5), are known to be "uncontrollable" and "unstabilizable" in the context of purely hyperbolic dynamics [41].

The reminder of this paper is devoted to the proofs of the main theorems. While we shall provide all the main points of the arguments involved, several technical details will be referred to other manuscripts.

2.   Proofs

2..1   Main Lemma and the proof of Theorem 1.1.

The key element in the proofs of the main Theorems is the following Lemma quantifying the singular behavior of the operator $e^{At}B$.

Lemma 2.1. *Under the Assumption 1 with $\gamma = 0$ the following singular estimate is valid*

$$(2.1) \qquad |e^{At}Bu|_H \leq C_T \frac{|u|_U}{t^{2r}}; \quad 2r < 1; \ 0 < t \leq T.$$

Remark 2.1. We note that the singular estimate in the Lemma is typical for analytic semigroups. The novelty in our case is the that the semigroup e^{At} is not analytic.

***Proof of Lemma* 2.1.**   For simplicity of the notation we shall consider clamped or hinged boundary conditions. The case of free boundary conditions is treated similarly with appropriate modifications in the definitions of the underlying operators.

Since $B : U \to \mathcal{D}(A^*)'$ is bounded, the operator $e^{At}B$ is bounded from U into $\mathcal{D}(A^*)'$ for all $t \geq 0$. Our goal is to show that this operator is bounded from $U \to H$, for $t > 0$ with the prescribed singularity at $t = 0$. To this end, we denote $y \equiv e^{At}Bu; u \in U, t > 0$. A priori we have that $y \in \mathcal{D}(A^*)'$ and $y = (z, z_t, w, w_t, \theta)$ satisfies the system

$$z_{tt} = c^2 \Delta z \quad \text{in} \quad \Omega \times (0, \infty);$$
$$z = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty);$$
$$(2.2) \qquad \frac{\partial}{\partial \nu} z + D z_t = w_t \quad \text{on} \quad \Gamma_0 \times (0, \infty);$$
$$(2.3) \qquad z(0) = 0, \qquad z_t(0) = 0 \ \ in \ \Omega;$$

$$(2.4) \qquad \left. \begin{array}{c} w_{tt} + \mathcal{A}w = \mathcal{A}_0\theta - \rho z_t \\ \theta_t + \mathcal{A}_0\theta = \Delta w_t \end{array} \right\} \quad \text{on} \quad \Gamma_0 \times (0, \infty);$$
$$w(0) = 0, \ w_t(0) = \mathcal{B}u; \ \theta(0) = 0.$$

An equivalent semigroup formulation of the system (2.2), (2.4) with either clamped or hinged boundary conditions is

$$(2.5) \qquad \left\{ \begin{array}{c} z_{tt} + A_N z - A_N N w_t + A_N N D N^* A_N z_t = 0; \text{ on } \mathcal{D}(A_N)' \\ w_{tt} + \mathcal{A}w - \mathcal{A}_0\theta + \rho N^* A_N z_t = 0; \text{ on } \mathcal{D}(\mathcal{A}^{1/2})' \\ \theta_t + \mathcal{A}_0\theta = \Delta w_t \\ z(0) = z_t(0) = w(0) = 0, \ w_t(0) = \mathcal{B}u; \ \theta(0) = 0. \end{array} \right.$$

where we have used the notation of Section 1.2.

In what follows we find convenient to use variational formulation for the first equation in (2.5). This is given by

$$(2.6) \qquad \begin{array}{c} (z_{tt}, \phi)_{L_2(\Omega)} + c^2(\nabla z, \nabla\phi)_{L_2(\Omega)} + (DN^*A_N z_t, N^*A_N\phi)_{L_2(\Gamma_0)} \\ -(w_t, N^*A_N\phi)_{L_2(\Gamma_0)} = 0, \ \phi \in H^1(\Omega); \phi = 0 \text{ on } \Gamma_1. \end{array}$$

Moreover, by using the variation of parameters formula we rewrite the last two equations in (2.5) in the form

$$(2.7) \quad v(t) \equiv \begin{pmatrix} w(t) \\ w_t(t) \\ \theta(t) \end{pmatrix} = e^{-A_v t} \begin{pmatrix} 0 \\ \mathcal{B}u \\ 0 \end{pmatrix} + \rho \int_0^t e^{-A_v(t-s)} \begin{pmatrix} 0 \\ N^*A_N z_t(s) \\ 0 \end{pmatrix} ds.$$

Since $\gamma = 0$, it is well known [35, 29] that $-A_v$ generates an analytic semigroup on the space H_v, which in this case reduces to

$$H_v = \mathcal{D}(\mathcal{A}^{1/2}) \times L_2(\Gamma_0) \times L_2(\Gamma_0).$$

Moreover, $e^{-A_v t}$ is a contraction. From Assumption 1 we have that

$$\mathcal{A}^{-r}\mathcal{B} : U \to L_2(\Gamma_0)$$

is bounded for $r < 1/2$. Since

$$\mathcal{D}(A_v) = \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}_0)$$

and A_v is invertible and generates an analytic semigroup, an interpolation formula from [10] applies and gives

$$(2.8) \qquad \mathcal{D}(A_v^\beta) = \mathcal{D}(\mathcal{A}^{\frac{1+\beta}{2}}) \times \mathcal{D}(\mathcal{A}^{\frac{\beta}{2}}) \times \mathcal{D}(\mathcal{A}_0^\beta); \ \ 0 \le \beta \le 1.$$

This implies

$$(2.9) \qquad |A_v^{-\beta} \begin{pmatrix} 0 \\ \mathcal{B}u \\ 0 \end{pmatrix}|_{H_v} \leq C|\mathcal{A}^{-\frac{\beta}{2}}\mathcal{B}u|_{L_2(\Gamma_0)}; \ 0 \leq \beta < 1$$

and, for $\beta \geq 2r; \ r < 1/2$,

$$(2.10) \qquad |A_v^{-\beta} \begin{pmatrix} 0 \\ \mathcal{B}u \\ 0 \end{pmatrix}|_{H_v} \leq C|u|_U$$

where in the last step we have used the Assumption 1. The analyticity of the semigroup associated with $-A_v$ then gives

$$|e^{-A_v t}Bu|_{H_v} = |e^{-A_v t}A_v^{2r}A_v^{-2r} \begin{pmatrix} 0 \\ \mathcal{B}u \\ 0 \end{pmatrix}|_{H_v} \leq \frac{C_T}{t^{2r}}|u|_U; \ 0 < t \leq T.$$

Combining this with (2.7) yields the estimate

$$(2.11) \qquad |v(t)|_{H_v} \leq C\frac{C|u|_U}{t^{2r}} + C\int_0^t |N^*A_N z_t|_{L_2(\Gamma_0)}dt.$$

Our next step is to estimate the trace of z_t. To accomplish this we shall use equation (2.2). Applying the variational equation (2.6) with $\phi = z_t$ (recall $z_t \in H^1(\Omega); \ z \in \mathcal{D}(A_z)$) integrating by parts and recalling positivity of the damping operator D we obtain

$$
\begin{aligned}
&|A_N^{1/2}z(t)|^2_{L_2(\Omega)} + |z_t(t)|^2_{L_2(\Omega)} + \delta \int_0^t |N^*A_N z_t|^2_{L_2(\Gamma_0)}ds \\
(2.12) \quad &\leq C\int_0^t |(w_t, N^*A_N z_t)_{L_2(\Gamma_0)}|ds \\
&\leq \epsilon \int_0^t |N^*A_N z_t|^2_{L_2(\Gamma_0)}ds + C_\epsilon \int_0^t |w_t|^2_{L_2(\Gamma_0)}ds.
\end{aligned}
$$

Taking ϵ sufficiently small, and noting that δ is positive we obtain

$$(2.13) \qquad \int_0^t |N^*A_N z_t|^2_{L_2(\Gamma_0)}ds \leq C\int_0^t |w_t|^2_{L_2(\Gamma_0)}ds \leq C\int_0^t |v|^2_{H_v}ds.$$

Combining (2.11) with (2.13) and applying Gronwall's inequality with L_1 kernel gives

$$(2.14) \qquad |w_t(t)|_{L_2(\Gamma_0)} \leq |v(t)|_{H_v} \leq \frac{C_T}{t^{2r}}|u|_U; \ \ 0 < t \leq T.$$

If $r < 1/4$, the above estimate when inserted into (2.12) gives the desired estimate for the variable z, hence proving the Lemma.

In order to obtain appropriate estimate for the z variable with $r \geq 1/4$, we are in a need of the second part of the Assumption 1. Indeed, by exploiting the additional coercivity of the damping operator D w obtain a stronger version of the inequality in (2.12) which now reads:

$$
\begin{aligned}
(2.15) \quad & |A_N^{1/2} z(t)|^2_{L_2(\Omega)} + |z_t(t)|^2_{L_2(\Omega)} + \delta \int_0^t |N^* A_N z_t|^2_{\mathcal{D}(\mathcal{A}^{r_0})} ds \\
& \leq C \int_0^t |(w_t, N^* A_N z_t)_{L_2(\Gamma_0)}| ds \\
& \leq \epsilon \int_0^t |N^* A_N z_t|^2_{\mathcal{D}(\mathcal{A}^{r_0})} ds + C_\epsilon \int_0^t |w_t|^2_{\mathcal{D}(\mathcal{A}^{r_0})'} ds.
\end{aligned}
$$

Taking, as before, ϵ sufficiently small we obtain ($r_0 \geq 0$):

$$
\begin{aligned}
(2.16) \quad & |A_N^{1/2} z(t)|^2_{L_2(\Omega)} + |z_t(t)|^2_{L_2(\Omega)} + \int_0^t |N^* A_N z_t|^2_{\mathcal{D}(\mathcal{A}^{r_0})} ds \\
& \leq C \int_0^t |w_t|^2_{\mathcal{D}(\mathcal{A}^{r-1/4+\epsilon})'} ds; \quad \epsilon = r_0 - r + 1/4 > 0.
\end{aligned}
$$

The remaining task is to obtain the estimate for the $L_2(0,T)$ norm of $|w_t|^2_{\mathcal{D}(\mathcal{A}^{r-1/4+\epsilon})'}$ To accomplish this, we shall use more of interpolation theory.

$$
\begin{aligned}
(2.17) \quad & \left|A_v^{-2r+1/2-2\epsilon} e^{-A_v t} \begin{pmatrix} 0 \\ \mathcal{B}u \\ 0 \end{pmatrix}\right|_{H_v} \\
& = |A_v^{1/2-2\epsilon} e^{-A_v t}|_{\mathcal{L}(H_v)} \left|A_v^{-2r} \begin{pmatrix} 0 \\ \mathcal{B}u \\ 0 \end{pmatrix}\right|_{H_v} \leq C \frac{|u|_U}{t^{1/2-2\epsilon}},
\end{aligned}
$$

where we have used (2.10).

From (2.9), (2.17), (2.16) and standard semigroup estimate applied to equation (2.7) we infer

$$
(2.18) \quad |w_t(t)|_{\mathcal{D}(\mathcal{A}^{r-1/4+\epsilon})'} \leq |A_v^{-2r+1/2-2\epsilon} v(t)|_{H_v} \leq \frac{C}{t^{1/2-2\epsilon}} |u|_U
$$

$$
+ C\sqrt{t}\left[\int_0^t |N^* A_N z_t|^2_{L_2(\Gamma_0)} dt\right]^{1/2} \leq \frac{C}{t^{1/2-2\epsilon}} |u|_U + C\sqrt{t}\left[\int_0^t |w_t|^2_{\mathcal{D}(\mathcal{A}^{r-1/4+\epsilon})'} ds\right]^{\frac{1}{2}}.
$$

By using Gronwall's inequality with L_1 kernel we conclude that

$$
(2.19) \quad |w_t(t)|_{\mathcal{D}(\mathcal{A}^{r-1/4+\epsilon})'} \leq |A_v^{-2r+1/2-2\epsilon} v(t)|_{H_v} \leq \frac{C_T}{t^{1/2-2\epsilon}} |u|_U; \ t \leq T.
$$

Hence

(2.20) $$\int_0^t |w_t(t)|^2_{\mathcal{D}(\mathcal{A}^{r-1/4+\epsilon})'} ds \le C_T |u|^2_U.$$

The estimate in (2.20) when inserted into (2.16) gives

(2.21) $$|A_N^{1/2} z(t)|^2_{L_2(\Omega)} + |z_t(t)|^2_{L_2(\Omega)} \le C|u|^2_U,$$

which combined with (2.14) gives the inequality stated in the Lemma. ∎

As mentioned before, the singular estimate in Lemma 2.1 is a critical ingredient for the proof of Theorem 1.1. This is needed in order to show that the gain operator $B^*P(t)$ and the function $B^*r(t)$ are bounded, in spite of the *unboundedness of B*. Once this is proved, the rest of the proof follows by the arguments, which are known in the literature and based on optimization [27, 10, 31, 21]. In what follows for the convenience of the reader we shall outline the main steps leading to the boundedness of the operator B^*P and the function B^*r.

Lemma 2.2.

(2.22) $$B^*P \in \mathcal{L}(H \to C([0,T];U)).$$

Proof. We introduce the following control-to-state operator $L_\tau : L_2(\tau, T; U) \to L_2(\tau, T; H)$ defined by

$$L_\tau u(t) \equiv \int_\tau^t e^{A(t-s)} Bu(s) ds.$$

Due to the singular estimate in Lemma 2.1, it is immediate to show that L_τ is a bounded operator on L_p spaces. This is to say

(2.23) $$L_\tau \in \mathcal{L}(L_p(\tau, T; U) \to L_p(\tau, T; H)); \quad 1 \le p \le \infty$$
$$L_\tau \in \mathcal{L}(C([\tau, T]; U) \to C([\tau, T]; H)).$$

By exploiting more singular estimates in Lemma 2.1 together with Young's inequality we are able to show [27] more regularity for L_τ. These are given below.

(2.24)
$$L_\tau \in \mathcal{L}(L_2(\tau, T; U) \to C([\tau, T]; H)) \text{ for } 2r < 1/2;$$

$$L_\tau \in \mathcal{L}(L_p(\tau, T; U) \to C([\tau, T]; H)) \text{ for } p \ge 2, \ 2r + \tfrac{1}{p} < 1;$$

$$L_\tau \in \mathcal{L}(L_p(\tau, T; U) \to L_{\bar{p}}(\tau, T; H))$$

$$\text{for } 2r \ge 1/2, \ p \ge 2, \ 2r + \tfrac{1}{p} \ge 1, \ \bar{p} = \tfrac{p}{(2r-1)p+1} > p.$$

It is important to notice that in the last case discussed above we always have $\bar{p} - p > 0$, a fact which reflects certain smoothing mechanism on L_p spaces. The above property will allow us to show that the evolution corresponding to "unforced" problem (with $F = 0$) is continuous in time. To see this let $\Phi(t, \tau)$ denote the evolution operator corresponding to the optimization problem with $F = 0$. It is well known [27, 10] that $\Phi(t, \tau)$ admits the following representation:

$$\Phi(t, \tau) = [I + L_\tau L_\tau^* R^* R]^{-1} e^{A(\cdot - \tau)},$$

where we always have that

$$[I + L_\tau L_\tau^* R^* R]^{-1} e^{A(\cdot - \tau)} \in \mathcal{L}(H \to L_2(\tau, T; H)).$$

By using properties in (2.24) together with the "boost trap" argument as in [27] one shows that

(2.25) $\Phi(\cdot, \tau) : H \to C([\tau, T]; H)$ is bounded $\forall\, 0 \le \tau \le T$.

The Riccati operator $P(t) : H \to H$, which is bounded, positive and selfadjoint, admits the following explicit representation in terms of the evolution $\Phi(t, \tau)$ [27].

$$P(t) = \int_t^T e^{A^*(s-t)} R^* R \Phi(s, t) ds.$$

This representation together with the continuity properties of the evolution $\Phi(t, \tau)$ and Lemma 2.1 allows us to conclude the assertion in Lemma 2.2. Indeed, we evaluate:

$$|B^* P(t) x|_U \le \int_t^T |B^* e^{A^*(s-t)}|_{\mathcal{L}(H \to U)} |R^* R \Phi(s, t) x|_H ds$$
(2.26)
$$\le C \int_t^T \frac{1}{(s-t)^{2r}} |\Phi(s, t) x|_H ds \le C_T \int_t^T \frac{1}{(s-t)^{2r}} ds |x|_H \le C_T |x|_H,$$

where we have used singular estimate in Lemma 2.1 together with the fact that $2r < 1$. ∎

Lemma 2.3. *With $A_P(t) \equiv A - BB^* P(t)$ we have:*
$A_P(t)$ *generates on H a strongly continuous evolution operator $\Phi(\tau, t)$.*

Proof. In order to prove the Lemma it suffices to establish unique solvability in $C([\tau, T]; H)$ of the following integral equation:

$$y(t) = e^{A(t-\tau)} x - \int_\tau^t e^{A(t-s)} BB^* P(s) y(s) ds$$

with a given initial condition $x \in H$. This can be accomplished by applying contraction principle. To see this, it is enough to notice the estimate

$$\int_\tau^t |e^{A(t-s)} BB^* P(s) y(s)|_H ds \le C \int_\tau^t \frac{1}{(t-s)^{2r}} |B^* P(s)|_{\mathcal{L}(H \to U)} |y(s)|_H ds$$

(2.27) $\qquad \le C \int_\tau^t \frac{1}{(t-s)^{2r}} |y(s)|_H ds \le C(t-\tau)^{1-2r} |y|_{C([\tau,T],H)}.$

The above estimate allows to apply contraction principle on some small interval (τ, T_0). This local result can be extended to a global one (with $T > 0$ arbitrary) in finitely many steps. ∎

The next key point is to establish singular estimate for the evolution $\Phi(t, \tau)$. This is given below.

Lemma 2.4.

(2.28) $\qquad\qquad |\Phi(t,\tau)B|_{\mathcal{L}(H \to U)} \le C_T \frac{1}{(t-\tau)^{2r}}; \; t > \tau.$

Proof. We will estimate $y(t) \equiv \Phi(t,\tau)Bu$, which solves the following integral equation

(2.29) $\qquad\qquad y(t) = e^{A(t-\tau)} Bu - \int_\tau^t e^{A(t-s)} BB^* P(s) y(s) ds.$

In order to establish solvability of this equation (with a given $u \in U$) we will use singular spaces Z_r, introduced in the context of studying the same problem for the analytic case [27, 10, 31]:

$$Z_r \equiv \Big\{ f \in C((\tau,T],H); \sup_{\tau \le t \le T} |f(t)(t-\tau)^{2r}|_H \le \infty \Big\}.$$

By applying fixed point argument together with the results of Lemma 2.1, Lemma 2.2, and repeating the same line of arguments as in [27], one proves the existence of an unique solution $y(t) \in Z_r$ satisfying (2.29). This, in turn, implies the estimate sated in the Lemma. ∎

Our final step is to establish the regularity of function r. Indeed, since $A_P(t)$ generates strongly continuous evolution,

$$r_t = -A_P^*(t)r - P(t)F: \; r(T) = 0$$

admits an unique solution $r \in C([0,T];H)$ with any $F \in L_1(0,T;H)$. We need to establish stronger regularity properties for $r(t)$.

Lemma 2.5.

(2.30)
$$B^* r \in L_2(0, T; U); \quad F \in L_2(0, T; H),$$
$$B^* r \in C([0, T]; U); \quad F \in C([0, T]; H).$$

Proof. By applying variation of parameters formula together with the estimate in Lemma 2.4 we obtain

(2.31)
$$|B^* r(t)|_H \leq \int_t^T |B^* \Phi^*(t, s) P(s) F(s)|_U ds$$

$$\leq C \int_t^T \frac{1}{(t-s)^{2r}} |P|_{\mathcal{L}(H \to C([0,T];H))} |F(s)|_H ds \leq C \int_t^T \frac{1}{(t-s)^{2r}} |F(s)|_H ds.$$

Since $2r < 1$, the conclusion in Lemma 2.5 follows now from convolution properties. ∎

The statement of Theorem 1.1 follows by applying regularity properties established in Lemma 2.2 and Lemma 2.5 together with a generic optimization argument presented in [21, 19].

Remark 2.2. Note that in the case when $r \geq 1/4$, the control operator B is not "admissible" [40, 38]. It is in this case, where a stronger "unbounded" damping operator D is needed. This phenomenon has no counterpart for structurally damped problems, where there is no need for any damping on the wave component and the singular estimate of Lemma 2.1 holds with $D = 0$ [21].

2..2 Proof of Theorem 1.2.

Since the singular estimate of Lemma 2.1 is no longer valid in this "purely" hyperbolic case (and when $2r \geq 1/2$), our argument relies on the "admissibility" condition being satisfied for this hyperbolic model. The key result in this direction is the following

Lemma 2.6. *Let $\gamma > 0$ and let the damping operator D be bounded and coercive on $L_2(\Gamma_0)$. With the control operator given by (1.5) we have the following inequality.*

(2.32)
$$\int_0^T |B^* e^{A^* t} x|_U^2 dt \leq C |x|_H^2.$$

Proof. The proof of this lemma is based on the following PDE inequality, of interest in its own.

Lemma 2.7 [45]. *Let w, θ be a solution of the following thermoelastic problem:*

$$\mathcal{M}_\gamma w_{tt} + \mathcal{A}w + \mathcal{A}_0\theta = 0, \quad \theta_t + \mathcal{A}\theta = -\mathcal{A}_0 w_t,$$

(2.33)

$$w(0) = w_0 \in \mathcal{D}(\mathcal{A}^{1/2}), \; w_t(0) = w_1 \in \mathcal{D}(\mathcal{M}_\gamma^{1/2}), \; \theta(0) = \theta_0 \in L_2(\Gamma_0)$$

for the clamped or hinged boundary conditions and

$$\mathcal{M}_\gamma w_{tt} + \mathcal{A}[w + \mathcal{G}_1\theta|_{\partial\Gamma_0} + \lambda\mathcal{G}_2\theta|_{\partial\Gamma_0}] + \mathcal{A}_0\theta = 0, \; \theta_t + \mathcal{A}\theta = \Delta w_t,$$

(2.34)

$$w(0) = w_0 \in \mathcal{D}(\mathcal{A}^{1/2}), \; w_t(0) = w_1 \in \mathcal{D}(\mathcal{M}_\gamma^{1/2}), \; \theta(0) = \theta_0 \in L_2(\Gamma_0)$$

for free boundary conditions. Then, the following estimates are valid

$$\int_0^T |\nabla w_t(\xi_i, t)|^2 dt \leq C\left[|w_0|^2_{\mathcal{D}(\mathcal{A}^{1/2})} + |w_1|^2_{\mathcal{D}(\mathcal{M}_\gamma^{1/2})} + |\theta_0|^2_{L_2(\Gamma_0)}\right]$$

$$\text{for } \xi_i \in \Gamma_0, \; \dim \Gamma_0 = 1;$$

(2.35)

$$\int_0^T |\nabla w_t(\cdot, t)|^2_{L_2(\xi_i)} dt \leq C\left[|w_0|^2_{\mathcal{D}(\mathcal{A}^{1/2})} + |w_1|^2_{\mathcal{D}(\mathcal{M}_\gamma^{1/2})} + |\theta_0|^2_{L_2(\Gamma_0)}\right]$$

$$\text{for } \xi_i \in \Gamma_0; \; \dim \Gamma_0 = 2.$$

Remark 2.3. Note that the estimate in Lemma above does not follow from the interior regularity of the solutions. Indeed, the velocity w_t is in $H^1(\Gamma_0)$. Thus the evaluation of the gradients ∇w_t on a manifold which is of one dimension less than Γ_0 [curve (resp point)] is not well defined. Nevertheless the stated regularity does hold true and results from the "hidden" regularity of traces corresponding to Kirchhoff equations. This fact was first observed in [43, 42] and later generalized to the two-dimensional case in [46].

To complete the proof of Lemma 2.6 the estimate in (2.33) needs to be propagated onto the entire structural acoustic system. This is done in the same way as in [12, 11]. ■

In order to cope with the infinite horizon nature of the problem, the following stability estimate established in [32, 24, 25] is critical.

Lemma 2.8. *Let the Assumption* 2 *be satisfied. Moreover, we assume that the damping operator D is bounded and coercive on $L_2(\Gamma_0)$. Then, the semi-group corresponding to A is exponentially stable on H. This is to say, there exist constants $C, \omega > 0$ such that*

$$|e^{At}|_{\mathcal{L}(H)} \leq Ce^{-\omega t}; \ t \geq 0.$$

From Lemma 2.6 we infer that the operator B is admissible. From Lemma 2.8 we know that the Finite Cost Condition as well as Detectability Condition are satisfied. This allows us to apply the conclusions of abstract Theorem in [19] to complete the proof of Theorem 1.2.

Remark 2.4. We note that strong coercivity of the damping operator D is essential in proving stability estimate in Lemma 2.8. In fact, it was shown in [2] that structural acoustic model is only strongly stable, but *not uniformly* stable, without the added dissipation on the boundary of the wave component.

References

[1] G. Avalos and I. Lasiecka, *Differential Riccati equation for the active control of a problem in structural acoustics*, JOTA **91** (1996), 695–728.

[2] G. Avalos and I. Lasiecka, *The strong stability of a semigroup arising from a coupled hyperbolic/parabolic system*, Semigroup Forum **57** (1997), 278–292.

[3] A.V. Balakrishnan, Applied Functional Analysis, Springer Verlag 1975.

[4] H.T. Banks, R.J. Silcox and R.C. Smith, *The modeling and control of acoustic/structure interaction problems via piezoceramic actuat ors:* 2*-D numerical examples*, ASME Journal of Vibration and Acoustics **2** (1993), 343–390.

[5] H.T. Banks and R. Smith, *Active control of acoustic pressure fields using smart material technology in flow control*, IMA, (M. Gunzburger, ed.) **68** Springer Verlag 1995.

[6] H.T. Banks and R.C. Smith, *Well-Posedness of a model for structural acoustic coupling in a cavity enclosed by a thin cylindrical shell*, Journal of Mathematical Analysis and Applications **191** (1995), 1–25.

[7] H.T. Banks, R.C. Smith and Y. Wang, *The modeling of piezoceramic patch interactions with shells, plates and beams*, Quaterly of Applied Mathematics **53**, 2 (1995), 353–381.

[8] V. Barbu, I. Lasiecka and R. Triggiani, *Extended Algebraic Riccati Equations arising in hyperbolic dynamics with unbounded controls*, Nonlinear Analysis, to appear.

[9] J. Beale, *Spectral properties of an acoustic boundary condition*, Indiana Univ. Math. J. **9** (1976), 895–917.

[10] A. Bensoussan, G. Da Prato, M.C. Delfour and S.K. Mitter, Representation and Control of Infinite Dimensional Systems, vol. II, Birkhauser, Boston 1993.

[11] M. Camurdan and R. Triggiani, Control Problems in Noise Reduction: The case of two coupled hyperbolic equations, in: Proceedings of SPIE's 4-th Annual Symposium on Smart Structures and Materials, Mathematic and Control in Smart Structures 1997.

[12] M. Camurdan and R. Triggiani, *Sharp Regularity of a coupled system of a wave and Kirchhoff equation with point control, arising in noise reduction*, Diff and Integral Equations 1998.

[13] E.F. Crawley and E. H. Anderson, *Detailed models of piezoceramic actuation of beams*, in: Proc. of AIAA Conference 1989.

[14] E.F. Crawley and J. de Luis, *Use of piezoelectric actuators as elements of intelligent structures*, AIAA Journal **25** (1987), 1373–1385.

[15] E.K. Dimitriadis, C.R. Fuller and C.A. Rogers, *Piezoelectric Actuators for Distributed Noise and Vibration Excitation of Thin Plates*, Journal of Vibration and Acoustics **13** (1991), 100–107.

[16] F. Flandoli, *Riccati equations arising in boundary control problems with distributed parameters*, SIAM J. Control **22** (1984), 76–86.

[17] F. Flandoli, I. Lasiecka and R. Triggiani, *Algebraic Riccati Equations with non-smoothing observations arising in hyperbolic and Euler-Bernoulli boundary control problems*, Annali di Matematica Pura et. Applicata **153** (1988), 307–382.

[18] P. Grisvard, *Characterization de quelques espaces d'interpolation*, Archive rational Mechanics and Analysis **26** (1967), 40–63.

[19] I. Lasiecka, NSF-CBMS Lecture Notes on Mathematical Control Theory of Coupled PDE's, SIAM, Philadelphia 2000.

[20] J. Lagnese, Boundary Stabilization of Thin Plates, SIAM, Philadelphia 1989.

[21] I. Lasiecka, *Mathematical control theory in structural acoustic problems*, Mathematical Models and Methods in Applied Sciences **8** (1998), 1119–1153.

[22] I. Lasiecka, *Boundary stabilization of a 3-dimensional structural acoustic model*, Journal de Math. Pure et Appli. **78** (1999), 203–232.

[23] I. Lasiecka, *Uniform decay rates for full von Karman system of dynamic elasticity with free boundary conditions and partial boundary dissipation*, Communications in PDE **24** (1999), 1801–1849.

[24] I. Lasiecka and C. Lebiedzik, *Decay rates in nonlinear structural acoustic models with thermal effects and without a damping on the interface*, Preprint, 1999.

[25] I. Lasiecka and C. Lebiedzik, *Boundary stabilizability of nonlinear structural acoustic models with thermal effects on the interface*, C.R. Acad. Sci. Paris, serie II b, (2000), 1–6.

[26] I. Lasiecka and R. Triggiani, *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems, Continuous Theory and Approximation Theory*, Springer Verlag, LNCIS **164**, 1991.

[27] I. Lasiecka and R. Triggiani, *Riccati differential equations with unbounded coefficients and nonsmooth terminal condition-the case of analytic semigroups*, SIAM J. Math. Analysis **23** (2) (1992), 448–481.

[28] I. Lasiecka and R. Triggiani, *Analyticity and lack thereof, of thermoelastic semigroups*, ESAIM **4** (1998), 199–222.

[29] I. Lasiecka and R. Triggiani, *Analyticity of thermo-elastic semigroups with free B.C.*, Annali di Scuola Normale Superiore XXVII (1998), 457–482.

[30] I. Lasiecka and R. Triggiani, *Two direct proofs on the analyticity of the S.C. semigroup arising in abstract thermo-elastic equations*, Advances in Differential Equations **3** (1998), 387–416.

[31] I. Lasiecka and R. Triggiani, Control Therory for Partial Differential Equations, Cambridge University Press, Cambridge 1999.

[32] C. Lebiedzik, *Exponential stability in structural acoustic models with thermoelasticity*, Dynamics of Continous, Discrete and Impulsive Systems, 1999.

[33] H.C. Lester and C.R. Fuller, *Active control of propeller induced noise fields inside a flexible cylinder*, in: Proc. of AIAA Tenth Aeroacoustics Conference, Seattle, WA 1986.

[34] J.L. Lions and E. Magenes, Non-homogenous Boundary Value Problems and Applications, Springer Verlag, New York 1972.

[35] Z. Liu and M. Renardy, *A note on the equation of thermoelastic plate*, Applied Math. Letters **8** (1995), 1–6.

[36] P.M. Morse and K.U. Ingard, Theoretical Acoustics, McGraw-Hill, New York 1968.

[37] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer Verlag, New York 1986.

[38] G. Da Prato, I. Lasiecka and R. Triggiani, *A Direct Study of Riccati Equations arising in Hyperbolic Boundary Control Problems*, Journal of Differential Equations **64** (1986), 26–47.

[39] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations. recent progress and open questions*, SIAM Review **20** (1978), 639–739.

[40] D.L. Russell, *Mathematical models for the elastic beam and their control-theoretic properties*, Semigroups Theory and Applications, Pitman Research Notes **152** (1986), 177–217.

[41] R. Triggiani, *Lack of exact controllability for wave and plate equations with finitely many boundary controls*, Differential and Integral Equations **4** (1991), 683–705.

[42] R. Triggiani, *Interior and boundary regularity of the wave equation with interior point control*, Differential and Integral Equations **6** (1993), 111–129.

[43] R. Triggiani, *Regularity with interior point control, part* II: *Kirchhoff equations*, Journal of Differential Equations **103** (1993), 394–420.

[44] R. Triggiani, *The Algebraic Riccati Equations with Unbounded Coefficients; Hyperbolic Case Revisited*, Contemporary Mathematics: Optimization Methods in PDE's, AMS, Providence **209** (1997), 315–339.

[45] R. Triggiani, *Sharp regularity of thermoelastic plates with point controls*, Abstract and Applied Analysis, to appear.

[46] M. Tucsnak, *Control of plates vibrations by means of piezoelectric actuators*, Discrete and Continuous Dynamical Systems **2** (1996), 281–293.

[47] G. Weiss and H. Zwart, *An example in LQ optimal control*, Systems and Control Letters **33** (1998), 339–349.