SIGNAL RECONSTRUCTION FROM GIVEN PHASE OF THE FOURIER TRANSFORM USING FEJÉR MONOTONE METHODS

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Abstract

The aim is to reconstruct a signal function $x \in L_2$ if the phase of the Fourier transform $\hat{x}$ and some additional a-priori information of convex type are known. The problem can be described as a convex feasibility problem. We solve this problem by different Fejér monotone iterative methods comparing the results and discussing the choice of relaxation parameters. Since the a-priori information is partly related to the spectral space the Fourier transform and its inverse have to be applied in each iterative step numerically realized by FFT techniques. The computation uses MATLAB routines.

Keywords: signal reconstruction, convex feasibility problem, projection onto convex sets, Fejér monotone iterative methods, Fourier transforms.

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1 Introduction

Often signals are disturbed and the receiver gets only a partial information. So the question arises wether the signal can be reconstructed. Besides, usually the reciever has a-priori information about the signal. This means that the signal x belongs to some subsets $M_i \subset H (i = 1, \cdots, m)$ of a certain signal space H. Hence, a so-called feasibility problem has to be solved to determine appropriate candidates for the signal x. Often the subsets are convex and closed. Then Fejér monotone iterative methods can be used for

the solution. The investigation of these methods started already in the 50th of our century. A well-known paper of the first period is [3]. The advantage of the methods is that already weak assumptions ensure convergence. The sections 2 and 3 summarize some basic concepts and results of my papers [7–10] devoted to this subject. Reviews about the state-of-the-art are given e.g. in [1] and in [2]. Signal reconstruction within this framework was already investigated in several papers. Here we follow the line of [12] and [6], where projection type methods are used. We suppose signals with known phase, limited carrier and range. This leads to a convex feasibility problem. We present convergence results for a more general class of iterative methods than in [6] and some reconstruction tests which supply new insights partly different from those given in [6]. The numerical results are obtained by use of MATLAB routines on a PC.

2   The general model and a solution concept

Let H be a (complex) Hilbert space. We consider closed convex subsets $M_i(i = 1, \cdots, m)$ of H which satisfy

$$M := \bigcap_{i=1}^{m} M_i \neq \emptyset$$

and look for elements $x^* \in M$. This is the so-called *convex feasibility problem.* The problem can be treated by the following

Philosophy:

- determine *set-valued* mappings

$$T_{i,k} : Q \mapsto \mathbb{P}(Q) \ (i = 1, \cdots, m, \ k \in \mathbb{N})$$

  with *fixed point sets* M_i and a domain Q satisfying $M_i \subseteq Q \subseteq H$,

- construct combined mappings T_k with *fixed point set* M,

- use the general iterative scheme:

$$x_{k+1} \in T_k x_k \quad , \quad x_0 \in Q \quad , \quad k \in \mathbb{N}.$$

We will choose for $T_{i,k}$ so-called *Fejér monotone mappings* w.r.t. M_i, especially the *metric projectors* $P_i : H \mapsto M_i$ or *relaxations*

$$T_{i,k} = (1 - \lambda_{i,k})E + \lambda_{i,k}P_i \quad , \quad |1 - \lambda_{i,k}| < 1$$

of P_i. Well-known and important *construction principles* for T_k are (see e.g. [1])

- *sequential* or *successive* combination of $T_{i,k}$:

$$T_k := T_{m,k}T_{m-1,k}\cdots T_{1,k},$$

- *parallel* or *simultaneous* combination of $T_{i,k}$:

$$T_k := \gamma_{1,k}\,T_{1,k} + \gamma_{2,k}\,T_{2,k} + \cdots + \gamma_{m,k}T_{m,k},$$

$$\gamma_{i,k} \geq 0 \quad , \quad \gamma_{1,k} + \gamma_{2,k} + \cdots + \gamma_{m,k} = 1.$$

3   Convergence of Fejér monotone methods

We consider a Hilbert space H and a closed convex subset Q.

Definition 31. Let $\alpha > 0$ be a parameter and M a nonempty subset of Q. The mapping $T : Q \mapsto \mathbb{P}(Q)$ is said to be α-*strongly Fejér monotone* w.r.t. M if

$$\|y - x\|^2 - \|z - x\|^2 \geq \alpha \|y - z\|^2 \qquad \forall x \in M,\ \forall y \in Q,\ \forall z \in Ty$$

$$y \notin Ty \quad \forall y \in Q \setminus M.$$

Remark 32. If the magnitude of α is not essential, the parameter will be dropped in the notation. Sometimes the limit case $\alpha = 0$ is also of interest. Such mappings T are called *regularly Fejér monotone* w.r.t. M. For $\alpha \geq 0$ the fixed point set of T is just M. Besides, M is necessarily convex and closed. Moreover, all fixed points are simple, i.e. $Tx = \{x\}$ for all $x \in M$ (see e.g. [8]). Similar concepts are partly denoted in the literature by other terms. Strongly Fejér monotone operators $T : Q \mapsto Q$ are called in [1, p. 372] *strongly attracting*.

Definition 33. Let $\alpha > 0$ be a parameter and M a nonempty subset of Q. The operator $T : Q \mapsto Q$ with the complement $T' = E - T$ (E identity) is said to be α-*strongly nonexpansive* w.r.t. M if

$$\|y - x\|^2 - \|Ty - Tx\|^2 \geq \alpha \|T'y - T'x\|^2 \quad \forall x \in Q,\ \forall y \in Q$$

and M is the fixed point set of T.

Remark 34. Again α can be dropped in the notation. For the limit case $\alpha = 0$ the operators T are called *regularly nonexpansive* w.r.t. M. These operators are just usual nonexpansive operators with the fixed point set M. It is easy to see that α-strongly nonexpansive operators T w.r.t. M are also α-strongly Fejér monotone w.r.t. M for all $\alpha \geq 0$. Besides, these operators T are special nonexpansive operators such that their complement $T' = E - T$ is demiclosed. For $\alpha > 0$ they are reasonable wanderers and consequently asymptotically regular (see [9, p. 89]). Further, it turns out that 1-strongly nonexpansive means just *firmly nonexpansive* (see [10, p. 721]). If $T \in L(Q)$ is a linear operator, then the concepts in Definitions 31 and 33 coincide (see [10, p. 713]).

Definition 35. Let M be a nonempty subset of Q. Then the mapping $T : Q \mapsto \mathbb{P}(Q)$ is said to be *normal* w.r.t. M if

$$\|y - z\| \to 0 \quad \Longrightarrow \quad \rho(y, M) \to 0 \qquad \forall y \in Q \setminus M, \forall z \in Ty,$$

where $\rho(\cdot, \cdot)$ means the distance functional.

The above concept is developed in [7], where however the complement $E - T$ is called zero-normal. Evidently, T is normal w.r.t. M if there is a constant $\beta > 0$ such that

$$\|y - z\| \geq \beta \, \rho(y, M) \qquad \forall y \in Q \setminus M, \ \forall z \in Ty.$$

We will use the following simple convergence result (see [9, p. 89] and [7]) which is completely sufficient for our applications. Generalizations can be found in [1].

Theorem 36. *Let be $M \neq \emptyset$. Under the assumptions*

   a) *T strongly Fejér monotone w.r.t. M,*

   b) *$T' = E - T$ demiclosed (E identity)*

the iterative method (x_k) defined by $x_{k+1} \in Tx_k$ converges for all $x_0 \in Q$ weakly to an element x^* in M. If instead of b) the condition

   b') *T is normal w.r.t. M*

is fulfilled, the method (x_k) converges even strongly.

Remark 37. Let M be the solution set of a convex feasibility problem with the subsets M_i. If the mappings $T_{i,k}$ are strongly Fejér monotone (strongly nonexpansive) w.r.t. M_i, then the sequential and parallel combinations T_k are strongly Fejér monotone (strongly nonexpansive) w.r.t. the intersection M. Also the normality of such mappings $T_{i,k}$ w.r.t. M_i is transfered to the mapping T_k w.r.t. M. If the corresponding parameters α_k do not accumulate to 0, then the T_k can be replaced by the union mapping T which is again strongly Fejér monotone w.r.t. M. Hence, using the method $x_{k+1} \in T_k x_k$, the above convergence result can be applied (see [9, p. 94]).

4 Signal reconstruction from given phase

Here we use the complex Hilbert space $H = L_2(\mathbb{R}^n, \mathbb{C})$ of quadratically integrable functions x containing n-dimensional complex-valued *signals*. The subspace of real-valued signals is denoted by $L_2(\mathbb{R}^n, \mathbb{R})$. Sometimes we use in both cases the short form L_2. Further, $B(x_0, r)$ means a ball with midpoint x_0 and radius r. The mapping $F : L_2(\mathbb{R}^n, \mathbb{C}) \mapsto L_2(\mathbb{R}^n, \mathbb{C})$ represents the unitary Fourier(-Plancherel) transform defined by the limit

$$(Fx)(\omega) = \lim_{r \to \infty} (2\pi)^{-n/2} \int_{B(0,r)} x(t) \, e^{-i\omega \cdot t} \, dt.$$

in L_2. Signals have two dual faces, namely

- the spatial face $x = x(t)$,

- the spectral face $\hat{x} = \hat{x}(\omega) = (Fx)(\omega)$.

The latter has the polar representation:

$$\hat{x}(\omega) = r_x(\omega) \exp(i\phi_x(\omega)),$$

where $r_x(\omega)$ is the *magnitude* and $\phi_x(\omega)$ is the *phase* of x. The scalar product is independent on the used face:

$$\langle x_1, x_2 \rangle = \int x_1(t) x_2^*(t) \, dt = \int \hat{x_1}(t) \hat{x_2}^*(t) \, dt = \langle \hat{x_1}, \hat{x_2} \rangle.$$

Now we consider some usual *a-priori informations* about signals. Signals with carriers in the bounded closed convex set $I \subset \mathbb{R}^n$ create the set

$$M_I = \{x : x(t) = 0 \text{ for } t \notin I\}.$$

This set is a linear subspace and has the orthogonal complement

$$M_I^\perp = \{x : x(t) = 0 \text{ for } t \in I\}.$$

The linear orthogonal projector onto M_I is

(1)
$$(P_I x)(t) = \begin{cases} x(t) & \text{if } t \in I, \\ 0 & \text{if } t \notin I. \end{cases}$$

Signals with ranges in the closed convex set $J \subseteq \mathbb{C}$ form the set

$$M^J = \{x : x(t) \in J \text{ for all } t\}.$$

This set is convex and closed. If P^J is the projector in $\mathbb{C}$ onto J, the extension $(P^J x)(t) = P^J x(t)$ just supplies the projector in L_2 onto M^J. The intersection

$$M_I^J = M_I \cap M^J$$

is a closed convex set, too. Obviously, the corresponding projector P_I^J satisfies $P_I^J = P_I P^J$. If we restrict ourselves to $L_2(\mathbb{R}^n, \mathbb{R})$ and $J = [c, d]$ then we have

(2)
$$(P_I^J x)(t) = \begin{cases} x(t) & \text{if } t \in I, \ c \leq x(t) \leq d, \\ c & \text{if } t \in I, \ x(t) < c, \\ d & \text{if } t \in I, \ d < x(t), \\ 0 & \text{if } t \notin I. \end{cases}$$

Signals with given phase ϕ are contained in

$$M_\phi = \{x : \phi_x(\omega) = \phi(\omega)\}.$$

This set is a closed convex cone. The projector onto M_ϕ is

(3)
$$(P_\phi x)(t) = F^{-1}(\cos_+ \psi_x \, r_x e^{i\phi})(t) \quad , \quad \psi_x = \phi_x - \phi,$$

where $+$ denotes the positive part of a real function (see [6, p. 287]). Now we consider the following

Reconstruction problem: Knowing the phase ϕ of a signal x (by measurement), its carrier I and an inclusion J of its range (by a-priori information), determine this signal, i.e. look for

$$x \in M = M_I^J \cap M_\phi.$$

This problem is solvable, but not uniquely. Namely the intersection will contain apart from a certain signal x at least certain multiples of it. In the *regular case*, which is described in [5] and shortly repeated in [6], M is for $J = \mathbb{C}$ just a one-dimensional subspace of L_2 and for bounded J only a bounded part of it. Hence, without additional information, one can reconstruct only a scaled version of the original signal. Since one restriction relates to the spectral face and the others to the spatial face of the signal the iterations will contain a permanent use of Fourier transform and its inverse. Observing Theorem 36 and Remark 37 we get especially for the methods

$$m1 \; : \; x_{k+1} \in T_\phi T_I^J x_k,$$

$$m2 \; : \; x_{k+1} \in T_I^J T_\phi x_k,$$

$$m3 \; : \; x_{k+1} \in \gamma T_I^J + (1 - \gamma)T_\phi, \quad 0 < \gamma < 1$$

two convergence results. In the first case the assumptions a) and b) of Theorem 36 are fulfilled, in the second the assumptions a) and b').

Theorem 41. *If T_I^J is strongly nonexpansive w.r.t. M_I^J and T_ϕ is strongly nonexpansive w.r.t. M_ϕ, then each of the general methods $m1$, $m2$ and $m3$, starting with an arbitrary signal x_0, converges weakly to a signal x^* in $M = M_I^J \cap M_\phi$.*

Theorem 42. *If T_I^J is Fejér monotone and normal w.r.t. M_I^J and T_ϕ is Fejér monotone and normal w.r.t. M_ϕ then each of the general methods $m1$, $m2$ and $m3$, starting with an arbitrary signal x_0, converges strongly to a signal x^* in $M = M_I^J \cap M_\phi$.*

Now we determine Fejér monotone mappings with respect to M_I^J and M_ϕ. Then we can apply our convergence results. We use the fact that in L_2 the relation $x(t) \geq 0$ a.e. implies $\|x\| \geq 0$.

Theorem 43. *The mapping $T_I^J : L_2(\mathbb{R}^n, \mathbb{C}) \mapsto L_2(\mathbb{R}^n, \mathbb{C})$ defined by*

$$(T_I^J y)(t) = \begin{cases} P_I^J y(t) + q(y, t)(y(t) - P_I^J y(t)) & \text{if} \quad t \in I, \\ q(y, t)y(t) & \text{if} \quad t \notin I, \end{cases}$$

$$q : L_2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R}^n \mapsto \mathbb{C}, \quad q(y, t) \subseteq B\left(\frac{\alpha}{1 + \alpha}, \frac{1}{1 + \alpha}\right) \setminus \{1\}$$

is α-strongly Fejér monotone w.r.t. M_I^J. If $q(y, t) \notin B(1, \delta)$ for some $\delta > 0$ (independently on y and t) then T_I^J is also normal.

Remark 44. For $q(y, t) \subseteq \left[\frac{\alpha - 1}{1 + \alpha}, 1\right[$ and $J \subseteq \mathbb{R}$ the mapping T_I^J lets $L_2(\mathbb{R}^n, \mathbb{R})$ invariant.

If we choose $q(y, t) = 1 - \lambda_I$ for a number $\lambda_I \in]0, \frac{2}{1 + \alpha}]$, then the condition for q is fulfilled. We get the *scalar relaxation*

(4) $$(T_I^J y)(t) = (T_{I, \lambda_I}^J y)(t) = (1 - \lambda_I) y(t) + \lambda_I P_I^J y(t)$$

of the metric projector P_I^J onto M_I^J. For $\alpha \leq 1$ we can choose $q(y, t) = 0$ and $\lambda_I = 1$, respectively. Then $T_I^J = P_I^J$ arises which is 1-strongly Fejér monotone w.r.t. M_I^J. Finally, for $J = \mathbb{C}$ we have $M_I^J = M_I$, $P_I^J = P_I$ and

$$(T_I^J y)(t) = (T_I y)(t) = \begin{cases} y(t) & \text{if } t \in I, \\ q(y, t) y(t) & \text{if } t \notin I. \end{cases}$$

For $q(y, t) = 1 - \lambda_I$ we arrive at linear scalar relaxation T_{I, λ_I} of the orthogonal projector P_I. The mentioned scalar relaxations are even strongly nonexpansive.

Theorem 45. *The mapping* $T_\phi : L_2(\mathbb{R}^n, \mathbb{C}) \mapsto L_2(\mathbb{R}^n, \mathbb{C})$ *defined by*

$$(T_\phi y)(t) = F^{-1}(A_y \, r_y e^{i\phi})(t)$$

with $A_y \subset L_2(\mathbb{R}^n, \mathbb{C})$ *is* α*-strongly Fejér monotone w.r.t.* M_ϕ *if each* $a_y \in A_y$ *satisfies with the notation* $\psi_y = \phi_y - \phi$ *the conditions*

$$\cos \psi_y(\omega) \leq Re \, a_y(\omega),$$

$$\left| a_y(\omega) - \frac{\alpha}{1 + \alpha} e^{i\psi_y(\omega)} \right| \leq \frac{1}{1 + \alpha},$$

$$a_y(\omega) \neq e^{i\psi_y(\omega)} \ for \ y \notin M_\phi.$$

If $a_y(\omega) \notin B(e^{i\psi_y(\omega)}, \delta)$ *for some* $\delta > 0$ *(independently on* y *and* ω*) then* T_ϕ *is also normal.*

Remark 46. Evidently, the mapping T_ϕ is homogenious. If the elements a_y are functions with even real part and odd imaginary part (even magnitude and odd phase) then the mapping T_ϕ lets the real subspace $L_2(\mathbb{R}^n, \mathbb{R})$ invariant.

Often A_y will contain only one function. Then we will write simply a_y instead of $\{a_y\}$. The following special cases are of particular interest:

For $a_y = 1$ we get the elementary phase-correction operator used by Hayes, Lim and Oppenheim in [4] which conserves the magnitude:

$$(T_\phi y)(t) = (U_\phi y)(t) = F^{-1}(r_y e^{i\phi})(t).$$

The above Theorem shows that U_ϕ is only regularly Fejér monotone ($\alpha = 0$). But the convex combination of E and U_ϕ leads to

$$(T_\phi y)(t) = (T_{\phi,\lambda_\phi}^H y)(t) = \mu_\phi y(t) + \lambda_\phi U_\phi y(t)$$

(5)
$$= F^{-1}((\mu_\phi e^{i\psi_y} + \lambda_\phi)\, r_y e^{i\phi})(t)$$

with

$$\mu_\phi = 1 - \lambda_\phi\,,\ 0 < \lambda_\phi < 1.$$

Here we have $a_y = \mu_\phi e^{i\psi_y} + \lambda_\phi$. This operator is α-strongly Fejér monotone and even α-strongly nonexpansive with $\alpha = \frac{1-\lambda_\phi}{\lambda_\phi}$.

For the projector $T_\phi = P_\phi$ stated in (3) we get $a_y = \cos_+ \psi_y$. This shows that T_ϕ is 1-strongly Fejér monotone. Finally, the scalar relaxation

$$(T_\phi y)(t) = (T_{\phi,\lambda_\phi}^P y)(t) = \mu_\phi y(t) + \lambda_\phi P_\phi y(t)$$

(6)
$$= F^{-1}((\mu_\phi e^{i\psi_y} + \lambda_\phi \cos_+ \psi_y)\, r_y e^{i\phi})(t)$$

$$\mu_\phi = 1 - \lambda_\phi,\ 0 < \lambda_\phi < 2$$

of P_ϕ supplies $a_y = \mu_\phi e^{i\psi_y} + \lambda_\phi \cos_+ \psi_y$. This operator is α-strongly Fejér monotone and even α-strongly nonexpansive with $\alpha = \frac{2-\lambda_\phi}{\lambda_\phi}$.

5   Experimental results

5.1   Discretization and iterative methods

Signals $x(t)$ with carrier $I = [a, b]$ are discretized using n equidistant values $x(t_i)$ with $i = 0, \cdots, n-1$ in I. We add values $x(t_i) = 0$ with $i = n, \cdots, N-1$, where N is a potence of 2 satisfying $N \geq 2n - 1$. So we reach a unique reconstruction (see [5]). Practically, we consider $x(t)$ on an extended interval $I_e = [t_0, t_{N-1}]$ which has at least twice the length of I. The Fourier transform (FT) will be replaced by the discrete Fourier transform (DFT). Finally, the results are again restricted to I. This is also in accordance with the fact that the DFT approximates the FT only in the first halve of the interval.

In our tests we choose $a = 0$, $b = 50$, $n = 51$ and $N = 128$. Then we have $I = [0, 50]$ and $I_e = [0, 127]$ discretized with the constant step size $h = 1$. The signals are represented by vectors of length 128. We use the following

Iterative methods m1, m2 and m3 with operators (4), (5) and (6):

- **HLO**: method m1 based on the phase-correction of Hayes, Lim and Oppenheim with the operators $T_\phi = T^H_{\phi,\lambda_\phi}$ and $T^J_I = T_I = T_{I,\lambda_I}$,
- **POCS** (Projection onto convex sets): method m1 with the operators $T_\phi = T^P_{\phi,\lambda_\phi}$ and $T^J_I = T_I = T_{I,\lambda_I}$,
- **POCS-d**: method m1 with the operators $T_\phi = T^P_{\phi,\lambda_\phi}$ and $T^J_I = T^{[0,d]}_{I,\lambda_I}$,
- **PCOS** (POCS with reversed order of the operators): method m2 with $T^J_I = T_I = T_{I,\lambda_I}$ and $T_\phi = T^P_{\phi,\lambda_\phi}$,
- **oPCOS, pPCOS**: variants of PCOS, where the parameters are chosen by per-cycle optimization according to [6] in $\mathbb{R}$ and $[0, 2]$, respectively,
- **PPCS-γ** (Parallel projection onto convex sets with mean parameter γ): method m3 with $T^J_I = T_I = T_{I,\lambda_I}$ and $T_\phi = T^P_{\phi,\lambda_\phi}$.

Obviously, the method PCOS can be interpreted as a method POCS with modified starting signal. All methods are homogenious, i.e. we get $x'_k = cx_k$ for $x'_0 = cx_0$. Observing Theorem 41 all these methods converge if T^J_{I,λ_I} and T^P_{ϕ,λ_ϕ} are used with relaxation parameters in a closed subinterval of $(0, 2)$ as well as T^H_{ϕ,λ_ϕ} in a closed subinterval of $(0, 1)$. Namely, then the occuring operators are all strongly nonexpansive. In this special case the experimental results show that one can use for the third operator also the greater interval $(0, 2)$. Now we present some results of reconstruction obtained by MATLAB files on a PC. We compare the mentioned methods under different aspects.

5.2   Influence of relaxation parameters

We use the original signal given in $I = [0, 50]$ by

$$x^1(t) = \frac{1}{2}(1 - \frac{t}{50})(1 + \cos\frac{\pi t}{10})$$

and the starting signal

$$x^1_0(t) \equiv 1$$

in the extended interval $I_e = [0, 127]$. This original supplies also the prescribed phase $\phi_{x^1}(\omega) = \phi(\omega)$. Table 1 contains the relative errors after

50 steps of iteration and the corresponding initial values $x_{50}(0)$. Naturally the errors do not relate to $x_{50}(t)$ but to $x_{50}^*(t) = x^1(0)\,x_{50}(t)/x_{50}(0)$ with the same initial value as $x^1(t)$. The methods with parameter optimization are listed outside the tables since here the parameters are determined automatically. Here and in the following the smallest error for each method is underlined. The smallest error altogether is set off by bold print. The results show that high overrelaxation (both parameters in [1.5, 2]) leads for all methods to the best results. Although the author of [6] asserts that POCS supplies better results than HLO we cannot verify this statement. On the contrary, for our test signals HLO is always slightly better than POCS. POCS can be improved by appropriate range restriction (POCS-2) or by per-cycle optimization (oPCOS , pPCOS). Although method POCS-1 ensures the correct initial value the reconstruction quality is often bader than for method POCS-2. Hence, the choice of d for POCS-d requires some care. The parallel method works only with rather slow convergence. We assume that the angle between the linear hulls of M_I^J and M_ϕ is small. This would explain the positive effect of overrelaxation and the bad performance of parallel methods. The priority of HLO is possibly based on the small parameter $\alpha = 0$ of strongness. Other methods with greater α can compensate this disadvantage only by greater relaxation parameters. Table 2 shows that the initial values of the iterates increase monotonically with the relaxation parameters. Further the growth in the POCS variants is dampened with regard to HLO. This seems to be also a hint to the hidden relaxation potential of HLO.

Table 1

rounded relative errors after $k = 50$ steps

start $x_0 \equiv 1$	method				
parameters	HLO	POCS	POCS-2	POCS-1	PPCS-1/2
(0.5,0.5)	1.2E-1	1.7E-1	1.6E-1	3.7E-1	4.2E-1
(1,1)	5.9E-3	9.7E-3	8.4E-3	3.8E-2	2.4E-1
(1.5,1)	5.2E-4	1.1E-3	8.7E-4	7.3E-3	1.9E-1
(1,1.5)	6.1E-4	1.1E-3	8.7E-4	7.4E-3	1.9E-1
(1.5,1.5)	<u>1.3E-8</u>	7.7E-8	5.1E-8	1.5E-5	1.4E-1
(1.7,1.7)	1.3E-8	<u>1.5E-8</u>	**5.0E-9**	<u>1.0E-8</u>	1.1E-1
(2,2)	3.0E-1	2.7E-1	1.0E-1	1.2E-1	<u>7.8E-2</u>

oPCOS: 1.4E-4   ,   pPCOS: 1.4E-4

Table 2

initial values of $x_{50}(t)$

start $x_0 \equiv 1$	method				
parameters	HLO	POCS	POCS-2	POCS-1	PPCS-1/2
(0.5,0.5)	2.22	1.83	1.83	1	1.71
(1,1)	2.48	1.92	1.92	1	1.80
(1.5,1)	2.70	1.92	1.92	1	1.82
(1,1.5)	2.64	1.92	1.92	1	1.82
(1.5,1.5)	3.17	1.95	1.94	1	1.85
(1.7,1.7)	4.33	2.07	1.97	0.97	1.86
(2,2)	7.3E15	3.46	1.9174	1	1.88

oPCOS: 2.32   ,   pPCOS: 2.32

5.3 Influence of starting signals

Again we use the original signal $x(t) = x^1(t)$ and compare the starting signals:

$$s1: \ r_{x_0}(\omega) \equiv 1, \quad \phi_{x_0}(\omega) = \phi_x(\omega) = \phi(\omega)$$
$$s2: \ x_0(t) = \cos(t), \quad s3: \ x_0(t) = 1 - \sin(t/2)$$
$$s4: \ x_0(t) \equiv 1$$

Table 3 contains the relative errors after $k = 50$ steps for the standard parameter choice (1,1). It turns out that the starting signals are already arranged after decreasing quality. The explanation seems to be simple. The first already considers the given phase. The second and third contain trigonometric functions as the original. The last is quite indifferent and uses no additional information.

Table 3

rounded relative errors after $k = 50$ steps

parameters (1,1)	method			
starting signals	HLO	POCS	POCS-2	PPCS-1/2
s1	<u>2.6E-3</u>	<u>3.7E-3</u>	**6.9E-5**	<u>8.3E-2</u>
s2	3.7E-3	6.3E-3	6.9E-3	1.3E-1
s3	3.9E-3	9.8E-3	8.5E-3	2.3E-1
s4	5.9E-3	9.7E-3	8.4E-3	2.4E-1

The methods oPCOS and pPCOS show the same tendency.

5.4 Influence of signal type

We use the following original signals given in $I = [0, 50]$ by

$$x^1(t) = \frac{1}{2}\Big(1 - \frac{t}{50}\Big)\Big(1 + \cos \frac{\pi t}{10}\Big),$$

$$x^2(t) = \frac{1}{2}\Big(1 + \cos \frac{\pi t}{30}\Big), \ x^3(t) = \frac{1}{2}\Big(1 + \cos \frac{\pi t}{20}\Big), \ x^4(t) = \frac{1}{2}\Big(1 + \cos \frac{\pi t}{10}\Big),$$

$$x^5(t) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 25 \\ 0.5 & \text{if} \quad 25 < x \le 50 \end{cases},$$

$$x^6(t) = \Big|\cos \frac{t+6}{20} \sin \frac{t+6}{18}\Big|, \ x^7(t) = (t+5)\, e^{-t/20}.$$

We start with the signal $x_0(t) \equiv 1$ in $I_e = [0, 127]$ and work with the standard parameters (1,1). The relative errors after $k = 50$ steps are listed in Table 4. The reconstruction results are good up to satisfactory. The ranking of the methods does not essentially change for the considered signals. Generally one can expect that the reconstruction quality decreases for periodical functions if the period becomes smaller. Surprisingly HLO and POCS show a different behaviour in the case of the signals $x^2(t)$, $x^3(t)$ and $x^4(t)$. Here the best results occur for the middle period (see underlined values). Especially for the more problematic signals x^4 up to x^7 the parallel method shortens its disadvantage. The choice of d in POCS-d influences the results for smaller d.

Table 4

rounded relative errors after $k = 50$ steps

parameters (1,1)	method					
signal	HLO	POCS	POCS-2	POCS-1	PPCS-1/2	o/pPCOS
x^1	5.9E-3	9.7E-3	8.4E-3	3.8E-2	2.4E-1	1.4E-4
x^2	5.3E-3	7.7E-3	7.7E-	4.0E-2	3.0E-2	3.2E-3
x^3	<u>4.4E-4</u>	<u>6.5E-4</u>	<u>6.5E-4</u>	6.2E-3	8.6E-2	**1.1E-6**
x^4	2.7E-1	5.3E-1	5.3E-1	6.7E-1	9.8E-1	2.3E-1
x^5	7.9E-3	1.1E-2	1.1E-2	<u>2.5E-5</u>	<u>2.7E-2</u>	4.8E-3
x^6	6.6E-2	8.9E-2	8.9E-2	1.0E-1	1.5E-1	5.4E-2
x^7	8.9E-2	1.1E-1	1.1E-2	1.0E-2	1.9E-1	7.8E-2

We have the methods also tested with complex-valued signals of one variable and with real-valued signals of two variables. Apart from the greater computation amount we observed no essential difficulties. The excellent possibilities of MATLAB to graphic representation allow a vivid imagination of the approximation process and a detailed study of its local behaviour.

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