Non-deterministic linear hypersubstitutions

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Abstract

A non-deterministic hypersubstitution maps operation symbols to sets of terms of the corresponding arity. A non-deterministic hypersubstitution of type $\tau$ is said to be linear if it maps any operation symbol to a set of linear terms of the corresponding arity. We show that the extension of non-deterministic linear hypersubstitutions of type $\tau$ map sets of linear terms to sets of linear terms. As a consequence, the collection of all non-deterministic linear hypersubstitutions forms a monoid. Non-deterministic linear hypersubstitutions can be applied to identities and to algebras of type $\tau$.

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1. Introduction

In 2008, K. Denecke, P. Glubudom and J. Koppitz [3] studied non-deterministic hypersubstitutions and considered the extensions of such mappings. They also showed that the set of all non-deterministic hypersubstitutions forms a monoid under a certain binary operation.

The concept of linear terms has a long history as old as the concept of terms. In 2012, M. Couceiro and E. Lehtonen [2] gave a sufficient and necessary condition that a set of operations is the set of linear term operations of some algebra.

In this paper, we define non-deterministic linear hypersubstitutions and we show that the set of all non-deterministic linear hypersubstitutions forms a monoid. Let $n \geq 1$ be a natural number. Let $X_n = \{x_1, \ldots, x_n\}$ be an $n$-element set. The set $X_n$ is called an alphabet and its elements are called variables. Let
\{f_i : i \in I\} be the set of operation symbols, indexed by the set \(I\). The sets \(X_n\) and \(\{f_i : i \in I\}\) have to be disjoint. To every operation symbol \(f_i\), we assign a natural number \(n_i \geq 1\), called the arity of \(f_i\). As in the definition of algebra, the sequence \(\tau = (n_i)_{i \in I}\) of all the arities is called the type. With this notation for operation symbols and variables, we can define the terms of type \(\tau\), (see also [5]).

The \(n\)-ary terms of type \(\tau\) are defined in the following inductive way:

(i) Every variable \(x_i \in X_n\) is an \(n\)-ary term.

(ii) If \(t_1, \ldots, t_{n_i}\) are \(n\)-ary terms and \(f_i\) is an \(n_i\)-ary operation symbol, then \(f_i(t_1, \ldots, t_{n_i})\) is an \(n\)-ary term.

(iii) The set \(W_\tau(X_n) = W_\tau(x_1, \ldots, x_n)\) of all \(n\)-ary terms is the smallest set which contains \(x_1, \ldots, x_n\) and is closed under finite application of (ii).

We denote by \(W_\tau(X)\) the set of all terms of type \(\tau\) over the countably infinite alphabet \(X\), that is,

\[
W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n).
\]

Let \(t\) be a term. We denote the set of variables occurring in the term \(t\) by \(\text{var}(t)\).

A term in which each variable occurs at most once, is said to be linear. For a formal definition of \(n\)-ary linear terms we replace condition (ii) in the definition of terms by a slightly different condition.

**Definition** [2]. An \(n\)-ary linear term of type \(\tau\) is defined in the following inductive way:

(i) For any \(j \in \{1, \ldots, n\}\), \(x_j \in X_n\) is an \(n\)-ary linear term (of type \(\tau\)).

(ii) If \(t_1, \ldots, t_{n_i}\) are \(n\)-ary linear terms and if \(\text{var}(t_j) \cap \text{var}(t_k) = \emptyset\) for all \(1 \leq j < k \leq n_i\), then \(f_i(t_1, \ldots, t_{n_i})\) is an \(n\)-ary linear term.

(iii) The set \(W_\tau^\text{lin}(X_n)\) of all \(n\)-ary linear terms is the smallest set which contains \(x_1, \ldots, x_n\) and is closed under finite application of (ii).

The set of all linear terms of type \(\tau\) over the countably infinite alphabet \(X\) is defined by

\[
W_\tau^\text{lin}(X) := \bigcup_{n \geq 1} W_\tau^\text{lin}(X_n).
\]

The set \(W_\tau(X)\) of all terms of type \(\tau\) is closed under substitution. This is not true for linear terms as the following example shows: Let \(\tau = (2)\) and let \(f\) be a binary operation symbol. Then \(f(x_1, x_2)\) and \(f(x_2, x_1)\) are linear, but if we substitute
f(x_1, x_2) for x_1 and f(x_2, x_1) for x_2 in f(x_1, x_2), we obtain f(f(x_1, x_2), f(x_2, x_1)), which is not a linear.

One of the most interesting operations on terms is the superposition. Let \( W_\tau(X_n) \) and \( W_\tau(X_m) \) be the set of all \( n \)-ary and \( m \)-ary terms, respectively. Then the superposition

\[
S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)
\]
is defined inductively as follows:

(i) \( S^n_m(x_j, t_1, \ldots, t_n) := t_j, x_j \in X_n \) and \( t_i \in W_\tau(X_m) \).

(ii) \( S^n_m(f_i(s_1, \ldots, s_{m_i}), t_1, \ldots, t_{n_i}) := f_i(S^n_m(s_1, t_1, \ldots, t_{n_i}), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_{n_i})) \).

We can extend the superposition operation \( S^n_m \) to sets of terms by the following:

Let \( m, n \) be natural numbers. We define

\[
\hat{S}^n_m : \mathcal{P}(W_\tau(X_n)) \times (\mathcal{P}(W_\tau(X_m)))^n \to \mathcal{P}(W_\tau(X_m))
\]
inductively as follows. Let \( B \in \mathcal{P}(W_\tau(X_n)), B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X_m)) \).

(i) If \( B = \{x_j\} \) for \( 1 \leq j \leq n \), then \( \hat{S}^n_m(\{x_j\}, B_1, \ldots, B_n) := B_j \).

(ii) If \( B = \{f_i(t_1, \ldots, t_{n_i})\} \) and if we suppose that the sets \( \hat{S}^n_m(\{t_j\}, B_1, \ldots, B_n) \) for \( 1 \leq j \leq n_i \) are already defined, then

\[
\hat{S}^n_m(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n) := \{f_i(r_1, \ldots, r_{n_i}) : r_j \in \hat{S}^n_m(\{t_j\}, B_1, \ldots, B_n), 1 \leq j \leq n_i\}.
\]

(iii) If \( B \) is an arbitrary non-empty subset of \( W_\tau(X_n) \), we define

\[
\hat{S}^n_m(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \hat{S}^n_m(\{b\}, B_1, \ldots, B_n).
\]

If one of the sets \( B, B_1, \ldots, B_n \) is empty, we define \( \hat{S}^n_m(B, B_1, \ldots, B_n) = \emptyset \).

Let \( \tau = (\tau_i)_{i \in I} \) be a type and let \( (f_i)_{i \in I} \) be an indexed set of operation symbols of type \( \tau \). Any mapping

\[
\sigma : \{f_i : i \in I\} \to \mathcal{P}(W_\tau(X))
\]
with \( \sigma(f_i) \subseteq W_\tau(X_n) \) for \( i \in I \) is called a non-deterministic hypersubstitution of type \( \tau \) [3]. For short we write non-deterministic hypersubstitution as nd-hypersubstitution. Every nd-hypersubstitution \( \sigma \) of type \( \tau \) induces a mapping \( \hat{\sigma} : \mathcal{P}(W_\tau(X)) \to \mathcal{P}(W_\tau(X)) \) by the following inductive definition [3]:

(i) \( \hat{\sigma}[\emptyset] := \emptyset \),
(ii) $\hat{\sigma}[\{x\}] := \{x\}$ for every variable $x \in X$.

(iii) For $t = f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X)$ we set

$$\hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}] := \hat{\sigma}^n_m(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}])$$

if we inductively assume that $\hat{\sigma}[\{t_j\}], 1 \leq j \leq n_i$ are already defined. Here $n_i$ is the arity of $f_i$.

(iv) $\hat{\sigma}[B] := \bigcup \{\hat{\sigma}[\{t\}] : t \in B \subseteq W_\tau(X)\}$.

We denote by $Hyp_{nd}(\tau)$ the set of all non-deterministic hypersubstitutions of type $\tau$.

In [3], the authors used the mapping $\hat{\sigma}$ for a nd-hypersubstitution $\sigma$ on the set $Hyp_{nd}(\tau)$ to define a binary operation

$$\circ_{nd} : Hyp_{nd}(\tau) \times Hyp_{nd}(\tau) \to Hyp_{nd}(\tau)$$

by $\sigma_1 \circ_{nd} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Hyp_{nd}(\tau)$. The nd-hypersubstitution $\sigma_{id}$ with $\sigma_{id}(f_i) := \{f_i(x_1, \ldots, x_{n_i})\}$, for all $i \in I$, is an identity element. They have shown that the algebra $(Hyp_{nd}(\tau); \circ_{nd}, \sigma_{id})$ is a monoid.

2. Non-deterministic linear hypersubstitutions

Non-deterministic linear hypersubstitution (for short, nd-linear hypersubstitution) map operation symbols to sets of linear terms of the corresponding arity.

Formally, we define nd-linear hypersubstitutions in the following way:

**Definition.** A non-deterministic linear hypersubstitution of type $\tau$ is a mapping

$$\sigma : \{f_i \mid i \in I\} \to \mathcal{P}(W_\tau^{lin}(X))$$

with $\sigma(f_i) \subseteq W_\tau^{lin}(X_{n_i})$ for $i \in I$.

We denote $Hyp_{nd}^{lin}(\tau)$ by the set of all non-deterministic linear hypersubstitutions.

For the extension of an nd-linear hypersubstitution $\sigma$ the following holds:

**Lemma 1** [1]. For any linear hypersubstitution $\sigma$ and any linear term $t$ we have

$$\text{var}(t) \supseteq \text{var}(\hat{\sigma}[t]).$$

**Lemma 2.** For any nd-linear hypersubstitution $\sigma$ and any set of linear terms $T$ we have

$$\text{var}(T) \supseteq \text{var}(\hat{\sigma}[T]).$$
Proof. If $T$ is a one-element set, then we will give a proof by induction on the complexity of the linear term which forms the only element of the one-element set $T$.

1. If $T = \{x_j\}$, where $x_j \in X$, then

$$\text{var}(T) = \text{var}(\{x_j\}) = \text{var}(\hat{\sigma}[\{x_j\}]) = \text{var}(\hat{\sigma}[T]).$$

2. If $T = \{f_i(t_1, \ldots, t_{n_i})\}$ and we assume that

$$\text{var}(\{t_j\}) \supseteq \text{var}(\hat{\sigma}[\{t_j\}]),$$

for all $1 \leq j \leq n_i$, then

$$\text{var}(T) = \text{var}(\{f_i(t_1, \ldots, t_{n_i})\}) = \bigcup_{j=1}^{n_i} \text{var}(\{t_j\}) \supseteq \bigcup_{j=1}^{n_i} \text{var}(\hat{\sigma}[\{t_j\}]) \supseteq \text{var}(\hat{\sigma}^n(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}]))) = \text{var}(\hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}]) = \text{var}(\hat{\sigma}[T]).$$

3. If $T$ is an arbitrary non-empty subset of $W_{\tau}^{\text{lin}}(X)$, then

$$\text{var}(T) = \bigcup_{t \in T} \text{var}(\{t\}) \supseteq \bigcup_{t \in T} \text{var}(\hat{\sigma}[\{t\}]) = \text{var}(\bigcup_{t \in T} \hat{\sigma}[\{t\}]) = \text{var}(\hat{\sigma}[T]).$$

4. If $T$ is the empty set, then $\emptyset = \text{var}(T) = \text{var}(\hat{\sigma}[\emptyset]) = \text{var}(\emptyset) = \emptyset$.

Therefore we have $\text{var}(T) \supseteq \text{var}(\hat{\sigma}[T])$. □
Lemma 3. For a set of linear terms of the form $T = \{f_i(t_1, \ldots, t_{n_i})\}$ and an nd-linear hypersubstitution $\sigma$ we have

$$\text{var}(\hat{\sigma}[\{t_j\}]) \cap \text{var}(\hat{\sigma}[\{t_k\}]) = \emptyset$$

for all $1 \leq j < k \leq n_i$.

Proof. By the previous lemma we have $\text{var}(\{t_l\}) \supseteq \text{var}(\hat{\sigma}[\{t_l\}])$ for any $1 \leq l \leq n_i$ and thus

$$\emptyset = \text{var}(\{t_j\}) \cap \text{var}(\{t_k\}) \supseteq \text{var}(\hat{\sigma}[\{t_j\}]) \cap \text{var}(\hat{\sigma}[\{t_k\}]).$$

Therefore $\text{var}(\hat{\sigma}[\{t_j\}]) \cap \text{var}(\hat{\sigma}[\{t_k\}]) = \emptyset$. 

Proposition 4. The extension of any nd-linear hypersubstitution maps non-empty sets of linear terms to non-empty sets of linear terms.

Proof. Let $T$ be an element in $P(W_\text{lin}^\tau(X))$ and let $\sigma \in Hyp_{\text{lin}}^n(\tau)$.

1. If $T$ is a one-element set, then we will give a proof by induction on the complexity of the linear term which forms the only element of the one-element set $T$.

   (a) If $T = \{x_j\}$, where $x_j \in X$, then

   $$\hat{\sigma}[T] = \hat{\sigma}[\{x_j\}] = \{x_j\},$$

   is a set of linear terms.

   (b) If $T = \{f_i(t_1, \ldots, t_{n_i})\}$, by the previous lemma we have $\text{var}(\hat{\sigma}[\{t_j\}]) \cap \text{var}(\hat{\sigma}[\{t_k\}]) = \emptyset$ for all $1 \leq j < k \leq n_i$, and if we assume that $\hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}]$ are sets of linear terms, then

   $$\hat{\sigma}[T] = \hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}] = \hat{\mathcal{S}}^\tau_n(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}]),$$

   is a set of linear terms.

2. If $T$ is an arbitrary non-empty subset of $W_\text{lin}^\tau(X)$, then $\hat{\sigma}[T] = \bigcup_{t \in T} \hat{\sigma}[\{t\}]$ is a non-empty set of linear terms.

Thus, the extension of an nd-linear hypersubstitution maps non-empty sets of linear terms to non-empty sets of linear terms.
Since the extension of an nd-linear hypersubstitution of type $\tau$ maps $P(W^\text{lin}_\tau(X))$ to $P(W^\text{lin}_\tau(X))$ we may define a product $\sigma_1 \circ_{nd} \sigma_2$, by

$$\sigma_1 \circ_{nd} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2.$$  

Here $\circ$ is the usual composition of mappings. By the previous lemma $(\sigma_1 \circ_{nd} \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)]$ is a set of linear terms.

From the above facts we obtain the following theorem.

**Theorem 5.** The set of all nd-linear hypersubstitutions is a submonoid of the set of all nd-hypersubstitution. That is, $(\text{Hyp}_{nd}^{\text{lin}}(\tau), \circ_{nd}, \sigma_{id})$ is a submonoid of the monoid $(\text{Hyp}_{nd}^{nd}(\tau), \circ_{nd}, \sigma_{id})$.

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**References**


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