

## ENUMERATION OF $\Gamma$ -GROUPS OF FINITE ORDER

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### Abstract

The concept of  $\Gamma$ -semigroups is a generalization of semigroups. In this paper, we consider  $\Gamma$ -groups and prove that every  $\Gamma$ -group is derived from a group then, we give the number of  $\Gamma$ -groups of small order.

**Keywords:**  $\Gamma$ -semigroup,  $\Gamma$ -group.

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### 1. INTRODUCTION

The concept of  $\Gamma$ -semigroups was introduced by Sen in [14] and [15] that is a generalization of a semigroups. Many classical notions of semigroups have been extended to  $\Gamma$ -semigroups (see, for example, [6, 10, 13, 16] and [17]). Dutta and Adhikari have found operator semigroups of a  $\Gamma$ -semigroup to be a very effective tool in studying  $\Gamma$ -semigroups [5]. Recently, Davvaz *et al.* introduced the notion of  $\Gamma$ -semihypergroups as a generalization of semigroups, a generalization of semihypergroups and a generalization of  $\Gamma$ -semigroups [2, 8, 9].

The determination of all groups of a given order up to isomorphism is a very old question in group theory. It was introduced by Cayley who constructed the groups of order 4 and 6 in 1854, see [4]. In this paper, we prove that a  $\Gamma$ -group is derived from a group. Also, we give the number of  $\Gamma$ -groups of small order.

## 2. PRELIMINARIES

We begin this section by the definition of a  $\Gamma$ -semigroup.

**Definition** [14]. Let  $S$  and  $\Gamma$  be nonempty sets. Then  $S$  is called a  $\Gamma$ -semigroup if there exists a mapping  $S \times \Gamma \times S \rightarrow S$ , written  $(a, \gamma, b)$  by  $a\gamma b$ , such that satisfies the identities  $(a\alpha b)\beta c = a\alpha(b\beta c)$ , for all  $a, b, c \in S$  and  $\alpha, \beta \in \Gamma$ .

Let  $S$  be a  $\Gamma$ -semigroup and  $\alpha$  be a fixed element in  $\Gamma$ . We define  $a.b = a\alpha b$ , for all  $a, b \in S$ . It is easy to check that  $(S, .)$  is a semigroup and we denote this semigroup by  $S_\alpha$ .

Let  $A$  and  $B$  be subsets of a  $\Gamma$ -semigroup  $S$  and  $\Delta \subseteq \Gamma$ . Then  $A\Delta B$  is defined as follows

$$A\Delta B = \{a\delta b \mid a \in A, b \in B, \delta \in \Delta\}.$$

For simplicity we write  $a\Delta B$  and  $A\Delta b$  instead of  $\{a\}\Delta B$  and  $A\Delta\{b\}$ , respectively. Also, we write  $A\delta B$  in place of  $A\{\delta\}B$ .

Let  $S$  be an arbitrary semigroup and  $\Gamma$  any nonempty set. Define a mapping  $S \times \Gamma \times S \rightarrow S$  by  $a\alpha b = ab$ , for all  $a, b \in S$  and  $\alpha \in \Gamma$ . It is easy to see that  $S$  is a  $\Gamma$ -semigroup. Thus a semigroup can be considered to be a  $\Gamma$ -semigroup.

In the following some examples of  $\Gamma$ -semigroups are presented.

**Example 1.** Let  $S = \{i, 0, -i\}$  and  $\Gamma = S$ . Then  $S$  is a  $\Gamma$ -semigroup under the multiplication over complex number while  $S$  is not a semigroup under complex number multiplication.

**Example 2.** Let  $S$  be the set of all  $m \times n$  matrices with entries from a field  $F$  and  $\Gamma$  be a set of  $n \times m$  matrices with entries from  $F$ . Then  $S$  is a  $\Gamma$ -semigroup with the usual product of matrices.

**Example 3.** Let  $(S, \leq)$  be a totally ordered set and  $\Gamma$  be a nonempty subset of  $S$ . We define

$$x\gamma y = \max\{x, \gamma, y\},$$

for every  $x, y \in S$  and  $\gamma \in \Gamma$ . Then  $S$  is a  $\Gamma$ -semigroup.

**Example 4.** Let  $S = [0, 1]$  and  $\Gamma = \mathbb{N}$ . For every  $x, y \in S$  and  $\gamma \in \Gamma$  we define  $x\gamma y = \frac{xy}{\gamma}$ . Then, for every  $x, y, z \in S$  and  $\alpha, \beta \in \Gamma$ , we have

$$(x\alpha y)\beta z = \frac{xyz}{\alpha\beta} = x\alpha(y\beta z).$$

This means that  $S$  is a  $\Gamma$ -semigroup.

A nonempty subset  $T$  of a  $\Gamma$ -semigroup  $S$  is said to be a  $\Gamma$ -subsemigroup of  $S$  if  $TTT \subseteq T$ .

**Definition.** A nonempty subset  $I$  of  $\Gamma$ -semigroup  $S$  is called a left (right)  $\Gamma$ -closed subset if  $STI \subseteq I$  ( $ITS \subseteq I$ ). A  $\Gamma$ -semigroup  $S$  is called a left (right) simple  $\Gamma$ -semigroup if it has no proper left (right)  $\Gamma$ -closed subset. Also,  $S$  is called a simple  $\Gamma$ -semigroup if it has no proper  $\Gamma$ -closed subset both left and right.

### 3. ENUMERATION OF $\Gamma$ -GROUPS OF FINITE ORDER

**Definition.** A  $\Gamma$ -semigroup  $S$  is called a  $\Gamma$ -group if  $S_\alpha$  is a group, for every  $\alpha \in \Gamma$ .

**Example 5.** Let  $S = \{a, b, c, d, e, f\}$  and  $\Gamma = \{\alpha, \beta\}$ . Define the operations  $\alpha$  and  $\beta$  as the following tables

$\alpha$	$a$	$b$	$c$	$d$	$e$	$f$
$a$	$b$	$c$	$d$	$e$	$f$	$a$
$b$	$c$	$d$	$e$	$f$	$a$	$b$
$c$	$d$	$e$	$f$	$a$	$b$	$c$
$d$	$e$	$f$	$a$	$b$	$c$	$d$
$e$	$f$	$a$	$b$	$c$	$d$	$e$
$f$	$a$	$b$	$c$	$d$	$e$	$f$

$\beta$	$a$	$b$	$c$	$d$	$e$	$f$
$a$	$c$	$d$	$e$	$f$	$a$	$b$
$b$	$d$	$e$	$f$	$a$	$b$	$c$
$c$	$e$	$f$	$a$	$b$	$c$	$d$
$d$	$f$	$a$	$b$	$c$	$d$	$e$
$e$	$a$	$b$	$c$	$d$	$e$	$f$
$f$	$b$	$c$	$d$	$e$	$f$	$a$

Then  $S$  is a  $\Gamma$ -group. One can see that  $f$  and  $e$  are the neutral elements of  $S_\alpha$  and  $S_\beta$ , respectively.

**Theorem 6.** Let  $S$  be a  $\Gamma$ -semigroup. Then  $S$  is a simple  $\Gamma$ -semigroup if and only if  $S_\alpha$  is a group, for every  $\alpha \in \Gamma$ .

**Proof.** Let  $S$  be a simple  $\Gamma$ -semigroup and  $\alpha \in \Gamma$ , we show that  $S_\alpha$  is a group. Let  $I = a\alpha S$ , where  $a \in S$ . Then,  $I$  is a right  $\Gamma$ -closed subset of  $S$ , indeed

$$ITS = (a\alpha S)\Gamma S \subseteq a\alpha S = I.$$

Since  $S$  has no proper right  $\Gamma$ -closed subset, we have  $I = a\alpha S = S$ . Similarly, we can prove that  $S\alpha a = S$ . Therefore,  $S_\alpha$  is a group.

Conversely, let  $I \neq \emptyset$  be a left  $\Gamma$ -closed subset of  $S$ ,  $s \in S$  and  $a \in I$ . Since  $S_\alpha$  is a group, there exists  $t \in S$  such that  $s = t\alpha a \subseteq S\alpha I \subseteq I$ . So  $S = I$ . Similarly, we can prove that  $S$  has no proper right  $\Gamma$ -closed subset. Therefore,  $S$  is simple. ■

**Corollary 7.** *Let  $S$  be a  $\Gamma$ -semigroup. If  $S_\alpha$  is a group, for some  $\alpha \in \Gamma$ , then  $S_\beta$  is a group, for every  $\beta \in \Gamma$ .*

**Proof.** Since  $S_\alpha$  is a group, previous theorem implies that  $S$  is a simple  $\Gamma$ -group. Thus, for every  $\beta \in \Gamma$ ,  $S_\beta$  is a group. ■

**Corollary 8.** *Let  $S$  be a  $\Gamma$ -semigroup. If  $S_\alpha$  is a group, for some  $\alpha \in \Gamma$ , then  $S$  is a  $\Gamma$ -group.*

**Proof.** By Corollary 7, it is trivial. ■

**Theorem 9.** *Let  $S$  be a  $\Gamma$ -group and  $\alpha, \beta \in \Gamma$ . Then there exists  $b \in S$  such that  $x\beta y = xab\alpha y$ , for every  $x, y \in S$ .*

**Proof.** It is sufficient to put  $b = e_\alpha \beta e_\alpha$ , where  $e_\alpha$  is the neutral element of  $S_\alpha$ . Then, for every  $x, y \in S$ , we have

$$\begin{aligned} x\beta y &= (x\alpha e_\alpha)\beta(e_\alpha\alpha y) \\ &= x\alpha(e_\alpha\beta e_\alpha)\alpha y \\ &= xab\alpha y. \end{aligned}$$

■

By the previous theorem, we conclude that every  $\Gamma$ -group is derived from a group. Therefore, if  $S$  is a  $\Gamma$ -group, then we can consider  $(S, \cdot)$  as a group and  $\Gamma \subseteq S$ , so  $x\alpha y$  is a product in  $(S, \cdot)$ , for every  $x, y \in S$  and  $\alpha \in \Gamma$ . Also, Theorem 9 implies that the groups  $S_\alpha$  and  $S_\beta$  are isomorphic, for every  $\alpha, \beta \in \Gamma$ .

**Definition.** Let  $S$  be a  $\Gamma$ -group and  $S'$  be a  $\Gamma'$ -group. If there exist mappings  $\varphi_\gamma : S \rightarrow S'$ , for every  $\gamma \in \Gamma$ , and  $f : \Gamma \rightarrow \Gamma'$  such that

$$\varphi_\gamma(xy) = \varphi_\gamma(x)f(\gamma)\varphi_\gamma(y),$$

for all  $x, y \in S$ , then we say  $(\{\varphi_\gamma\}_{\gamma \in \Gamma}, f)$  is a homomorphism between  $S$  and  $S'$ . Also, if  $f$  and  $\varphi_\gamma$ , for every  $\gamma \in \Gamma$ , are bijections, then  $(\{\varphi_\gamma\}_{\gamma \in \Gamma}, f)$  is called an isomorphism, and  $S$  and  $S'$  are called isomorphic.

**Lemma 10.** *Let  $S$  be a  $\Gamma$ -group and  $S'$  be a  $\Gamma'$ -group. Then  $S$  and  $S'$  are isomorphic if and only if  $S$  and  $S'$  are isomorphic group and  $|\Gamma| = |\Gamma'|$ .*

**Proof.** If  $S$  and  $S'$  are isomorphic, then by the previous definition, for every  $\alpha \in \Gamma$ , the groups  $S_\alpha$  and  $S'_{\alpha'}$  are isomorphic where  $f : S \rightarrow S'$  is a bijection and  $f(\alpha) = \alpha'$ . ■

**Theorem 11.** *The number of  $\Gamma$ -groups of order  $n$  is  $nk$ , up to isomorphism, where  $k$  is the number of isomorphism classes of groups of order  $n$ .*

**Proof.** Suppose that  $(S, \cdot)$  is a group and  $\Gamma$  and  $\Gamma'$  be two subsets of  $S$  such that  $|\Gamma| = |\Gamma'|$ . Then by previous lemma, there exists only one  $\Gamma$ -group derived from  $(S, \cdot)$ , up to isomorphism. So, for every  $m \leq n$  there exists only one  $\Gamma$ -group, where  $\Gamma$  is a subset of  $S$  such that  $|\Gamma| = m$ . Thus, the number of  $\Gamma$ -groups derived from  $(S, \cdot)$  is  $n$ , up to isomorphism. Therefore, if there exist  $k$  groups of order  $n$ , then we have  $nk$   $\Gamma$ -groups of order  $n$ , up to isomorphism. ■

**Corollary 12.** *Suppose that  $n > 1$  is an integer with decomposition into primes as  $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ . If  $n$  is prime to*

$$\prod_{j=1}^r (p_j^{e_j} - 1)$$

*and  $e_j \leq 2$ , then the number of  $\Gamma$ -groups of order  $n$  is  $n2^m$ , where  $m$  is the number of  $j$ 's with  $e_j = 2$ .*

**Proof.** By a result of Rédei [12], all such groups of order  $n$  are abelian. Thus, the number of isomorphism types of abelian groups of order  $n$  is given by

$$\prod_{j=1}^r p(e_j) = 2^m,$$

where  $p(e_j)$  is the number of partitions of  $e_j \leq 2$  and  $p(1) = 1, p(2) = 2$ . The proof is completed by applying Theorem 11. ■

The case  $m = 0$  of the Corollary 12 was studied by Szele [18]. In connection with this, Erdős [7] showed that the number of  $n \leq x$  such that  $(n, \varphi(n)) = 1$  ( $\varphi(n)$  is Euler's phi function) is asymptotic to

$$\frac{e^{-\gamma} x}{\log \log \log x}$$

where  $\gamma$  is Euler's constant. For additional results on the asymptotic of  $n \leq x$  satisfying Rédei's condition and asymptotic enumeration of finite abelian groups see [1, 11, 19].

In the following table we give the number of  $\Gamma$ -groups of order less than 30.

<i>Order</i>	<i>Number of <math>\Gamma</math> – groups</i>	<i>Order</i>	<i>Number of <math>\Gamma</math> – groups</i>
1	1	16	224
2	2	17	17
3	3	18	90
4	8	19	19
5	5	20	100
6	12	21	42
7	7	22	44
8	40	23	23
9	18	24	360
10	20	25	50
11	11	26	52
12	60	27	135
13	13	28	112
14	28	29	29
15	15	30	120

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