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SOME PROPERTIES OF THE ZERO DIVISOR GRAPH OF A COMMUTATIVE RING

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Abstract

Let $\Gamma(R)$ be the zero divisor graph for a commutative ring with identity. The k-domination number and the 2-packing number of $\Gamma(R)$, where R is an Artinian ring, are computed. k-dominating sets and 2-packing sets for the zero divisor graph of the ring of Gaussian integers modulo n, $\Gamma(\mathbb{Z}_n[i])$, are constructed. The center, the median, the core, as well as the automorphism group of $\Gamma(\mathbb{Z}_n[i])$ are determined. Perfect zero divisor graphs $\Gamma(R)$ are investigated.

Keywords: automorphism group of a graph, center of a graph, core of a graph, k-domination number, Gaussian integers modulo n, median of a graph, 2-packing, perfect graph, and zero divisor graph.

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1. Introduction

The zero divisor graph of a commutative ring R was first introduced by Beck [11] whose primary interest was coloring a graph with vertex set R, and two vertices are adjacent if their product is zero. Anderson and Livingston [8] restricted the vertex set to $Z^*(R)$, the set of nonzero zero divisors of R, this graph is denoted by $\Gamma(R)$. Since then, $\Gamma(R)$, as well as other types of graphs associated with rings, have been extensively studied by many authors. For a survey of zero divisor graphs, the reader may refer to [5]. Abu Osba *et al.* [2] introduced the zero divisor graph for the ring of Gaussian integers modulo n, in which the number of vertices, the diameter, and the girth are found, as well as a complete characterization of the n for which $\Gamma(\mathbb{Z}_n[i])$ is Eulerian or planar. Further properties of $\Gamma(\mathbb{Z}_n[i])$ are investigated in [3]. The complement of $\Gamma(\mathbb{Z}_n[i])$ is studied in [1]. On the other hand, Nazzal and Ghanem investigated the line graph of $\Gamma(\mathbb{Z}_n[i])$ and $\overline{\Gamma(\mathbb{Z}_n[i])}$, extensively in [25] and [18] respectively. A formula for the degree of each vertex in $\Gamma(\mathbb{Z}_n[i])$ is given in [18].

The set of Gaussian integers, denoted by, $\mathbb{Z}[i]$, is defined by $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z} \text{ and } i = \sqrt{-1}\}$. Clearly, $\mathbb{Z}[i]$ is a ring under the usual complex operations. A Gaussian prime integer is a unit multiple of one of the following: $1+i$, or a prime integer q in $\mathbb{Z}$ which is congruent to 3 (mod 4), or $a+bi$, $a-bi$, where $a^2+b^2 = p$ and p is a prime integer in $\mathbb{Z}$ which is congruent to 1 (mod 4). Throughout this paper, p and p_j denote prime integers which are congruent to 1 modulo 4, while q and q_j denote prime integers which are congruent to 3 modulo 4. Let $\langle n\rangle$ be the principal ideal generated by n in $\mathbb{Z}[i]$, where n is a natural number greater than 1, and let $\mathbb{Z}_n = \{0, 1, 2, 3, 4, \ldots, n-1\}$ be the ring of integers modulo n. The factor ring $\mathbb{Z}[i]/\langle n\rangle$ is isomorphic to $\mathbb{Z}_n[i] = \{a + bi : a, b \in \mathbb{Z}_n\}$. Obviously, $\mathbb{Z}_n[i]$ with addition and multiplication modulo n is a ring. This ring is called the ring of Gaussian integers modulo n. The zero divisor graph of a commutative ring R, denoted by $\Gamma(R)$, is the graph whose vertex set is the set of all nonzero zero divisors of R, denoted by $Z^*(R)$, and edge set $E(\Gamma(R)) = \{xy : x, y \in Z^*(R) \text{ and } xy = 0\}$. To avoid triviality throughout this paper, $\mathbb{Z}_n[i]$ will be different from $\mathbb{Z}_2[i]$ or $\mathbb{Z}_q[i]$, since $\Gamma(\mathbb{Z}_2[i])$ is K_1 and $\Gamma(\mathbb{Z}_q[i])$ is K_0.

For a connected graph G, the distance $d(u, v)$ between two vertices u and v is the minimum of the lengths of all $u-v$ paths of G. The eccentricity of a vertex v in G is the maximum distance from v to any vertex in G. The radius of G, $\operatorname{rad}(G)$, is the minimum eccentricity among the vertices of G. The open neighborhood of a vertex x in G is the set $N(x) = \{y : xy \in E(G)\}$ while the closed neighborhood of a vertex x in G is the set $N[x] = N(x) \cup \{x\}$. The minimum degree of G denoted by $\delta(G)$ is defined by $\min\{\deg(x) : x \in V(G)\}$.

In this article, we present our work as follows: in Section two, a complete characterization for the center, the median, and the core of $\Gamma(\mathbb{Z}_n[i])$ is given. In Section three, the k-domination and 2-packing numbers for $\Gamma(R)$, where R is a commutative Artinian ring with unity, are determined. In particular, k-dominating sets and 2-packing sets for $\Gamma(\mathbb{Z}_n[i])$ are constructed. In section four, the automorphism group of $\Gamma(\mathbb{Z}_n[i])$ is studied. Finally, in Section five, perfect zero divisor graphs are investigated. In particular perfect $\Gamma(\mathbb{Z}_n)$ and perfect $\Gamma(\mathbb{Z}_n[i])$ are studied.

2. The center, the median and the core of $\Gamma(\mathbb{Z}_n[i])$

The center of G is the set of all vertices of G with minimum eccentricity. For any vertex x of a connected graph G, the status of x, denoted by $s(x)$, is the sum of the distances from x to the other vertices of G. The set of vertices with minimal status is called the median of the graph. The center of $\Gamma(\mathbb{Z}_n[i])$ when n is a power of a prime is studied in [25], where it was shown that the center of $\Gamma(\mathbb{Z}_{2^m}[i])$ consists of one vertex, namely $\{(1+i)2^{m-1}\}$ and the center of $\Gamma(\mathbb{Z}_{q^m}[i])$ is the set $\{aq^{m-1}+bq^{m-1}i : a,b \in U(\mathbb{Z}_q)\}-\{0\}$. Note that $\mathbb{Z}_{p^m}[i] \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$, the center of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m})$ is the set $\{(x,y) : x,y \in Z(\mathbb{Z}_{p^m})\}-\{(0,0)\}$. Here we will find the center for the general case. Let n and m be positive integers. Let $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$ where each R_j is a commutative Artinian local ring with unity that is not a field and each F_j is a field. For each $j = 1,2,\ldots,m$, define the ideal $I_j = \{0\} \times \{0\} \times \cdots \times F_j \times \{0\} \times \cdots \times \{0\}$. Then the center of $\Gamma(R)$ is the set $C = J(R) \cup (\bigcup_{j=1}^{m} I_j) - \{(0,0,\ldots,0)\}$, where $J(R)$ is the Jacobson radical of R, [26].

If M_j is the maximal ideal of R_j, then $J(R) = M_1 \times M_2 \times \cdots \times M_n \times \{0\} \times \{0\} \times \cdots \times \{0\}$. Now, let us study the maximal ideals of the factors in the Artinian decomposition of $\mathbb{Z}_n[i]$ where $n = 2^m \prod_{j=1}^{r} p_j^{r_j} \prod_{j=1}^{s} q_j^{s_j} \prod_{j=1}^{t} q_j$ with $s_j \geq 2$. The maximal ideal in $\mathbb{Z}_{2^m}[i]$ is $\langle 1+i \rangle$. For $\mathbb{Z}_{q^m}[i]$, the maximal ideal is $\langle q \rangle$, while $\mathbb{Z}_{p^m}[i] \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$ and the maximal ideal in $\mathbb{Z}_{p^m}$ is $\langle p \rangle$. Thus each $p_j^{r_j}$ in the decomposition of n gives rise to two factors in the Artinian decomposition of $\mathbb{Z}_n[i]$. If the Artinian decomposition of $\mathbb{Z}_n[i] = R_1 \times R_2 \times \cdots \times R_l \times F_1 \times F_2 \times \cdots \times F_t$, then $R_1 = \mathbb{Z}_{2^m}[i]$ if n is even. Otherwise, R_j is either of the form $\mathbb{Z}_{q^m}[i]$ or $\mathbb{Z}_{p^m}$ and $F_j = \mathbb{Z}_q[i]$ for some q. Thus if l is the number of local rings in the Artinian decomposition of $\mathbb{Z}_n[i]$, then $J(R) = \{(z_1,z_2,\ldots,z_l,0,0,\ldots,0) : z_j \in Z(R_j)\}$. Thus, the following theorem is obtained.

Theorem 2.1. *If the Artinian decomposition of* $\mathbb{Z}_n[i] = R_1 \times R_2 \times \cdots \times R_l \times F_1 \times F_2 \times \cdots \times F_t$*, where n is divisible by at least two distinct primes, then the center of* $\Gamma(\mathbb{Z}_n[i])$ *is given by* $C = \{(z_1,z_2,\ldots,z_l,0,\ldots,0) : z_j \in Z(R_j)\} \cup (\bigcup_{j=1}^{t} I_j) - \{(0,0,\ldots,0)\}$*, where* $I_j = \{0\} \times \{0\} \times \cdots \times \mathbb{Z}_{q_j}[i] \times \{0\} \times \cdots \times \{0\}$.

If $n = p$ or $n = q_1q_2$, then $\Gamma(\mathbb{Z}_n[i])$ is complete bipartite and $ecc(v) = 2$ for each vertex in $\Gamma(\mathbb{Z}_n[i])$. Hence, the center of $\Gamma(\mathbb{Z}_n[i])$ is $V(\Gamma(Z_n[i]))$; i.e., in this case, $V(\Gamma(\mathbb{Z}_n[i]))$ is self-centered.

The eccentricity of each vertex in $\Gamma(\mathbb{Z}_n[i])$, when n is a power of a prime, is determined in [25]. If $n \neq q_1q_2$, and n is divisible by at least two distinct primes, then, $\operatorname{diam}(\Gamma(\mathbb{Z}_n[i])) = 3$, [3], this together with the above theorem give the eccentricity of each vertex in $\Gamma(\mathbb{Z}_n[i])$ when n is divisible by at least two distinct primes.

Corollary 2.2. *If n is divisible by at least two distinct primes, $n \neq q_1q_2$ and $v \in V(\Gamma(\mathbb{Z}_n[i]))$, then $\operatorname{ecc}(v) = 2$, if $v \in C$, otherwise $\operatorname{ecc}(v) = 3$, where C is the center of $\Gamma(\mathbb{Z}_n[i])$.*

The cardinality of the center of $\Gamma(\mathbb{Z}_n[i])$, when n is divisible by at least two distinct primes, could easily be computed using appropriate formulas for the cardinality of each $Z(R_j)$ given in [2].

Corollary 2.3. *The cardinality of the center of $\Gamma(\mathbb{Z}_n[i])$ is*

(1) 1, *if* $n = 2^m$,

(2) $q^2 - 1$, *if* $n = q^m$,

(3) $p^{2m-2} + 2p^{m-1} - 2p^{m-1} - 1$, *if* $n = p^m$, $m \geq 2$.

(4) $(\prod_{j=1}^{l} |Z(R_j)|)+(\prod_{j=1}^{t} q_j)-1$, *if* $\mathbb{Z}_n[i] = R_1\times R_2\times\cdots\times R_l\times F_1\times F_2\times\cdots\times F_t$, *where n is divisible by at least two distinct primes.*

The relationship between the center and the median of $\Gamma(R)$ is investigated by Redmond [26], who proved that if R is a finite commutative ring with unity that is not an integral domain, then the median and the center of $\Gamma(R)$ are equal if the radius of $\Gamma(R)$ is at most 1, and the median is a subset of the center if the radius is 2. Now, we study the median of $\Gamma(\mathbb{Z}_n[i])$.

Theorem 2.4.

(1) *The median of $\Gamma(\mathbb{Z}_{2^m}[i])$ is $\{(1+i)2^{m-1}\}$ and the median of $\Gamma(\mathbb{Z}_{q^m}[i])$ is $\{\alpha q^{m-1} + \beta q^{m-1} i : \alpha, \beta \in U(\mathbb{Z}_q)\} - \{0\}, m \geq 2$.*

(2) *The median of $\Gamma(\mathbb{Z}_p[i])$* ***is*** *$Z^*(\mathbb{Z}_p[i])$.*

(3) *if $n = q_1q_2\ldots q_t, t \geq 2$ and $q_1 < q_2 < \cdots < q_t$, then the median of $\Gamma(\mathbb{Z}_n[i])$* ***is the set*** *$\{(u, 0, 0, \ldots, 0) : u \in U(\mathbb{Z}_{q_1}[i])\}$.*

(4) *Let $\mathbb{Z}_n[i] = R_1 \times R_2 \times \cdots \times R_l \times F_1 \times F_2 \times \cdots \times F_t, l \geq 1$, be the Artinian decomposition of $\mathbb{Z}_n[i]$. Let $S = \{s : 1 < s < l, |Z^*(R_s)| \prod_{j\neq s} |R_j| = \max\{|Z^*(R_k)| \prod_{j\neq k} |R_j| : 1 \leq j, k \leq l\}\}$. If $\mathbb{Z}_n[i]$ is not local, then the median of $\Gamma(\mathbb{Z}_n[i])$ is given by $\{y : y = (y_j)_{j=1}^{l+t}$ where if $j \in S$, then $y_j = uz, u \in U(R_j)$ and, $\operatorname{ann}(z) = Z(R_j)$, otherwise, $y_j = 0\}$.*

Proof. (1) If R is local, then $\Gamma(R)$ has a vertex or set of vertices each of which is adjacent to all other vertices. Thus, in this case, the median is equal to the center. So, the median of $\Gamma(\mathbb{Z}_{2^m}[i])$ is $\{(1+i)2^{m-1}\}$, and the median of $\Gamma(\mathbb{Z}_{q^m}[i])$ is $\{\alpha q^{m-1} + \beta q^{m-1} i : \alpha, \beta \in U(\mathbb{Z}_q)\} - \{(0)\}, m \geq 2$, [25].

(2) If $\mathbb{Z}_n[i]$ is not local, then from [3] the radius of $\Gamma(\mathbb{Z}_n[i])$ is 2. Note that if x is in the center of $\Gamma(\mathbb{Z}_n[i])$, then the eccentricity of x is 2. Hence, $s(x) = \deg(x) + 2(|Z^*(R)| - \deg(x) - 1)$, thus $s(x) = 2|Z^*(R)| - \deg(x) - 2$. Therefore, vertices in the median of $\Gamma(\mathbb{Z}_n[i])$ are precisely those vertices of the center of maximum degree in $\Gamma(\mathbb{Z}_n[i])$. Since $\Gamma(\mathbb{Z}_p[i])$ is regular graph then the median of $\Gamma(\mathbb{Z}_p[i]))$ **is** $Z^*(\mathbb{Z}_p[i])$.

(3) The result holds by the argument in the proof of (2) and the fact that vertices of maximum degree are vertices in the given set.

(4) Since R_j is local, it contains an element z, such that $ann(z) = Z(R_i)$. Again, using similar argument to the proof of (2) and then finding vertices of maximum degree in $\Gamma(\mathbb{Z}_n[i])$, the result holds. ■

The chromatic number of a graph G, $\chi(G)$, is the minimum number k such that G can be colored using k different colors with no two adjacent vertices having the same color. The clique number, $\omega(G)$, of a graph G is the maximum order among the complete subgraphs of G. A graph G is a core if any homomorphism from G to itself is an automorphism. Also, a subgraph H of G is called a core of G if H is a core itself, and there is a homomorphism from G to H. If R is a ring such that its chromatic number and clique number coincide, i.e., $\chi(\Gamma(R)) = \omega(\Gamma(R))$. Then, the core of $\Gamma(R)$ is the maximal clique in $\Gamma(R)$, [16]. On the other hand, $\chi(\Gamma(\mathbb{Z}_n[i]))$, and $\omega(\Gamma(\mathbb{Z}_n[i]))$, when n is a power of a prime are computed in, [3] and [18] respectively. Furthermore, the maximal clique, when n is a power of a prime, is determined in [18] comparing the results in the two papers, we see that, $\chi(\Gamma(\mathbb{Z}_n[i]))$, and $\omega(\Gamma(\mathbb{Z}_n[i]))$ are equal, and so we get,

Corollary 2.5. *The core of $\Gamma(\mathbb{Z}_n[i])$ is the maximal complete subgraph of $\Gamma(\mathbb{Z}_n[i])$ induced by the following set,*

(1)

$$S = \bigcup_{\lceil \frac{m}{2} \rceil \leq k,j \leq m} \{\alpha 2^j + \beta 2^k i : \alpha \in U(\mathbb{Z}_{2^{m-j}}), \beta \in U(\mathbb{Z}_{2^{m-k}})\} - \{0\}, \text{ if } m \text{ is even},$$

and $S \cup 2^{\lfloor \frac{m}{2} \rfloor}$ if m is odd, where $n = 2^m, m \geq 2$.

(2)

$$S = \bigcup_{\lceil \frac{m}{2} \rceil \leq k,j \leq m} \{\alpha q^j + \beta q^k i : \alpha \in U(\mathbb{Z}_{q^{m-j}}), \beta \in U(\mathbb{Z}_{q^{m-k}})\} - \{0\}, \text{ if } m \text{ is even},$$

and $S \cup q^{\lfloor \frac{m}{2} \rfloor}$ if m is odd, where $n = q^m, m \geq 2$.

(3)

$$S = \bigcup_{\lceil \frac{m}{2} \rceil \leq k,j \leq m} \{(\alpha p^j, \beta p^k) : \alpha \in U(\mathbb{Z}_{p^{m-j}}), \beta \in U(\mathbb{Z}_{p^{m-k}})\} - \{(0,0)\}, \text{ if } m \text{ is even},$$

and $S \cup (p^{\lfloor \frac{m}{2} \rfloor}, p^{\lceil \frac{m}{2} \rceil})$ *if* m *is odd, where* $n = p^m, m \geq 2$.

Since $\Gamma(\mathbb{Z}_n[i])$ is a complete graph if and only if $n = q^2$, [2], the following corollary is obtained.

Corollary 2.6. $\Gamma(\mathbb{Z}_n[i])$ *is a core if and only if* $n = q^2$.

3. Multiple domination and 2-packing of $\Gamma(R)$

The domination number of the zero divisor graph of a commutative Artinian ring, with identity that is not a domain and with radius at most one is also one. While if the radius is 2, then the domination number of $\Gamma(R)$ is equal to the number of factors in the Artinian decomposition of R [16]. A subset S of the vertex set $V(G)$ of a graph G is a 2-packing if for each $u, v \in S$, $N[u] \cap N[v] = \phi$. The 2-packing number, $\rho(G)$, is the cardinality of a maximum packing. A subset D of the vertex set $V(G)$ of a graph G is a dominating set in G if each vertex of G, not in D, is adjacent to at least one vertex of D. The minimum cardinality of all dominating sets in G, $\gamma(G)$, is called the domination number of G. A set D is a k-dominating set for a graph G, if each vertex in $V(G)\backslash D$ is dominated by at least k vertices in D. The minimum cardinality of a k-dominating set is denoted by $\gamma_k(G)$. A set D is a k-tuple dominating set for a graph G, if each vertex in $V(G)$ is dominated by at least k vertices in D, the minimum cardinality of a k-tuple dominating set is denoted by $\gamma_{\times k}(G)$, [19].

The next theorem gives the k-domination number $\Gamma(R)$, where R is a commutative Artinian ring with unity that is not a domain.

Theorem 3.1. *Let R be a commutative Artinian ring with unity that is not a domain, $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$. Suppose that if $n \geq 1$, then $k \leq |\mathrm{center}(\Gamma(R_j))|, j = 1, \ldots, n$, and if $m \geq 1$, then $k \leq |F_j{}^*|$, $j = 1, \ldots, m$. Then the k-domination number is equal to $k(m+n)$.*

Proof. If R is local and $k \leq |\ \mathrm{center}(\Gamma(R))|$, then since each vertex in the center of $\Gamma(R)$ dominates all other vertices, we have $\gamma_k(\Gamma(R)) = k$. Now, if R is not local, let $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$. Let $Y_j = \{y_{jt}\}_{t=1}^k$ where $y_{jt} = (0, 0, \ldots, 0, x_{jt}, 0, \ldots, 0)$ such that $x_{jt} \in \mathrm{center}(\Gamma(R_j))$ and $j = 1, \ldots, n$ and $Z_s = \{z_{st}\}_{t=1}^k$, where $z_{st} = (0, 0, \ldots, 0, u_{st}, 0, \ldots, 0)$ such that $u_{st} \in F_s{}^*$ and $s = 1, \ldots, m$.

Let $D = (\bigcup_{j=1}^n Y_j) \cup (\bigcup_{s=1}^m Z_s)$. Clearly D is a k-dominating set of $\Gamma(R)$. So, $\gamma_k(\Gamma(R)) \leq k(m+n)$.

Now, let $t_j = (1, 1, \ldots, 1, 0, 1, \ldots, 1)$ where 0 is in the i^{th} position. Then $N(t_i) = \{(0, 0, \ldots, 0, x_i, 0, \ldots, 0)$, where $x_i \in R_i{}^*$, if $i \leq l$ and $x_i \in F_i{}^*$ otherwise$\}$. Thus, any k-dominating set $\acute{D}$ of $\Gamma(R)$ must contain k vertices of $N[t_i]$ for each i.

Note that $N[t_j] \cap N[t_l] = \phi$, for $l \neq j$. Thus, if $\acute{D}$ is a k-dominating set, then $|\acute{D}| \geq k(m+n)$, hence equality holds. ■

Since each vertex in the suggested k-dominating set D is also k-dominated by D, we have the following corollary

Corollary 3.2. *Let R be a commutative Artinian ring with unity that is not a domain, $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$. Suppose that if $n \geq 1$, then $k \leq |\text{center}(\Gamma(R_j))|$, $j = 1, \ldots, n$, and if $m \geq 1$, then $k \leq |F_j{}^*|$, $j = 1, \ldots, m$. Then $\gamma_{\times k}(\Gamma(R)) = k\gamma(\Gamma(R))$ which is k times the number of factors in the Artinian decomposition of R.*

Corollary 3.3. *Let R be a commutative Artinian ring with unity that is not a domain, then the 2-packing number $\rho(\Gamma(R)) = \gamma(\Gamma(R))$.*

Proof. If R is local, then $\Gamma(R)$ has a vertex which is adjacent to every other vertex, so $\rho(\Gamma(R)) = 1$. Now, assume R is not local, let $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$ where $n + m \geq 2$ is the Artinian decomposition of R. Let $t_i = (1, 1, \ldots, 1, 0, 1, \ldots, 1)$ where 0 is in the i^{th} position. Since $N[t_j] \cap N[t_l] = \phi$, for $j \neq l$. Thus, the set $T = \{t_i : i = 1, 2, \ldots, n+m\}$ is a 2-packing set with $\gamma(\Gamma(R))$ vertices. On the other hand, for any graph G, $\rho(G) \leq \gamma(G)$, (Theorem 2.13, [19]) and so, equality holds. ■

Next, we move to the multiple domination and 2-packing of $\Gamma(\mathbb{Z}_n[i])$. The domination number of $\Gamma(\mathbb{Z}_n[i])$, $n = \prod_{j=1}^{m} \pi_j^{r_j}$, where $\pi_j' s$ are Gaussian prime integers and $r_j' s$ are positive integers, was determined in [25] to be the number m of distinct prime factors of n, if n is odd and $m-1$ if n is even. Besides, it was shown that a minimum dominating set for $\Gamma(\mathbb{Z}_n[i])$ is $D = \{P_j = \pi_1^{r_1}\pi_2^{r_2}\ldots\pi_j^{r_j-1}\ldots\pi_m^{r_m} : j = 1, 2, \ldots, m\}$. Note that for each $j = 1, 2, \ldots, m$, the vertex π_j is only adjacent to P_j in D, and $N(\pi_j) = \{u\ P_j : u \in U(\mathbb{Z}_n)\}$ besides, $\delta(\Gamma(\mathbb{Z}_n[i]))$ =deg(π_r), for some $r \leq m$. On the other hand $N[\pi_j] \cap N[\pi_l] = \phi$ for $j \neq l$. Thus, $\gamma_k(\Gamma(\mathbb{Z}_n[i]) \geq k\gamma(\Gamma(\mathbb{Z}_n[i]))$. Since the collection of k-unit multiples of each P_j, where $j = 1, 2, \ldots, m$, also k-dominates $V(\Gamma(\mathbb{Z}_n[i]))$, we have $\gamma_k(\Gamma(\mathbb{Z}_n[i])) \leq k\gamma(\Gamma(\mathbb{Z}_n[i]))$. Thus the following theorem is obtained.

Theorem 3.4. *If $n = \prod_{j=1}^{m} \pi_j^{r_j}$, where $\pi_j' s$ are Gaussian prime integers and $r_j' s$ are positive integers, and $k \leq \delta(\Gamma(\mathbb{Z}_n[i]))$, then*

(1) $\gamma_k((\Gamma(\mathbb{Z}_n[i])) = k\gamma(\Gamma(\mathbb{Z}_n[i])) = \begin{cases} km, & \text{if } n \text{ is odd;} \\ k(m-1), & \text{if } n \text{ is even.} \end{cases}$

(2) $\gamma_{\times k}(\Gamma(\mathbb{Z}_n[i])) = \gamma_k(\Gamma(\mathbb{Z}_n[i])) = k\gamma(\Gamma(\mathbb{Z}_n[i]))$.

Graphs with equal 2-packing number and domination number are of special importance, in the same spirit of the above theorem, the following result is obtained.

Theorem 3.5. *If G is a connected graph for which $\rho(G) = \gamma(G)$, then $\gamma_k(G) = k\gamma(G)$, where $k \leq \delta(G)$.*

The domination number of the complement of $\Gamma(R)$ where $R = R_1 \times R_2$ and R_1 and R_2 are commutative rings with unity is 2, [18]. The following theorem gives the k-domination number of $\overline{\Gamma(R)}$.

Theorem 3.6. *Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with unity such that $|reg(R_i)| \geq k$, for $i = 1, 2$. Then $\gamma_k(\overline{\Gamma(R)}) = k\gamma(\overline{\Gamma(R)}) = 2k$.*

Proof. Let $u_1, u_2, \ldots, u_k \in \mathrm{reg}(R_1)$ and $v_1, v_2, \ldots, v_k \in \mathrm{reg}(R_2)$. The set $S = \{(u_i, 0), (0, v_i) : i = 1, \ldots, k\}$ is a k-dominating set with $2k$ vertices. Suppose that there is a k-dominating set B with less than $2k$ vertices, then there exists $u \in \mathrm{reg}(R_1)$ or $v \in \mathrm{reg}(R_2)$ such that either $(u, 0)$ or $(0, v)$ is dominated by less than k vertices in B. Thus the result holds. ■

The Cartesian product of two graphs G and H, $G\square H$, is the graph whose vertex set is $V(G) \times V(H)$ and two vertices in $G\square H$ are adjacent if they are equal in one coordinate and adjacent in the other coordinate [19]. A graph G is said to satisfy Vizing's conjecture if for any graph H, we have $\gamma(G\square H) \geq \gamma(G)\gamma(H)$. A graph G, for which $\rho(G) = \gamma(G)$, satisfies Vizing's conjecture, [20]. Another class of graphs which satisfies Vizing's conjecture are graphs with domination number 2, [17]. Thus the following corollary is obtained.

Corollary 3.7.

(1) *If R is a commutative Artinian ring with unity that is not a domain, then $\Gamma(R)$ satisfies Vizing's conjecture.*

(2) *If $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with unity, then $\overline{\Gamma(R)}$ satisfies Vizing's conjecture.*

4. THE AUTOMORPHISM GROUP OF $\Gamma(\mathbb{Z}_n[i])$

A graph automorphism f of a graph G is a bijection $f : G \to G$ which preserves adjacency. The set $Aut(G)$ of all graph automorphisms of G forms a group under the usual composition of functions called the automorphism group of G.

We are going to determine the automorphism group for $\Gamma(\mathbb{Z}_n[i])$, where n is a power of a prime integer. We use the notation wr to denote the wreath product.

Theorem 4.1.

(1) $Aut(\Gamma(\mathbb{Z}_{2^m}[i])) \cong \prod_{j=1}^{2m-1} S_{2^{2m-j-1}}$, $m \geq 2$.

(2) $Aut(\Gamma(\mathbb{Z}_{q^2}[i])) \cong S_{q^2-1}$, *and* $Aut(\Gamma(\mathbb{Z}_{q^m}[i])) \cong \prod_{j=1}^{2m-1} S_{n_k}, m \geq 3$

where $n_k = 2(q-1)(q^{2m-1} - q^{m+k-2}) - (q-1)^2 q^{2m-2k-2}$.

(3) $Aut(\Gamma(\mathbb{Z}_p[i])) \cong S_{p-1} wr S_2$. *And for* $m \geq 2$, $Aut(\Gamma(\mathbb{Z}_{p^m}[i]))$

$$\cong \Big[\Big(\prod_{k=1}^{m-1} S_{n_{mk}} \times \prod_{k=1}^{m-1} (S_{n_{0k}}) wr S_2\Big] \times \Big[\prod_{j=1}^{m-2} \prod_{k=j+1}^{m-1} S_{n_{jk}} wr S_2\Big)\Big] \times \prod_{k=1}^{m-1} S_{n_{kk}}, n_{kj}$$
$$= (p-1)^2 p^{2m-k-j-2}.$$

Proof. (1) From [3], we have $\Gamma(\mathbb{Z}_{2^m}[i]) \cong \Gamma(\mathbb{Z}_{2^{2m}})$. Thus the result holds by Theorem 31 of [22].

(2) Since $\Gamma(\mathbb{Z}_{q^2}[i]) = K_{q^2-1}$, then Aut $(\Gamma(\mathbb{Z}_{q^2}[i])) \cong S_{q^2-1}$. Now, we study the case $n = q^m, m \geq 3$. Let $A_{kj} = \{\alpha q^j + \beta q^k i : \alpha \in U(\mathbb{Z}_{q^{m-j}}), \beta \in U(\mathbb{Z}_{q^{m-k}}), 1 \leq j,k \leq m\}$. Clearly, the sets A_{kj}, not both $k,j = m$, partition $V(\Gamma(\mathbb{Z}_{q^m}[i]))$. Note that $N(\alpha q^r + \beta q^s i) = \bigcup_{k,j \geq m-min(r,s)} A_{kj}$. For $k = 1,\ldots,m$. Let $N_k = \{\alpha q^r + \beta q^s i : \alpha \in U(\mathbb{Z}_{q^{m-r}}), \beta \in U(\mathbb{Z}_{q^{m-s}})$ and $\min(r,s) = k\}$, then two vertices in $\Gamma(\mathbb{Z}_{q^m}[i])$ have the same neighborhood, and the same degree, if and only if they belong to the same set N_k. Let $n_k = |N_k|$, then easy calculations gives $n_k = 2(q-1)(q^{2m-1} - q^{m+k-2}) - (q-1)^2 q^{2m-2k-2}$. Thus, Aut $(\Gamma(\mathbb{Z}_{q^m}[i])) \cong \prod_{k=1}^{m} S_{n_k}$.

(3) $\Gamma(\mathbb{Z}_p[i]) = K_{p-1,p-1}$ and hence $\mathrm{Aut}(\Gamma(\mathbb{Z}_p[i])) \cong S_{p-1}\mathrm{wr}S_2$. Now, we study the Automorphism group of $\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m})$. Let $A_{kj} = \{(\alpha p^j, \beta p^k) : \alpha \in U(\mathbb{Z}_{p^{m-j}}), \beta \in U(\mathbb{Z}_{p^{m-k}}), 0 \leq j,k \leq m\}$. The sets A_{kj}, not both $k,j = m$ or 0, partition $V(\Gamma(\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}))$. Two vertices have the same neighborhood if and only if they belong to the same set A_{kj}, and two vertices x and y have the same degree if and only if $x,y \in A_{jk} \cup A_{kj}$. Let $n_{kj} = |A_{kj}|$, then $n_{kj} = (p-1)^2 p^{2m-k-j-2}$. For each $k = 1,\ldots,m-1$, vertices in each A_{00} permute to give $S_{n_{k0}}$ and vertices in each A_{km} permute to give $S_{n_{km}}$. Let G_1 be the graph induced by $(\bigcup_{k=1}^{m-1} A_{mk}) \cup (\bigcup_{k=1}^{m-1} A_{0k})$. The automorphism group of G_1 is $(\prod_{k=1}^{m-1} S_{n_{mk}}) \times (\prod_{k=1}^{m-1} S_{n_{k0}})$. Another copy of G_1 is induced by $(\bigcup_{k=1}^{m-1} A_{km}) \cup (\bigcup_{k=1}^{m-1} A_{k0})$. Therefore, $\mathrm{Aut}(2G_1) \cong (\prod_{k=1}^{m-1} S_{n_{mk}} \times \prod_{k=1}^{m-1} S_{n_{0k}})$ wr S_2. For $1 \leq j < k \leq m-1$, the set A_{kj} induces a graph which is isomorphic to $\langle A_{jk} \rangle$. If $j < \lceil \frac{m}{2} \rceil$, then A_{kj} is an independent set, otherwise, $\langle A_{kj} \rangle$ is a complete graph. Thus, $\mathrm{Aut}(\Gamma(\mathbb{Z}_{p^m}[i])) \cong [(\prod_{k=1}^{m-1} S_{n_{mk}} \times \prod_{k=1}^{m-1} S_{n_{0k}})$ wr $S_2] \times [\prod_{j=1}^{m-2} \prod_{k=j+1}^{m-1} (S_{n_{jk}}$ wr $S_2)] \times \prod_{k=1}^{m-1} S_{n_{kk}}$. ■

5. PERFECT ZERO DIVISOR GRAPHS

A graph G is perfect if the chromatic number of every induced subgraph H equals the size of the largest clique of that subgraph, i.e., for every $H \subseteq G$, $\omega(H) = \chi(H)$, otherwise G is called an imperfect graph. In 1960, Berge [12] formulated two conjectures about perfect graphs. The week perfect graph conjecture which states that a graph is perfect if and only if its complement is perfect. This conjecture was proved in 1972 by Lovász, [23]. The second conjecture is the strong perfect graph conjecture and states that a graph is perfect if and only if it contains, as an induced subgraph, neither an odd cycle of length at least five nor its complement, it was not until 2002 that this conjecture was settled by Chudnovsky *et al.*, [15].

In this section we investigate which zero divisor graphs are perfect. In particular, we study when the zero divisor graphs $\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_n[i])$ are perfect.

A graph G is called P_4-free if it does not contain a P_4 as an induced subgraph. And a graph is called slightly triangulated if it contains no induced odd cycle of length at least five and every induced subgraph H contains a vertex whose neighborhood in H does not contain a P_4. A graph G is called a murky graph if it contains no C_5, P_6 or $\overline{P_6}$ as an induced subgraph.

The following theorem provides some tools for proving that a graph is perfect.

Theorem 5.1.

(1) *If G is a P_4-free graph, then G is perfect,* [13].

(2) *If G is slightly triangulated graph, then G is perfect,* [24].

(3) *If G is murky graph, then G is perfect,* [21].

Theorem 5.2.

(1) *If $n = t^m$, t is prime, then $\Gamma(\mathbb{Z}_n)$ is perfect.*

(2) *If $n = 2^m$, then $\Gamma(\mathbb{Z}_n[i])$ is perfect.*

(3) *If $n = q^m$, then $\Gamma(\mathbb{Z}_n[i])$ is perfect.*

Proof. (1) The graph $\Gamma(\mathbb{Z}_{t^m})$ is p_4-free graph. To show this, let $v_1 - v_2 - v_3 - v_4$ be an induced P_4 subgraph of $\Gamma(\mathbb{Z}_{t^m})$. Then $v_j = a_j t^{m_j}$ where $(a_j, t) = 1$ and $m_j + m_{j+1} \geq m$ for $j = 1, 2, 3, 4$. Hence, $m_1 \leq m_4$ gives $m \leq m_1 + m_2 \leq m_2 + m_4$ and $m_1 \geq m_4$ gives $m_1 + m_3 \geq m_3 + m_4 \geq m$. Thus, v_2 and v_4 are adjacent or v_1 and v_3 are adjacent, a contradiction.

(2) Since, $\Gamma(\mathbb{Z}_{2^m}[i]) \cong \Gamma(\mathbb{Z}_{2^{2m}})$, from part (1) the result holds.

(3) Let $v \in \Gamma(\mathbb{Z}_{q^m}[i])$. Then $0 \neq v = \alpha q^a + \beta q^b i$, were $\alpha, \beta \in U(\mathbb{Z}_q)$ and $N(v) = \{cq^r + dq^s i : r, s \geq m - min\{a, b\}\}$, [25]. Now assume that $v_1 - v_2 - v_3 - v_4$ is an induced P_4 subgraph of $\Gamma(\mathbb{Z}_{q^m}[i])$. Then $v_j = \alpha_j q^{a_j} + \beta_j q^{b_j} i$, where $\alpha_j, \beta_j \in U(\mathbb{Z}_q)$. So, v_1 and v_3 are adjacent or v_2 and v_4 are adjacent, a contradiction. So $\Gamma(\mathbb{Z}_{q^m}[i])$ is P_4 free graph and hence it is perfect. ■

Chiang-Hsieh *et al.*, [14] proved that if (R, m) is local ring such that $|R| = t^n$, where t is prime, and $m^{n-1} \neq 0$ then $\Gamma(R) \cong \Gamma(\mathbb{Z}_{t^n})$. So, $\Gamma(R)$ is a perfect graph. This together with the above theorem gives the following corollary.

Corollary 5.3. *If (R, m) is local ring such that $|R| = t^n$, where t is prime and $m^{n-1} \neq 0$, then $\Gamma(R)$ is perfect.*

Theorem 5.4. *Let $R = R_1 \times R_2$. Then*

(1) *If R_1 and R_2 are integral domains, then $\Gamma(R)$ is perfect.*

(2) *If R_1 is an integral domain and R_2 is non-integral domain such that for every $a, b \in Z(R_2)$ where a, b need not be distinct, $ab = 0$, then $\Gamma(R)$ is perfect.*

(3) *If R_1 and R_2 are non-integral domains such that for every $a, b \in Z(R_i)$, where a, b need not be distinct, $ab = 0$, then $\Gamma(R)$ is perfect.*

Proof. (1) Since $\Gamma(R)$ is acomplete bipartite graph it is perfect

(3) Assume that R_1 and R_2 are non-integral domain such that for every $a, b \in Z(R_i)$, where a, b need not be distinct, $ab = 0$. Then $\Gamma(R)$ is a slightly triangulated graph. To show this, note that any induced P_4 graph of $\Gamma(R)$ is one of the following,

$(a, v) - (b, 0) - (0, c) - (u, 0)$ where $a, b \in Z^*(R_1), c \in {R_2}^*$ and $u \in U(R_1), v \in U(R_2)$,

$(a, v) - (b, 0) - (0, c) - (u, d)$ where $a, b \in Z^*(R_1), c, d \in Z^*(R_2)$ and $u \in U(R_1), v \in U(R_2)$,

$(u, x) - (0, y) - (z, 0) - (0, v)$ where $x, y \in Z^*(R_2), z \in {R_1}^*$ and $u \in U(R_1), v \in U(R_2)$,

or

$(u, x) - (0, y) - (z, 0) - (w, v)$ where $x, y \in Z^*(R_2), z, w \in Z^*(R_1)$ and $u \in U(R_1), v \in U(R_2)$.

So, $\Gamma(R)$ has no induced cycle C_n, for $n \geq 5$. Also, for every $(t, s) \in V(\Gamma(R))$, $N((t, s))$ does not contain a P_4 subgraph, since $N((t, s)) \cap \{(r, v) : r \in Z(R_1), v \in U(R_2)\} = \phi$ or $N((t, s)) \cap \{(u, r) : r \in Z(R_2), u \in U(R_1)\} = \phi$. ■

Corollary 5.5.

(1) *For $n = t_1^{m_1} t_2^{m_2}$, where t_1, t_2 are primes and $1 \leq m_1, m_2 \leq 2$, $\Gamma(\mathbb{Z}_n)$ is perfect.*

(2) *For $n = p$ or $n = p^2$ or $n = 2q^{m_1}$ or $n = q_1^{m_1} q_2^{m_2}$, where $1 \leq m_1, m_2 \leq 2$, $\Gamma(\mathbb{Z}_n[i])$ is perfect.*

Proof. (2) Note that, $\mathbb{Z}_{p^m}[i] \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$ and $\mathbb{Z}_{2q^m}[i] \cong \mathbb{Z}_4 \times \mathbb{Z}_{q^m}[i]$. ■

Theorem 5.6. *If $R = R_1 \times R_2 \times R_3$, where R_1, R_2 and R_3 are integral domains, then $\Gamma(R)$ is a perfect graph.*

Proof. Assume that $R = R_1 \times R_2 \times R_3$, where R_1, R_2 and R_3 are integral domains. Then it easy to verify that any P_4 path of $\Gamma(R)$ is one of the following,

$$(a_1, a_2, 0) - (0, 0, b_3) - (c_1, 0, 0) - (0, d_2, d_3),$$

$$(a_1, a_2, 0) - (0, 0, b_3) - (0, c_2, 0) - (d_1, 0, d_3),$$

or

$$(a_1, 0, a_3) - (0, b_2, 0) - (c_1, 0, 0) - (0, d_2, d_3),$$

where $a_i, b_i, c_i, d_i \in R_i^*$. So $\Gamma(R)$ has no induced cycle $C_n, n \geq 5$ and there is no vertex $v \in \Gamma(R)$ such that $N(v)$ contains a P_4. Hence $\Gamma(R)$ is a slightly triangulated graph and thus $\Gamma(R)$ is perfect. ■

Corollary 5.7.

(1) *For* $n = t_1t_2t_3$, *where* t_1, t_2, t_3 *are primes,* $\Gamma(\mathbb{Z}_n)$ *is perfect.*

(2) *For* $n = qp$ *or* $n = q_1q_2q_3$, $\Gamma(\mathbb{Z}_n[i])$ *is perfect.*

If R is a product of four integral domains, then $\Gamma(R)$ is not a slightly triangulated graph since $N((0,0,0,1))$ contains $(1,1,0,0) - (0,0,1,0) - (1,0,0,0) - (0,1,1,0)$ as an induced subgraph. However, the next theorem shows that $\Gamma(R)$ is perfect.

Theorem 5.8. *If* $R = \prod_{i=1}^{4} R_i$ *where* R_i *is an integral domain for* $i = 1, 2, 3$ *and* 4, *then* $\Gamma(R)$ *is perfect.*

Proof. Let $a = (a_i), b = (b_i), c = (c_i), d = (d_i), e = (e_i), f = (f_i) \in Z^*(R)$. Suppose that $a - b - c - d - e$ is an induced P_5 subgraph of $\Gamma(R)$. Then a has at least two non-zero components. So, we have the following two cases:

Case 1. a has exactly two non-zero components, say a_1 and a_2. Then $b = (0, 0, b_3, 0)$ or $b = (0, 0, 0, b_4)$ implies that $ed \neq 0$ since $bd \neq 0$ and $be \neq 0$, a contradiction while $b = (0, 0, b_3, b_4)$, gives $c = (c_1, 0, 0, 0)$ or $c = (0, c_2, 0, 0)$. Let $c = (c_1, 0, 0, 0)$, then $d = (0, d_2, d_3, 0)$ and $e = (e_1, 0, 0, e_4)$ or $d = (0, d_2, 0, d_4)$ and $e = (e_1, 0, e_3, 0)$.

Case 2. a has exactly three non-zero components, say a_1, a_2 and a_3. Then $b = (0, 0, 0, b_4)$. Since $bd \neq 0be$, we have $d_4 \neq 0 \neq e_4$. So, $ed \neq 0$, a contradiction.

Now, it is easy to verify that $\Gamma(R)$ has no C_5 or P_6 as an induced subgraph. Moreover $\overline{\Gamma(R)}$ has no induced P_6 path. To show this let $a - b - c - d - e - f$ be an induced path of $\overline{\Gamma(R)}$. Then we have three cases:

Case 1. a has exactly one non-zero component, say a_1. Then $b_1 \neq 0$. If b has only two non zero components say $b = (b_1, b_2, 0, 0)$, then $c = (0, c_2, c_3, c_4)$ gives $e = (0)$ since $be = ec = 0$ and $c = (0, c_2, c_3, 0)$ or $c = (0, c_2, 0, c_4)$ gives $d = (0, 0, d_3, d_4)$ and $f = (0)$, a contradiction.

And if b has three non-zero components, say $b = (b_1, b_2, b_3, 0)$, then $bd = 0$ and $cd \neq 0$ gives $c_4 \neq 0$. But $ec = eb = 0$ implies that $e = (0)$, a contradiction.

Case 2. a has exactly two non-zero components, say a_1 and a_2. Then $ae = be = ce = ac = 0$ and $ab \neq 0 \neq bc$, yields $e = (0)$, a contradiction.

Case 3. a has exactly three non zero components, say a_1, a_2 and a_3. Then $ac = 0$ and $ad = 0$ implies that $c = (0, 0, 0, c_4)$ and $d = (0, 0, 0, d_4)$. Since $ed \neq 0$ we have $ec \neq 0$, a contradiction. So, $\Gamma(R)$ is a murky graph and hence it is perfect. ■

Corollary 5.9.

(1) *For $n = t_1t_2t_3t_4$, where t_1, t_2, t_3, t_4 are primes, $\Gamma(\mathbb{Z}_n)$ is perfect.*

(2) *For $n = q_1q_2p$ or $n = q_1q_2q_3q_4$ or $n = p_1p_2$, $\Gamma(\mathbb{Z}_n[i])$ is perfect.*

Theorem 5.10. *If S is an integral domain ring and $R = \mathbb{Z}_{p^m} \times S$, then $\Gamma(R)$ is a perfect graph.*

Proof. Let $u, v \in U(S)$ and $w \in U(\mathbb{Z}_{p^m})$. Then any induced P_4 path of $\Gamma(R)$ has one of the following forms:

$(p^s, 0) - (0, u) - (p^t, 0) - (p^l, v)$ or $(p^s, u) - (p^t, 0) - (0, u) - (w, 0)$.

Any induced P_4 path of $\overline{\Gamma(R)}$ has one of the following forms:

$(0, u) - (p^s, v) - (p^r, 0) - (p^l, 0)$ or $(p^s, u) - (p^t, v) - (p^r, 0) - (p^l, 0)$.

So there is no C_5 or a P_6 induced subgraph of $\Gamma(R)$ or $\overline{\Gamma(R)}$. ■

Corollary 5.11.

(1) *For $n = t_1t_2^m$, where t_1, t_2 are primes, $\Gamma(\mathbb{Z}_n)$ is perfect.*

(2) *For $n = q_1q_2^m$ or $n = 2^mq$, $\Gamma(\mathbb{Z}_n[i])$ is perfect.*

Theorem 5.12. If $R = R_1 \times R_2 \times R_3$, where R_1 and R_2 are non-integral domains, then $\Gamma(R)$ is imperfect.

Proof. Let $x, y \in Z^*(R_1)$ and $a, b \in Z^*(R_2)$ such that $xy = 0$ in R_1 and $ab = 0$ in R_2. Then $(1, 0, 0) - (0, a, 1) - (x, c, 0) - (y, 0, 1) - (0, 1, 0)$ is an induced C_5 subgraph of $\Gamma(R)$. So $\Gamma(R)$ is imperfect. ■

Theorem 5.13. *If $R = R_1 \times R_2$ and $\Gamma(R_i)$ is an imperfect graph for $i = 1$ or 2, then $\Gamma(R)$ is imperfect.*

Proof. Suppose that $R = R_1 \times R_2$ and $\Gamma(R_1)$ is imperfect graph. Then $\Gamma(R_1)$ or $\overline{\Gamma(R_1)}$ has an induced odd cycle of length at least five. But $v_1 - v_2 - v_3 - \cdots - v_n$ is a cycle of length n of $\Gamma(R_1)$ ($\overline{\Gamma(R_1)}$) if and only if $(v_1, 0) - (v_2, 0) - (v_3, 0) - \cdots - (v_n, 0)$ is a cycle of length n of $\Gamma(R)$ ($\overline{\Gamma(R_1)}$). So $\Gamma(R)$ is imperfect. ■

Alternative proofs of part (1) of Theorem 5.4 and Theorem 5.6 could be obtained from Theorems 5.8 and 5.13 since $\prod_{i=1}^{4} R_i \cong \prod_{i=1}^{3} R_i \times R_4$ and $\prod_{i=1}^{3} R_i \cong \prod_{i=1}^{2} R_i \times R_3$. We conclude that $\prod_{i=1}^{3} R_i$ and $\prod_{i=1}^{2} R_i$ are perfect graphs.

As a consequence of Theorem 5.12, 5.13 , we have the following.

Theorem 5.14. *If $R = \prod_{i=1}^{n} R_i, n \geq 5$, then $\Gamma(R)$ is imperfect.*

Proof. By induction,

(i) For $n = 5$, $\Gamma(R)$ is imperfect graph since $R \cong (R_1 \times R_2) \times (R_3 \times R_4) \times R_5$.

(ii) Assume that $\prod_{i=1}^{m} R_i, m \geq 5$ is imperfect graph. Then, $\prod_{i=1}^{m+1} R_i \cong (\prod_{i=1}^{m} R_i) \times R_{m+1}$ is imperfect. ■

Corollary 5.15.

(1) *For $n = \prod_{i=1}^{m} {t_i}^{m_i}$, where $m \geq 5$, $\Gamma(\mathbb{Z}_n)$ is imperfect.*

(2) *For $n = 2^k q_1^{m_1} q_2^{m_2}$ where $m_1 \geq 2$, $n = p_1^{m_1} p_2^{m_2}$ where $m_1 \geq 2$, $n = q_1^{m_1} q_2^{m_2} q_3^{m_3}$ where $m_1, m_2 \geq 2$, $n = q_1^{m_1} q_2^{m_2} q_3^{m_3} q_4^{m_4}$ where $m_1 \geq 2$, $n = 2^k \prod_{i=1}^{m} q^{m_i} \prod_{i=1}^{l} p_i^{n_i}$, $n = 2^k \prod_{i=1}^{l} p_i^{m_i}$ where $l \geq 2$, $n = \prod_{i=1}^{l} p_i^{n_i}, l \geq 3$ or $n = \prod_{i=1}^{m} {t_i}^{m_i}$, where $m \geq 5$, $\Gamma(\mathbb{Z}_n[i])$ is imperfect.*

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