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THE CLIFFORD SEMIRING CONGRUENCES ON AN ADDITIVE REGULAR SEMIRING

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Abstract

A congruence ρ on a semiring S is called a (generalized)Clifford semiring congruence if S/ρ is a (generalized)Clifford semiring. Here we characterize the (generalized)Clifford congruences on a semiring whose additive reduct is a regular semigroup. Also we give an explicit description for the least (generalized)Clifford congruence on such semirings.

Keywords: additive regular semiring, skew-ring, trace, kernel, Clifford congruence.

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1. Introduction

A semiring $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both, the additive reduct $(S, +)$ and multiplicative reduct $(S, \cdot)$ are semigroups and connected by the ring like distributive laws. An element a of S is called an *additive idempotent* if $a + a = a$. We denote the set of all additive idempotents of a semiring S by E^+ or sometimes by $E^+(S)$. A subset $I \neq \emptyset$ of a semiring S is called a left ideal of S if $a + b, sa \in I$ for all $a, b \in I$ and $s \in S$. Right ideals are defined dually. I is said to be an ideal of S if it is both, a left and a right ideal of S. A semiring S is *additive regular* if its additive reduct $(S, +)$ is a regular semigroup, that is, if for all $a \in S$ there exists $x \in S$ such that $a = a + x + a$. An element x of a semiring S is called an *additive inverse* of $a \in S$ if $a = a + x + a$ and $x = x + a + x$. The set of all additive inverses of a is denoted by $V^+(a)$. A semiring S is called an *inverse semiring* if every element of S has a unique additive inverse, equivalently, the additive reduct $(S, +)$ is an inverse semigroup. A semiring S is called a *skew-ring* [6] if its additive reduct $(S, +)$ is a group. A semiring S is called an *idempotent semiring* if $a + a = a = a^2$ for every $a \in S$. An idempotent semiring with commutative addition is called a *b-lattice* [14]. By a subdirect product of two semirings S_1 and S_2 we mean a semiring S which is isomorphic to a subsemiring T of the direct product of S_1 and S_2 such that the projection maps of T into both, S_1 and S_2 are surjective.

Thus semirings are generalizations of both, distributive lattices and rings, and hence it is interesting to characterize the class of semirings which are subdirect products of a distributive lattice and a ring. Bandelt and Petrich [1] characterized the same for inverse semirings. In the same paper they described a construction, namely strong distributive lattices of rings, which is analogous to the construction of a strong semilattice of groups. Ghosh [4] improved the construction of strong distributive lattices of rings given by Bandelt and Petrich, and characterized the class of all semirings which are subdirect products of a distributive lattice and a ring. The additive reduct of every semiring which is a strong distributive lattice of rings is commutative. To get a noncommutative one, Sen, Maity and Shum [14] considered the semirings which are strong distributive lattices of skew-rings and a more general class of semirings, namely strong b-lattices of skew-rings. Such semirings are known as Clifford semirings and generalized Clifford semirings, respectively.

Sen, Ghosh and Mukhopadhayay [13] characterized the Clifford semiring congruences on an additive commutative and inverse semiring. Maity [9] improved this to inversive semirings S such that $E^+(S)$ is a bisemilattice. In this article we wish to characterize the Clifford semiring congruences on additive regular semirings.

It was recognized by Feigenbaum [3] that every congruence ρ on a regular semigroup S is uniquely determined by its kernel, $ker\rho$ and its trace, $tr\rho$. Pastijn and Petrich [11] characterized the least and the greatest congruences on a regular semigroup S with a given trace and kernel. For an inverse semigroup, and more generally for an orthodox semigroup S, the least Clifford congruence was described by Mills [10]. LaTorre [8] used the least group congruence on a regular semigroup given by Feigenbaum [2] to describe the least Clifford congruence ξ on a regular semigroup.

Following Pastijn and Petrich [11], we define two relations τ and κ on the lattice $\mathcal{C}(S)$ of all congruences on an additive regular semiring S, induced by trace and kernel, respectively. Equivalence classes determined by τ and κ are intervals with the greatest and the least elements. If ρ is a [generalized] Clifford semiring congruence on an additive regular semiring S, then the greatest element ρ_{max} in $\rho\tau$ is a [b-lattice]distributive lattice congruence and the greatest element ρ^{max} in $\rho\kappa$ is a ring congruence. Thus it follows that every Clifford semiring is a subdirect product of a distributive lattice and a ring. Such details are given in Section 3. In Section 4, we give explicit characterizations for the least Clifford semiring congruence and the least generalized Clifford semiring congruence, which is analogous to the characterization of the least Clifford congruence on a regular semigroup given by LaTorre.

2. Preliminaries

Sen, Ghosh and Mukhopadhyay [4, 13] used the term *Clifford semiring* to mean the semirings which are strong distributive lattices of rings. By the same phrase, Sen, Maity and Shum [14] called the semirings which are strong distributive lattices of skew-rings, and used the name *generalized Clifford semiring* for the semirings which are strong b-lattices of skew-rings. They also characterized both, the Clifford semirings and generalized Clifford semirings as equational classes. Here we follow the conventions of Sen, Maity and Shum.

Definition 2.1. A semiring is called a Clifford semiring if it is an inverse semiring satisfying following conditions: for every $a, b \in S$ and $a' \in V^+(a)$, $b' \in V^+(b)$,

$$a + a' = a' + a, \tag{2.1}$$

$$a(a + a') = a + a', \tag{2.2}$$

$$a(b + b') = (b + b')a, \tag{2.3}$$

$$a + a(b + b') = a, \tag{2.4}$$

$$\text{and } a + b = b \text{ implies that } a + a = a. \tag{2.5}$$

As a generalizations of Clifford semirings Sen, Maity and Shum [14] introduced generalized Clifford semirings which are defined by:

Definition 2.2. A semiring S is a generalized Clifford semiring if it is an inverse semiring satisfying the conditions (2.1), (2.2) and (2.5) for all $a, b \in S$ and $a' \in V^+(a)$, $b' \in V^+(b)$.

An inverse semiring is a [generalized] Clifford semiring if and only if it is a subdirect product of a [b-lattice] distributive lattice and a skew-ring [14, Theorem 2.7]. For detailed descriptions and examples of Clifford and generalized Clifford semirings we refer to [14].

Let S be a semiring. A congruence ρ on S is called a *Clifford* (*generalized*) *congruence* if S/ρ is a Clifford (generalized) semiring. We define b-lattice congruences and distributive lattice congruences similarly.

We denote the set of all congruences on S by $\mathcal{C}(S)$. Lallement's Lemma shows that the idempotents are well behaved in connection with homomorphisms of regular semigroups. A corresponding result for additive regular semirings is as follows:

Lemma 2.3 (Lallement's Lemma)**.** *Let S be an additive regular semiring and $\rho \in \mathcal{C}(S)$. Then $a\rho \in E^+(S/\rho)$ if and only if $a\rho e$ for some $e \in E^+(S)$.*

Pastijn and Petrich [11] characterized the congruences on a regular semigroup by their trace and kernel. Similarly, the following notions can be developed and results can be proved.

The trace of a congruence ρ on an additive regular semiring gives us how the additive idempotents are ρ-related and the kernel of ρ gives us the union of equivalence classes of all additive idempotents. Thus the trace and kernel of a congruence ρ is are defined by [12]:

$$\operatorname{tr}\rho = \rho \cap (E^+ \times E^+) \quad \text{and} \quad \ker\rho = \{a \in S : \ a\rho \in E^+(S/\rho)\}.$$

Since S is regular, by Lallement's Lemma,

$$\ker\rho = \{a \in S : \ a \ \rho \ e \ \text{ for some } \ e \in E^+\}$$

Lemma 2.4. *A congruence on an additive idempotent semiring S is uniquely determined by its kernel and its trace.*

Define two relations τ and κ on the lattice $\mathcal{C}(S)$ by: for $\rho, \sigma \in \mathcal{C}(S)$,

$$\rho\tau\sigma \quad \text{if} \quad \operatorname{tr}\rho = \operatorname{tr}\sigma \quad \text{and} \quad \rho\kappa\sigma \quad \text{if} \quad \ker\rho = \ker\sigma.$$

Then we have:

Lemma 2.5. *Let S be an additive regular semiring and $\rho \in \mathcal{C}(S)$. Then both, $\rho\tau$ and $\rho\kappa$ are intervals with the greatest and the least elements.*

For $\rho \in \mathcal{C}(S)$, we denote $\rho\tau = [\rho_{min}, \rho_{max}]$ and $\rho\kappa = [\rho^{min}, \rho^{max}]$.

Lemma 2.6. *Let S be an additive regular semiring. Then for every $\rho \in \mathcal{C}(S)$,*

$$\rho = \rho_{max} \cap \rho^{max}.$$

We use, whenever possible, the notations of Golan [5] and Howie [7].

3. Clifford and generalized Clifford Congruences

First we characterize ρ_{max} and ρ^{max} when ρ is a generalized Clifford congruence.

Lemma 3.1. *Let S be an additive regular semiring and ρ be a generalized Clifford congruence on S. Then*

(1) ρ_{max} *is given by:*

$$\rho_{\max} = \{(a,b) \in S \times S : \exists\, a' \in V^{+}(a),\, b' \in V^{+}(b) \ \text{ such that } (a+a')\ \rho\ (b+b')\}.$$

(2) ρ^{max} *is given by:*

$$\rho^{\max} = \{(a,b) \in S \times S : \exists\, b' \in V^{+}(b) \ \text{ such that } a+b' \in \ker\rho\}.$$

Proof. (1) It follows from the definition that $\rho_{\max}$ is reflexive and symmetric. Now let $a,b,c \in S$ such that $a\rho_{\max}b$ and $b\rho_{\max}c$. Then $(a+a')\rho(b+b')$ and $(b+b'')\rho(c+c')$ for some $a' \in V^{+}(a), b', b'' \in V^{+}(b)$ and $c' \in V^{+}(c)$. Since $(S/\rho,+)$ is an inverse semigroup, so $b'\rho b''$, and hence $(a+a')\rho(b+b')\rho(b+b'')\rho(c+c')$. Thus ρ_{max} is an equivalence relation. Let $a,b,c \in S$ and $a\rho_{max}b$. Then $(a+a')\rho(b+b')$ for some $a' \in V^{+}(a)$ and $b' \in V^{+}(b)$. Consider $c' \in V^{+}(c)$, $(a+c)' \in V^{+}(a+c)$ and $(b+c)' \in V^{+}(b+c)$. Since $(S/\rho,+)$ is an inverse semigroup, so $(a+c)'\rho(c'+a')$ and $(b+c)'\rho(c'+b')$. Then $((a+c)+(a+c)')\rho(a+c+c'+a')\rho(a+a'+c+c')$, since $(S/\rho,+)$ is a Clifford semigroup. Similarly $((b+c)+(b+c)')\rho(b+b'+c+c')$. Thus $((a+c)+(a+c)')\rho((b+c)+(b+c)')$ and hence $(a+c)\rho_{max}(b+c)$. Similarly, $(c+a)\rho_{max}(c+b)$. Now $ca' \in V^{+}(ca)$ and $a'c \in V^{+}(ac)$ implies that $ac\rho_{max}bc$ and $ca\rho_{max}cb$. Thus ρ_{max} is a congruence on S.

Also $\operatorname{tr}\rho = \operatorname{tr}\rho_{max}$. Now let ξ be a congruence on S such that $\operatorname{tr}\rho = \operatorname{tr}\xi$. Then for $a,b \in S$, $a\xi b$ implies that $(a+a')\xi(b+b')$ and so $(a+a')\rho(b+b')$. Thus $a\rho_{max}b$ and hence the result follows.

(2) Let $a,b \in S$ such that $a\rho^{max}b$. Then $a+b' \in \ker\rho$ and hence, by Lallement's Lemma, $a\rho + b'\rho = e\rho$ for some $e \in E^{+}$. Since $(S/\rho,+)$ is an inverse semigroup, so $b\rho + a'\rho = (a\rho + b'\rho)' = e\rho$ which implies that $b+a' \in \ker\rho$. Thus $b\rho^{max}a$, and so ρ^{max} is symmetric. Now let $a,b,c \in S$, and $a\rho^{max}b$ and $b\rho^{max}c$. Then there are $e,f \in E^{+}$ such that $(a+b')\rho e$ and $(b+c')\rho f$. Then $(a+c'+(c+b+b'+c'))\rho(a+b+b'+c')\rho(e+f)$. Since $(S/\rho,+)$ is an inverse semigroup, $(c+b'+b+c')\rho, (e+f)\rho \in E^{+}(S/\rho)$ and hence $(a+c'+(c+b+b'+c'+e+f))\rho(c+b+b'+c'+e+f)$. Then $(c+b'+b+c'+e+f)\rho \in E^{+}(S/\rho)$ implies that $(a+c')\rho \in E^{+}(S/\rho)$, since S/ρ is a generalized Clifford semiring. Hence $a+c' \in \ker\rho$, by the Lallement's Lemma, and so $a\rho^{max}c$. Thus ρ^{max} is an equivalence relation.

Let $a,b,c \in S$ and $a\rho^{max}b$. Then $(a+b')\rho e$ for some $e \in E^{+}$. Consider $b' \in V^{+}(b), c' \in V^{+}(c)$ and $(b+c)' \in V^{+}(b+c)$. Since $(S/\rho,+)$ is an inverse semigroup, so $(b+c)'\rho(c'+b')$. Then $(a+c+(b+c)')\rho(a+c+c'+b')\rho(a+b'+b+c+c'+b')\rho g$ for some $g \in E^{+}$. Thus $(a+c)\rho^{max}(b+c)$. Similarly $(c+a)\rho^{max}(c+b)$. Also $ac\rho^{max}bc$ and $ca\rho^{max}cb$ follows form the fact that $ce, ec \in E^{+}$, $b'c\rho(bc)'$ and $cb'\rho(cb)'$ for every $b' \in V^{+}(b), (bc)' \in V^{+}(bc)$ and $(cb)' \in V^{+}(cb)$. Thus ρ^{max} is a congruence on S.

Now $a \in \ker\rho^{max}$ implies that $a+e \in \ker\rho$ for some $e \in E^{+}$. Then there is $f \in E^{+}$ such that $(a+e)\rho f$ and so $a+(e+f)\rho(e+f)$. This implies that

$a \in \ker\rho$, since S/ρ is a generalized Clifford semiring. Thus $\ker\rho^{max} \subseteq \ker\rho$. Since $\rho \subseteq \rho^{max}$ reverse inclusion follows directly. Hence $\ker\rho = \ker\rho^{max}$. Let ξ be a congruence on S such that $\ker\rho = \ker\xi$, and $a, b \in S$ such that $a\xi b$. Then $(a+b')\xi(b+b')$ implies that $a+b' \in \ker\xi = \ker\rho$, and hence $a\rho^{max}b$. Thus $\xi \subseteq \rho^{max}$. ■

On a semiring S, we denote the least distributive lattice congruence on S by η, the least b-lattice congruence on S by ν and the least skew-ring congruence on S by σ. If ρ is a generalized Clifford congruence on S then $(S/\rho, +)$ is a Clifford semigroup and so $(a+e)\rho(e+a)$ for every $a \in S$ and $e \in E^+$.

Theorem 3.2. *Let S be an additive regular semiring and ρ be a congruence on S. Then the following statements are equivalent:*

(1) *ρ is a generalized Clifford congruence,*

(2) *ρ_{max} is a b-lattice congruence and ρ^{max} is a skew-ring congruence on S,*

(3) *$\rho_{max} = \rho \vee \nu$ and $\rho^{max} = \rho \vee \sigma$,*

(4) *$tr\rho = tr(\rho \vee \nu)$ and $ker\rho = ker(\rho \vee \sigma)$.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in S$, and consider $a' \in V^+(a)$, $b' \in V^+(b)$, $(a+b)' \in V^+(a+b)$ and $(b+a)' \in V^+(b+a)$. Since $(S/\rho, +)$ is an inverse semigroup, $(a+b)'\rho(b'+a')$ and $(b+a)'\rho(a'+b')$. Then $((a+b)+(a+b)')\rho(a+b+b'+a')\rho(b+b'+a+a')\rho(b+a+a'+b')\rho((b+a)+(b+a)')$ shows that $a+b\rho_{max}b+a$. Consider $(a+a)' \in V^+(a+a)$. Then $(a+a)'\rho(a'+a')$ which implies that $((a+a)+(a+a)')\rho(a+a+a'+a')\rho(a+a'+a+a') = a+a'$. Thus $(a+a)\rho_{max}a$. Now consider $a^{2'} \in V^+(a^2)$. Since $aa' \in V^+(a^2)$, so $a^{2'}\rho aa'$, and since ρ is a generalized Clifford congruence $a(a+a')\rho(a+a')$. Thus $(a^2+a^{2'})\rho(a^2+aa')\rho a(a+a')\rho(a+a')$ which implies that $a^2\rho_{max}a$. Therefore ρ_{max} is a b-lattice congruence on S.

Let $e, f \in E^+$. Since ρ is an inverse semigroup, so $(e+f)\rho \in E^+(S/\rho)$. Then $e+f \in \ker\rho$, by Lallement's Lemma. This implies that $e\rho^{max}f$. Thus ρ^{max} is a skew-ring congruence on the semiring S.

(2) $\Rightarrow$ (3) By our hypothesis, $\nu \subseteq \rho_{max}$. Also $\rho \subseteq \rho_{max}$. Hence $\nu \vee \rho \subseteq \rho_{max}$. Again $ker\rho_{max} \subseteq S = ker\nu = ker(\nu \vee \rho)$ and $tr\rho_{max} = tr\rho \subseteq tr(\nu \vee \rho)$ implies that $\rho_{max} \subseteq \nu \vee \rho$. Thus $\rho_{max} = \nu \vee \rho$. Similarly $\rho^{max} = \rho \vee \sigma$.

(3) $\Rightarrow$ (4) Follows directly.

(4) $\Rightarrow$ (1) Let $a \in S$. Then for any two $a', a'' \in V^+(a)$, $a'\nu a''$ implies that $(a+a')\nu(a+a'')$ and $(a'+a)\nu(a''+a)$. Then $tr\rho = tr(\rho \vee \nu)$ implies that $(a+a')\rho(a+a'')$ and $(a'+a)\rho(a''+a)$, and hence $a' = (a'+a+a')\rho(a''+a+a'') = a''$. Thus ρ is an inversive semiring congruence. Let $a \in S$. Then for all $a' \in V^+(a)$ both $a'\nu$ and $a\nu$ are inverses of $a\nu$ in S/ν. So $a'\nu a$, which implies that $(a+a')\,\nu\,a\,\nu\,(a'+a)$ that is $(a+a')\nu(a'+a)$. Consequently, $(a+a')\rho(a'+a)$.

Now $aa'\nu a^2$ implies that $(a^2+aa')\nu a^2\nu a\nu(a+a')$, that is $a(a+a')\nu(a+a')$, which implies that $a(a+a')\rho(a+a')$. Let $a, b \in S$ such that $a\rho+b\rho = b\rho$. Then $(a+b)\rho b$ that is $(a+b)+b' \in ker\rho = ker(\rho \vee \sigma)$. Since σ and hence $\rho \vee \sigma$ is skew-ring congruence and $b+b' \in E^+$, so $a \in ker(\rho\vee\sigma) = ker\rho$. Hence $a\rho+a\rho = a\rho$. Thus ρ is a generalized Clifford congruence on S. ■

Corollary 3.3. *On a semiring S the following conditions are equivalent:*

(i) *S is a generalized Clifford semiring.*

(ii) *For every $\rho \in \mathcal{C}(S), \rho_{max} = \rho \vee \nu$ and $\rho^{max} = \rho \vee \sigma$.*

Let S be a generalized Clifford semiring. Consider ε, the relation of equality on S. Then ε is a congruence on S and $\varepsilon_{max} = \nu$ and $\varepsilon^{max} = \sigma$. Hence $\varepsilon = \varepsilon_{max} \cap \varepsilon^{max}$ implies that S is a subdirect product of S/ν and S/σ. Thus every generalized Clifford semiring is a subdirect product of a b-lattice and a skew-ring [14, Theorem 2.7].

Theorem 3.4. *Let S be an additive regular semiring and ρ be a congruence on S. Then the following statements are equivalent:*

(1) *ρ is a Clifford congruence on S;*

(2) *ρ_{max} is a distributive lattice congruence on S and ρ^{max} is a skew-ring congruence on S;*

(3) *$\rho_{max} = \rho \vee \eta$ and $\rho^{max} = \rho \vee \sigma$;*

(4) *$tr\rho = tr(\rho \vee \eta)$ and $ker\rho = ker(\rho \vee \sigma)$.*

Proof. (1) $\Rightarrow$ (2) Every Clifford semiring is a generalized Clifford semiring. So ρ_{max} is a b-lattice congruence, by Theorem 3.2. Let a, $b \in S$ and $a' \in V^+(a)$, $b' \in V^+(b)$. Then $ab' \in V^+(ab)$ and $b'a \in V^+(ba)$, and we have $ab+ab' = a(b+b')\rho(b+b')a = ba+b'a$. Thus $ab\rho_{max}ba$. Let $(a+ab)' \in V^+(a+ab)$. Then $(a+ab)'\rho(ab'+a')$, since $(S/\rho,+)$ is an inverse semigroup. Hence $(a+ab)+(a+ab)'\rho(a+ab+ab'+a')\rho(a+a'+a(b+b'))\rho a+a'$ which implies that $(a+ab)\rho_{max}a$. Hence ρ_{max} is a distributive lattice congruence.

It follows from Theorem 3.2, that ρ^{max} is a skew-ring congruence.

(2) $\Rightarrow$ (3) Similar to Theorem 3.2.

(3) $\Rightarrow$ (4) Follows directly.

(4) $\Rightarrow$ (1) From Theorem 3.2 it follows that ρ is a generalized Clifford congruence. Let $a, b \in S$ and $b' \in V^+(b)$. Then $ab' \in V^+(ab)$ and $b'a \in V^+(ba)$ implies that $(ab+ab')\eta ab\eta ba\eta(ba+b'a)$. Since $tr\rho = tr(\rho \vee \eta)$, this implies that $a(b+b')\rho(b+b')a$. Now consider $(a+ab)' \in V^+(a+ab)$ and $a' \in V^+(a)$. Then $(a+ab+(a+ab)')\eta(a+ab)\eta a\eta(a+a')$ implies that $(a+ab+(a+ab)')\rho(a+a')$. Since $(S/\rho,+)$ is an inversive semigroup, so $(a+ab)'\rho(ab'+a')$. Hence $(a+ab+$

$(a+ab)')\rho(a+ab+ab'+a')$ which implies that $(a+a(b+b')+a')\rho(a+a')$ and so $(a+a(b+b')+a'+a)\rho(a+a'+a)=a$. Then $(a+a(b+b'))\rho(a+a(b+b')+a'+a)\rho a$. Thus ρ is a Clifford congruence. ■

Corollary 3.5. *On a semiring S the following conditions are equivalent:*

(i) *S is a Clifford semiring.*
(ii) *For every $\rho \in \mathcal{C}(S)$, $\rho_{max} = \rho \vee \eta$ and $\rho^{max} = \rho \vee \sigma$.*

This corollary implies that every Clifford semiring is a subdirect product of a distributive lattice and a skew-ring.

4. The least generalized Clifford and Clifford congruence

In [8] LaTorre described a construction for the least semilattice of group congruence on a regular semigroup. Let S be a regular semigroup. For $a \in S$, $V(a)$ denotes the set of all inverses of a. A subset T of S is called self-conjugate if $x'Tx \subseteq T$ for all $x \in S$ and all $x' \in V(x)$; T is called full if $E \subseteq T$. Let $\mathcal{C}$ denote the collection of all full, self-conjugate subsemigroups of S and let $U = \bigcap_{T\in\mathcal{C}} T$ be the least member in $\mathcal{C}$. In her doctoral dissertation, Feigenbaum [2] proved that

Lemma 4.1 ([2]; Theorems 4.1, 4.2)**.** *For each H in $\mathcal{C}$, the relation*

$$\beta_H = \{(a,b) \in S\times S : xa = by \textit{ for some } x,y \in H\}$$

is a group congruence on S.

The least group congruence on S is given by $\sigma = \beta_U$.

LaTorre considered the least semilattice congruence η on a regular semigroup S and $Y = S/\eta$. Then $S = \cup_{\alpha\in Y} S_\alpha$ is a semilattice Y of its η-classes S_α which is a regular semigroup for each $\alpha \in Y$. Let $U_\alpha = U \cap S_\alpha$ for each $\alpha \in Y$. Then U_α is a full, self-conjugate subsemigroup of S_α. Hence the relation $\beta_{U_\alpha} = \{(a,b) \in S\times S : xa = by \text{ for some } x,y \in U_\alpha\}$ is a group congruence on S_α. Let $\xi = \cup_{\alpha\in D}\beta_{U_\alpha}$. Then ξ is a congruence on S such that $S/\xi = \cup_{\alpha\in Y} S_\alpha/\beta_{U_\alpha}$ is a semilattice Y of groups $S_\alpha/\beta_{U_\alpha}$. Thus the relation ξ is given by:

$$a\xi b \text{ if } a,b \in S_\alpha \text{ and } a\beta_{U_\alpha} b \text{ for some } \alpha \in Y,$$
$$\text{equivalently, } a\xi b \text{ if } a\eta b \text{ and } xa = by \text{ for some } x,y \in U \cap a\eta.$$

LaTorre [8] proved that

Lemma 4.2 ([8], Theorem 1)**.** *Let S be a regular semigroup. Then ξ is the least semilattice of groups congruence on S.*

Here we give the least distributive lattice of skew-rings congruence ξ on an additive regular semiring in a slight modified form than that given by Lattore [8]. A subset I of S is called self-conjugate if $x'+I+x \subseteq I$ for all $x \in S$ and $x' \in V^+(x)$, and I is called full if $E^+ \subseteq I$. Let $\mathcal{C}$ denote the collection of all full, self-conjugate ideals of S and let $U = \bigcap_{T\in\mathcal{C}} T$ be the least member in $\mathcal{C}$.

The following result can be proved similar to [2]:

Lemma 4.3. *For each I in $\mathcal{C}$, the relation $\beta_I = \{(a,b) \in S \times S : x + a = b + y$ for some $x, y \in I\}$ is a skew-ring congruence on S.*

The least skew-ring congruence on S is given by $\sigma = \beta_U$.

Now we are ready to prove the main theorems of this section.

Theorem 4.4. *Let S be an additive regular semiring. Then the least generalized Clifford congruence ζ on S is given by:*

$$a\zeta b \text{ if and only if } a\nu b \text{ and } x + a = b + y \text{ for some } x, y \in U.$$

Proof. As in the proof of Theorem 1 [8], it can be proved that ζ is an additive congruence on S such that $(S/\zeta, +)$ is an inverse semigroup. Let $a, b \in S$ be such that $a\zeta b$ and $c \in S$. Then there are $x, y \in U$ such that $a\nu b$ and $x + a = b + y$. Then $ca\nu cb$. Also $cx + ca = cb + cy$. Since U is an ideal $cx, cy \in U$. Therefore $ca\zeta cb$. Similarly $ac\zeta bc$. Thus ζ is an inversive semiring congruence on S.

Let $a \in S$ and $a' \in V^+(a)$. Then $(a + a')\nu(a' + a)$. Also $x + (a + a') = (a' + a) + y$ where $x = a' + a, y = a + a' \in E^+ \subseteq U$, since U is full. Hence $(a+a')\zeta(a'+a)$. Again $a(a+a')\nu(a^2+aa')\nu(a+a')$ and $x+a(a+a') = a+a'+y$ where $x = a(a + a'), y = a + a' \in U$. Therefore $a(a + a')\zeta(a + a')$. Let $c, d \in S$ be such that $(c + d)\zeta d$. Then there are $x, y \in U$ such that $x + d = c + d + y$. Let $c' \in V^+(c), d' \in V^+(d)$. Then $(c+x+d+d'+c')+c = c+c+(d+y+d'+c'+c)$. Since U is a full subsemiring $x + d + d', c' + c \in U$. U is self-conjugate. So $c+x+d+d'+c', d+y+d' \in U$. This implies that $c+x+d+d'+c', d+y+d'+c'+c \in U$. Also $(c + c)\nu c$. Therefore $(c + c)\zeta c$. Thus ζ is a generalized Clifford congruence on S.

Note that $\zeta = \nu \cap \sigma$. Let ρ be a generalized Clifford congruence on S. Then Theorem 3.2 implies that ρ_{max} is a b-lattice congruence on S and ρ^{max} is a skew-ring congruence on S which implies that $\nu \subseteq \rho_{max}$ and $\sigma \subseteq \rho^{max}$. Therefore $\zeta = \nu \cap \sigma \subseteq \rho_{max} \cap \rho^{max} = \rho$, by Lemma 2.6. Hence ζ is the least generalized Clifford congruence on S. ■

Theorem 4.5. *Let S be an additive regular semiring. Then the least Clifford congruence ξ on S is given by:*

$$a\xi b \text{ if and only if } a\eta b \text{ and } x+a=b+y \text{ for some } x,y\in U.$$

Proof. Theorem 4.4 implies that ξ is a generalized Clifford congruence. Let $a,b\in S$ and $a'\in V^+(a)$. $(S/\eta,\cdot)$ commutative. So $b(a+a')\eta(a+a')b$. Taking $x=(a+a')b$ and $y=b(a+a')$ we get that $b(a+a')\xi(a+a')b$. From the absorptive property of distributive lattice it follows that $(b+b(a+a'))\eta b$. Let $b'\in V(b)$. Then $(b+b')+(b+b(a+a'))=b+(b'+b+b(a+a'))$ where $b+b', b'+b+b(a+a')\in U$. Therefore $(b+b(a+a'))\xi b$. Hence ξ is a Clifford congruence on S.

Note that $\xi=\eta\cap\sigma$. Let ρ be a Clifford congruence on S. Then Theorem 3.4 implies that ρ_{max} is a distributive lattice congruence on S and ρ^{max} is a skew-ring congruence on S which implies that $\eta\subseteq\rho_{max}$ and $\sigma\subseteq\rho^{max}$. Therefore $\xi=\eta\cap\sigma\subseteq\rho_{max}\cap\rho^{max}=\rho$. Hence ξ is the least Clifford congruence on S. ∎

The least skew-ring congruence σ on an inversive semiring S is as follows:

$$a\sigma b \text{ if and only if } a+e=b+e \text{ for some } e\in E^+.$$

Hence the description of the least Clifford congruence, given by Sen, Ghosh and Mukhopadhaya [13], on an additive commutative inversive semiring follows as a corollary:

Corollary 4.6 [13]**.** *Let S be an inversive semiring. Then the least Clifford congruence ξ on S is given by:*

$$a\xi b \text{ if and only if } a\eta b \text{ and } a+e=b+e \text{ for some } e\in E^+.$$

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