CHARACTERIZATIONS OF ORDERED Γ-ABEL-GRASSMANN’S GROUPOIDS

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Abstract

In this paper, we introduced the concept of ordered Γ-AG-groupoids, Γ-ideals and some classes in ordered Γ-AG-groupoids. We have shown that every Γ-ideal in an ordered Γ-AG**-groupoid S is Γ-prime if and only if it is Γ-idempotent and the set of Γ-ideals of S is Γ-totally ordered under inclusion. We have proved that the set of Γ-ideals of S form a semilattice, also we have investigated some classes of ordered Γ-AG**-groupoid and it has shown that weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular and (2, 2)-regular ordered Γ-AG**-groupoids coincide. Further we have proved that every intra-regular ordered Γ-AG**-groupoid is regular but the converse is not true in general. Furthermore we have shown that non-associative regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2, 2)-regular and strongly regular Γ-AG*-groupoids do not exist.

Keywords: ordered Γ-AG-groupoids, Γ-ideals, regular Γ-AG**-groupoids.

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1. Introduction

The concept of a left almost semigroup (LA-semigroup) [10] was first introduced by Kazim and Naseeruddin in 1972. In [7], the same structure is called a left invertive groupoid. Protić and Stevanović called it an Abel-Grassmann’s groupoid (AG-groupoid) [24].

An AG-groupoid is a groupoid $S$ whose elements satisfy the left invertive law $(ab)c = (cb)a$, for all $a, b, c \in S$. In an AG-groupoid, the medial law [10] $(ab)(cd) = (ac)(bd)$ holds for all $a, b, c, d \in S$. An AG-groupoid may or may not contain a left identity. In an AG-groupoid $S$ with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds for all $a, b, c, d \in S$. If an AG-groupoid contains a left identity, then by using medial law, we get $a(bc) = b(ac)$, for all $a, b, c \in S$.

The concept of ordered Γ-semigroups has been studied by many mathematicians, for instance, Chinram et al. [1], Hila et al. [2, 3, 4, 5, 6], Iampan [8, 9] and Kwon et al. [15, 16, 17, 18]. Also see [25].

In this paper, we have introduced the notion of ordered Γ-AG∗∗-groupoids. Here, we have explored all basic ordered Γ-ideals, which includes ordered Γ-ideals (left, right, two-sided) and some classes of ordered Γ-AG-groupoids.

Definition 1. Let $S$ and $\Gamma$ be two non-empty sets, then $S$ is said to be a Γ-AG-groupoid if there exists a mapping $S \times \Gamma \times S \rightarrow S$, written $(x, \gamma, y)$ as $x \gamma y$, such that $S$ satisfies the left invertive law, that is

\[(x \gamma y) \delta z = (z \gamma y) \delta x, \text{ for all } x, y, z \in S \text{ and } \gamma, \delta \in \Gamma.\]

Definition 2. A Γ-AG-groupoid $S$ is called a Γ-medial if it satisfies the medial law, that is

\[(x \alpha y) \beta (s \gamma t) = (x \alpha s) \beta (y \gamma t), \text{ for all } x, y, s, t \in S \text{ and } \alpha, \beta, \gamma \in \Gamma.\]

Definition 3. A Γ-AG-groupoid $S$ is called a Γ-AG∗∗-groupoid if it satisfy the following law:

\[x \alpha (y \beta z) = y \alpha (x \beta z), \text{ for all } x, y, z \in S \text{ and } \alpha, \beta \in \Gamma.\]

Definition 4. A Γ-AG∗∗-groupoid $S$ is called a Γ-paramedial if it satisfies the paramedial law, that is

\[(x \alpha y) \beta (s \gamma t) = (t \alpha s) \beta (y \gamma x), \text{ for all } x, y, s, t \in S \text{ and } \alpha, \beta, \gamma \in \Gamma.\]

If an AG-groupoid (without left identity) satisfies medial law, then it is called an AG∗-groupoid [20].
An AG-groupoid has been widely explored in [12, 13, 14, 21] and [24]. An AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup with wide applications in theory of flocks [23].

**Definition 5.** An AG-groupoid $S$ is called a $\Gamma$-AG$^*$-groupoid [20], if the following hold:

$$ (a\beta b)\gamma c = b\beta(a\gamma c), \text{ for all } a, b, c \in S \text{ and } \beta, \gamma \in \Gamma. $$

**Definition 6.** In an AG$^*$-groupoid $S$, the following law holds (see [24])

$$ (x_1\alpha x_2)\beta(x_3\gamma x_4) = (x_{p(1)}\alpha x_{p(2)})\beta(x_{p(3)}\gamma x_{p(4)}) \text{ for all } \alpha, \beta, \gamma \in \Gamma $$

where $\{p(1), p(2), p(3), p(4)\}$ means any permutation on the set $\{1, 2, 3, 4\}$. It is an easy consequence that if $S = STS$, then $S$ becomes a commutative $\Gamma$-semigroup.

An AG-groupoid may or may not contains a left identity. The left identity of an AG-groupoid allow us to introduce the inverses of elements in an AG-groupoid. If an AG-groupoid contains a left identity, then it is unique [21].

**Definition 7.** An ordered $\Gamma$-AG-groupoid (po-$\Gamma$-AG-groupoid) is a structure $(S, \Gamma, \leq)$ in which the following conditions hold:

(i) $(S, \Gamma)$ is a $\Gamma$-AG-groupoid.

(ii) $(S, \leq)$ is a poset.

(iii) For all $a, b$ and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for all $\beta \in \Gamma$.

Let $S$ be an ordered $\Gamma$-AG-groupoid. For $H \subseteq S$, we define

$$ [H] = \{t \in S \mid t \leq h \text{ for some } h \in H\}. $$

For $H = \{a\}$, usually written as $(a)$.

**Definition 8.** A non-empty subset $A$ of an ordered $\Gamma$-AG-groupoid $S$ is called a $\Gamma$-left (resp. $\Gamma$-right) ideal of $S$ if

(i) $STA \subseteq A$ (resp. $A\Gamma S \subseteq A$), and

(ii) If $a \in A$ and $b$ is in $S$ such that $b \leq a$, then $b \in A$.

**Definition 9.** A non-empty subset $A$ of an ordered $\Gamma$-AG-groupoid $S$ is called a (\Gamma\text{-two-sided}) ideal of $S$ if $A$ is both $\Gamma$-left and $\Gamma$-right ideal of $S$.

**Definition 10.** A $\Gamma$-ideal $P$ of an ordered $\Gamma$-AG-groupoid $S$ is called $\Gamma$-prime if for any two $\Gamma$-ideals $A$ and $B$ of $S$ such that $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. 
Definition 11. A Γ-ideal $I$ of an ordered Γ-AG-groupoid $S$ is called Γ-completely prime if for any two elements $a$ and $b$ of $S$ and $\beta \in \Gamma$ such that $a\beta b \in I$, then $a \in I$ or $b \in I$.

Definition 12. A Γ-ideal $P$ of an ordered Γ-AG-groupoid $S$ is said to be Γ-semiprime if $I^2 \subseteq P$ implies that $I \subseteq P$, for any Γ-ideal $I$ of $S$.

Definition 13. A Γ-AG-groupoid $S$ is said to be Γ-fully semiprime if every Γ-ideal of $S$ is Γ-semiprime. An ordered Γ-AG-groupoid $S$ is called Γ-fully prime if every Γ-ideal of $S$ is Γ-prime.

Definition 14. The set of Γ-ideals of an ordered Γ-AG-groupoid $S$ is called Γ-totally ordered under inclusion if for all Γ-ideals $A, B$ of $S$, either $A \subseteq B$ or $B \subseteq A$ and is denoted by Γ-ideal$(S)$.

2. Ideals in Ordered Γ-AG-groupoid

In this section we developed some results on ideals in ordered AG-groupoid.

Lemma 1. Let $S$ be an ordered Γ-AG-groupoid, then the following are true:

(i) $A \subseteq (A\lambda)$, for all $A \subseteq S$,

(ii) If $A \subseteq B \subseteq S$, then $(A\lambda) \subseteq (B\lambda)$,

(iii) $(A\lambda)(B\lambda) \subseteq (A\Gamma B\lambda)$ for all subsets $A, B$ of $S$,

(iv) $(A\lambda) = ((A\lambda)\lambda)$ for all $A \subseteq S$,

(v) For every Γ-left (resp. Γ-right) ideal or Γ-bi-ideal $T$ of $S$, $(T\lambda) = T$,

(vi) $((A\lambda)(B\lambda)) = (A\Gamma B\lambda)$ for all subsets $A, B$ of $S$.

Proof. It is same as in [11].

Lemma 2. $(S\Gamma a\lambda)$, $(a\Gamma S\lambda)$ and $(S\Gamma a\Gamma S\lambda)$ are a Γ-left, a Γ-right and a Γ-ideal of an ordered Γ-AG*-groupoid $S$ respectively, for all $a$ in $S$ such that $(S) = (S\Gamma S\lambda)$.

Proof. Let $a$ be any element of $S$. Then it has to be shown that $(S\Gamma a\lambda)$ is the Γ-left ideal of $S$. For this consider an element $x$ in $S\Gamma(S\Gamma a\lambda)$, then $x = y\gamma z$ for some $y$ in $S$ and $\gamma$ in $(S\Gamma a\lambda)$ where $z \leq s\beta a$ for some $s\beta a$ in $S\Gamma a$ and $\gamma, \beta \in \Gamma$. Since $S = S\Gamma S\lambda$ so let $y = y_1\delta y_2$ for some $\delta \in \Gamma$ and $y_1, y_2 \in S$, then by using (4) and (1), we have

$$x \leq y\gamma(s\beta a) = (y_1\delta y_2)\gamma(s\beta a) = (a\delta s)\gamma(y_2\beta y_1) = ((y_2\beta y_1)\delta s)\gamma a \subseteq S\Gamma a.$$ 

Which implies that $x$ is in $(S\Gamma a\lambda)$. For the second condition of $(S\Gamma a\lambda)$ to be Γ-left ideal let $x$ be any element in $(S\Gamma a\lambda)$, then $x \leq s\beta a$ for some $s\beta a$ in $S\Gamma a$. Let $y$ be
any other element of $S$ such that $y \leq x \leq s\beta a$, which implies that $y$ is in $(S\Gamma a]$. Hence $(S\Gamma a]$ is the $\Gamma$-left ideal of $S$. It is to be noted that $(a\Gamma S]$ and $(S\Gamma a\Gamma S]$ can be shown $\Gamma$-right and $\Gamma$-two-sided ideal respectively with an analogy to the proof of $(S\Gamma a]$ to be $\Gamma$-left ideal of $S$. ■

**Proposition 1.** If $S$ is an ordered $\Gamma$-$AG$-groupoid such that $S = S\Gamma S$, then every $\Gamma$-right ideal of $S$ is a $\Gamma$-ideal.

**Proof.** Let $I$ be a $\Gamma$-right ideal of an ordered $\Gamma$-$AG$-groupoid $S$. Let $x \in S\Gamma(I]$ which implies that $x = y\gamma z$ for some $y \in S$ and $z \in (I]$ where $z \leq i$ for some $i \in I$. Since $S = S\Gamma S$ so let $y = y_1\delta y_2$ for some $\delta \in \Gamma$ and $y_1, y_2 \in S$, then by (1), we get

$$x \leq y\gamma i = (y_1\delta y_2)\gamma i = (i\delta y_2)\gamma y_1 \subseteq (I\Gamma S)\Gamma S \subseteq I.$$

Which implies that $x \in (I]$ and the second condition of $(I]$ to be $\Gamma$-left ideal holds obviously. Hence $I$ is a $\Gamma$-ideal of $S$. ■

**Remark 1.**

1. If $(S) = (S\Gamma S]$ then every $\Gamma$-right ideal is also a $\Gamma$-left ideal and $S\Gamma I \subseteq I\Gamma S$.
2. If $I$ is a $\Gamma$-right ideal of $S$, then $S\Gamma I$ is a $\Gamma$-left and $I\Gamma S$ is a $\Gamma$-right ideal of $S$.

**Lemma 3.** If $I$ is a $\Gamma$-left ideal of an ordered $\Gamma$-$AG^{**}$-groupoid $S$, then $(a\Gamma I]$ is a $\Gamma$-left ideal of $S$.

**Proof.** Let $I$ be a $\Gamma$-left ideal of an ordered $\Gamma$-$AG^{**}$-groupoid $S$. Let $x \in S\Gamma(a\Gamma I]$ which implies that $x = y\beta z$ for some $y \in S$ and $z \in (a\Gamma I]$ where $z \leq a\gamma i$ for some $a\gamma i \in a\Gamma I$ and $\beta, \gamma \in \Gamma$. Then by using (3), we get

$$x \leq y\beta(a\gamma i) = a\beta(y\gamma i) \subseteq a\Gamma(S\Gamma I) \subseteq a\Gamma I.$$

Which implies that $x \in (a\Gamma I]$ and for the second condition of $(a\Gamma I]$ to be $\Gamma$-left ideal let $x$ be any element in $(a\Gamma I]$ then $x \leq a\gamma i$ for some $a\gamma i \in a\Gamma I$. Let $y$ be any other element of $S$ such that $y \leq x \leq a\gamma i$, which implies that $y$ is in $(a\Gamma I]$. Hence $(a\Gamma I]$ is a $\Gamma$-left ideal of $S$. ■

**Lemma 4.** Intersection of two $\Gamma$-ideals of an ordered $\Gamma$-$AG$-groupoid $S$ is a $\Gamma$-ideal.

**Proof.** Assume that $P$ and $Q$ are any two $\Gamma$-ideals of an ordered $\Gamma$-$AG$-groupoid $S$. Let $x \in S\Gamma(P \cap Q)$, then $x = y\beta z$ for some $\beta \in \Gamma$, $y \in S$ and $z \in (P \cap Q]$ where $z \in (P]$ which implies that $z \leq a$ for some $a \in P$ and also $z \in (Q]$ which
implies that \( z \leq b \) for some \( b \in Q \). Then by using (2), we have
\[
x \leq y\beta a \subseteq S \Gamma P \subseteq P \text{ and } x \leq y\beta b \subseteq S \Gamma Q \subseteq Q.
\]
This shows that \( x \in (P) \cap (Q) = (P \cap Q) \) and the second condition of \((P \cap Q)\) to be \( \Gamma \)-left ideal is obvious. Similarly \((P \cap Q)\) is a \( \Gamma \)-right ideal of \( S \). Hence \((P \cap Q)\) is a \( \Gamma \)-ideal of \( S \).

\begin{lemma}
If \( I \) is a \( \Gamma \)-left ideal of an ordered \( \Gamma \)-AG\(\ast\ast\)-groupoid \( S \), then \((I \Gamma I)\) is a \( \Gamma \)-ideal of \( S \).
\end{lemma}

\begin{proof}
Let \( I \) be a \( \Gamma \)-left ideal of an ordered \( \Gamma \)-AG\(\ast\ast\)-groupoid \( S \). Let \( x \in (I \Gamma I) \Gamma S \), then \( x = k \alpha s \) for some \( k \in (I \Gamma I) \) and \( s \in S \), where \( k \leq i\beta j \) for some \( i\beta j \in I \Gamma I \) and \( \alpha, \beta \in \Gamma \). Now by using (1), we get
\[
x \leq (i\beta j)\alpha s = (s\beta j)\alpha i \subseteq I \Gamma I.
\]
Now let \( x \in S \Gamma (I \Gamma I) \), then \( x = s \alpha k \) for some \( s \in S \) and \( k \in (I \Gamma I) \), where \( k \leq i\beta j \) for some \( i\beta j \in I \Gamma I \) and \( \alpha, \beta \in \Gamma \). Now by using (3), we get
\[
x \leq s \alpha(i\beta j) = i\alpha(s\beta j) \subseteq I \Gamma I.
\]
This implies that \( x \in (I \Gamma I) \) and for the second condition of \((I \Gamma I)\) to be a \( \Gamma \)-ideal, let \( x \) be any element in \((I \Gamma I)\) then \( x \leq i\beta j \) for some \( i\beta j \in I \Gamma I \). Let \( y \) be any other element of \( S \) such that \( y \leq x \leq i\beta j \), which implies that \( y \) is in \((I \Gamma I)\). Hence \((I \Gamma I)\) is a \( \Gamma \)-ideal of \( S \).
\end{proof}

\begin{remark}
If \( I \) is a \( \Gamma \)-left ideal of \( S \) then \((I \Gamma I)\) is a \( \Gamma \)-ideal of \( S \).
\end{remark}

\begin{proposition}
A proper \( \Gamma \)-ideal \((M)\) of an ordered \( \Gamma \)-AG\(\ast\ast\)-groupoid \( S \) is minimal if and only if \((M) = ((a\Gamma a) \Gamma M)\) for all \( a \in S \).
\end{proposition}

\begin{proof}
Let \((M)\) be the minimal \( \Gamma \)-ideal of \( S \), as \((M \Gamma M)\) is a \( \Gamma \)-ideal so \((M) = (M \Gamma M)\). Now let \( x \in ((a\Gamma a) \Gamma M) \Gamma S \) then \( x = yaz \) for some \( y \) in \((a\Gamma a) \Gamma M\) and \( z \) in \( S \), where \( y \leq (a\gamma a) \beta m \) for some \( (a\gamma a) \beta m \) in \((a\Gamma a) \Gamma M\) and \( \alpha, \beta, \gamma \in \Gamma \). Now by using (1) and (4), we have
\[
x \leq ((a\gamma a) \beta m) \alpha z = (z \beta m) \alpha (a\gamma a) = (a\beta a) \alpha (m \gamma z) \subseteq (a\Gamma a) \Gamma (M \Gamma S) \subseteq (a\Gamma a) \Gamma M.
\]
Which implies that \( x \in ((a\Gamma a) \Gamma M)\).
Now let \( x \in S((a\Gamma a) \Gamma M) \) then \( x = sat \) for some \( s \) in \( S \) and \( t \) in \((a\Gamma a) \Gamma M\), where \( t \leq (a\gamma a) \beta m \) for some \( (a\gamma a) \beta m \) in \((a\Gamma a) \Gamma M\) and \( \alpha, \beta, \gamma \in \Gamma \), then by
using (3), we have

\[ x \leq sa((a^\gamma a)\beta m) = (a^\gamma a)\alpha(s\beta m) \subseteq (a^\Gamma a)\Gamma(S\Gamma M) \subseteq (a^\Gamma a)\Gamma M. \]

Which implies that \( x \in ((a^\Gamma a)\Gamma M) \), and for the second condition of \(((a^\Gamma a)\Gamma M)\) to be a \( \Gamma \)-ideal let \( x \) be any element in \(((a^\Gamma a)\Gamma M)\) then \( x \leq (a^\gamma a)\beta m \) for some \( (a^\gamma a)\beta m \) in \((a^\Gamma a)\Gamma M\). Let \( y \) be any other element of \( S \) such that \( y \leq x \leq (a^\gamma a)\beta m \), which implies that \( y \) is in \((a^\Gamma a)\Gamma M\). Hence \(((a^\Gamma a)\Gamma M)\) a \( \Gamma \)-ideal of \( S \) contain in \([M]\) and as \([M]\) is minimal so \([M] = ((a^\Gamma a)\Gamma)\Gamma M\).

Conversely, assume that \([M] = ((a^\Gamma a)\Gamma M)\) for all \( a \in S \). Let \([A]\) be the minimal \( \Gamma \)-ideal properly contain in \([M]\) containing \( a \), then \([M] = ((a^\Gamma a)\Gamma M) \subseteq [A] \), which is a contradiction. Hence \([M]\) is a minimal \( \Gamma \)-ideal.

A \( \Gamma \)-ideal \( I \) of an ordered \( \Gamma \)-AG-groupoid \( S \) is called minimal if and only if it does not contain any \( \Gamma \)-ideal of \( S \) other than itself.

**Theorem 6.** If \( I \) is a minimal \( \Gamma \)-left ideal of an ordered \( \Gamma \)-AG\(^*\)-groupoid \( S \), then \(((a^\Gamma a)\Gamma(I\Gamma I))\) is a minimal \( \Gamma \)-ideal of \( S \).

**Proof.** Assume that \( I \) is a minimal \( \Gamma \)-left ideal of an ordered \( \Gamma \)-AG\(^*\)-groupoid \( S \). Now let \( x \in ((a^\Gamma a)\Gamma(I\Gamma I))\Gamma S \) then \( x = yaz \) for some \( y \) in \(((a^\Gamma a)\Gamma(I\Gamma I))\) and \( z \) in \( S \) where \( y \leq (a^\delta a)\delta(i\gamma j) \) for some \( i, j \) in \( I \) and \( \alpha, \beta, \gamma, \delta \in \Gamma \), then by using (1) and (4), we have

\[ x \leq ((a^\delta a)\beta(i\gamma j))\alpha z = (z\beta(i\gamma j))\alpha(a^\delta a) = (a^\beta a)\alpha((i\gamma j)\delta z) = (a^\delta a)\alpha((z\gamma j)\delta i) \subseteq (a^\Gamma a)\Gamma((S\Gamma(I\Gamma I)) \subseteq (a^\Gamma a)\Gamma(I\Gamma I). \]

Which implies that \( x \in ((a^\Gamma a)\Gamma(I\Gamma I)) \) and for the second condition of \(((a^\Gamma a)\Gamma(I\Gamma I))\) to be a \( \Gamma \)-ideal let \( x \) be any element in \(((a^\Gamma a)\Gamma(I\Gamma I))\) then \( x \leq (a^\delta a)\beta(i\gamma j) \) for some \( (a^\delta a)\beta(i\gamma j) \) in \(((a^\Gamma a)\Gamma(I\Gamma I))\). Which shows that \(((a^\Gamma a)\Gamma(I\Gamma I))\) is a \( \Gamma \)-right ideal of \( S \). Similarly \(((a^\Gamma a)\Gamma(I\Gamma I))\) is a \( \Gamma \)-left ideal so is \( \Gamma \)-ideal. Let \( H \) be a non-empty \( \Gamma \)-ideal of \( S \) properly contained in \(((a^\Gamma a)\Gamma(I\Gamma I))\). Define \( H' = \{r \in I : a^\psi r \in H\} \). Then \( a^\psi(s^\xi y) = s^\psi(a^\xi y) \in S\Gamma H \subseteq H \) imply that \( H' \) is a \( \Gamma \)-left ideal of \( S \) properly contained in \( I \). But this is a contradiction to the minimality of \( I \). Hence \(((a^\Gamma a)\Gamma(I\Gamma I))\) is a minimal \( \Gamma \)-ideal of \( S \).

**Theorem 7.** An ordered \( \Gamma \)-AG\(^*\)-groupoid \( S \) is \( \Gamma \)-fully prime if and only if every \( \Gamma \)-ideal is \( \Gamma \)-idempotent and \( \Gamma \)-ideal \((I\Gamma I)\) is \( \Gamma \)-totally ordered under inclusion.

**Proof.** Assume that an ordered \( \Gamma \)-AG\(^*\)-groupoid \( S \) is \( \Gamma \)-fully prime. Let \( I \) be the \( \Gamma \)-ideal of \( S \). Then by Lemma 5, \((I\Gamma I)\) becomes a \( \Gamma \)-ideal of \( S \) and obviously \( I\Gamma I \subseteq I \) and by Lemma 1, \((I\Gamma I) \subseteq (I)\). Now

\[ (I\Gamma I) \subseteq (I\Gamma I) \text{ yields } (I) \subseteq (I\Gamma I) \text{ and hence } \]
Let $P, Q$ be $\Gamma$-ideals of $S$ and $PTQ \subseteq P$, $PTQ \subseteq Q$ imply that $PTQ \subseteq P \cap Q$. Since $P \cap Q$ is $\Gamma$-prime, so $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$ which further imply that $P \subseteq Q$ or $Q \subseteq P$. Hence $\Gamma$-ideal($S$) is $\Gamma$-totally ordered under inclusion.

Converse is same as in [19].

If $S$ is an ordered $\Gamma$-AG$^{**}$-groupoid then the principal $\Gamma$-left ideal generated by $a$ is defined by $\langle a \rangle = S\Gamma a = \{s\gamma a : s \in S, \gamma \in \Gamma\}$, where $a$ is any element of $S$. Let $P$ be a $\Gamma$-left ideal of an ordered $\Gamma$-AG-groupoid $S$, $P$ is called $\Gamma$-quasi-prime if for $\Gamma$-left ideals $A, B$ of $S$ such that $A \Gamma B \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$. $P$ is called $\Gamma$-quasi-semiprime if for any $\Gamma$-left ideal $I$ of $S$ such that $I \Gamma I \subseteq P$, we have $I \subseteq P$.

**Theorem 8.** If $S$ is an ordered $\Gamma$-AG$^{**}$-groupoid, then a $\Gamma$-left ideal $P$ of $S$ is $\Gamma$-quasi-prime if and only if $a \Gamma (S \Gamma b) \subseteq P$ implies that either $a \in P$ or $b \in P$, where $a, b \in S$.

**Proof.** The proof is same as in [19].

**Corollary 1.** If $S$ is an ordered $\Gamma$-AG$^{**}$-groupoid, then a $\Gamma$-left ideal $P$ of $S$ is $\Gamma$-quasi-semiprime if and only if $a \Gamma (S \Gamma a) \subseteq P$ implies $a \in P$, for all $a \in S$.

**Proposition 3.** A $\Gamma$-ideal $I$ of an ordered $\Gamma$-AG-groupoid $S$ is $\Gamma$-prime if and only if it is $\Gamma$-semiprime and $\Gamma$-strongly irreducible.

**Proof.** The proof is obvious.

**Theorem 9.** Let $S$ be an ordered $\Gamma$-AG-groupoid and $\{P_i : i \in N\}$ be a family of $\Gamma$-prime ideals $\Gamma$-totally ordered under inclusion in $S$. Then $\cap P_i$ is a $\Gamma$-prime ideal.

**Proof.** The proof is same as in [19].

**Theorem 10.** For each $\Gamma$-ideal $I$ there exists a minimal $\Gamma$-prime ideal of $I$ in an ordered $\Gamma$-AG-groupoid $S$.

**Proof.** The proof is same as in [19].

**Definition 15.** An ordered $\Gamma$-AG-groupoid $(S, \Gamma, \leq)$ is called regular if $a \in ((a\Gamma S)\Gamma a]$ for every $a \in S$, or

1. For every $a \in S$ there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$.
2. $A \subseteq ((A\Gamma S)\Gamma A]$ for every $A \subseteq S$.

**Definition 16.** An ordered $\Gamma$-AG-groupoid $(S, \Gamma, \leq)$ is called weakly regular if $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for every $a \in S$, or
(1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$.
(2) $A \subseteq ((A\Gamma S)(\Gamma A)S)$ for every $A \subseteq S$.

**Definition 17.** An ordered $\Gamma$-AG-groupoid $(S, \Gamma, \leq)$ is called intra-regular if $a \in (\Gamma (a\delta a)\Gamma S)$ for every $a \in S$ and $\delta \in \Gamma$, or
(1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta(a\delta a))\gamma y$.
(2) $A \subseteq ((\Gamma(A\Gamma A))\Gamma S)$ for every $A \subseteq S$.

**Definition 18.** An ordered $\Gamma$-AG-groupoid $(S, \Gamma, \leq)$ is called right regular if $a \in (\Gamma (a\delta a)\Gamma S)$ for every $a \in S$ and $\delta \in \Gamma$, or
(1) For every $a \in S$ there exist $x \in S$ and $\beta, \delta \in \Gamma$ such that $a \leq (x\beta(a\delta a))\gamma a$.
(2) $A \subseteq (\Gamma(A\Gamma A))\Gamma S$ for every $A \subseteq S$.

**Definition 19.** An ordered $\Gamma$-AG-groupoid $(S, \Gamma, \leq)$ is called left regular if $a \in (\Gamma (a\delta a)\Gamma S)$ for every $a \in S$ and $\delta \in \Gamma$, or
(1) For every $a \in S$ there exist $x \in S$ and $\beta, \delta \in \Gamma$ such that $a \leq x\beta(a\delta a)$.
(2) $A \subseteq (\Gamma(A\Gamma A))\Gamma S$ for every $A \subseteq S$.

**Definition 20.** An ordered $\Gamma$-AG-groupoid $(S, \Gamma, \leq)$ is called left quasi regular if $a \in (\Gamma (a\delta a)\Gamma S)$ for every $a \in S$, or
(1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta(a\delta a))\gamma y$.
(2) $A \subseteq (\Gamma(A\Gamma A))\Gamma S$ for every $A \subseteq S$.

**Definition 21.** An ordered $\Gamma$-AG-groupoid $(S, \Gamma, \leq)$ is called strongly regular if $a \in (\Gamma (a\delta a)\Gamma S)$ and $a\beta x = x\beta a$ for every $a \in S$ and $\beta \in \Gamma$, or
For every \( a \in S \) there exist \( x \in S \) and \( \beta, \gamma \in \Gamma \) such that \( a \leq (a\beta x)\gamma a \) and \( a\beta x = x\beta a \).

(2) \( A \subseteq ((AG\Gamma)\Gamma A) \) for every \( A \subseteq S \).

**Example 1.** Let us consider an ordered AG-groupoid \( S = \{1, 2, 3\} \) in the following multiplication table.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Let us define \( \Gamma = \{\alpha, \beta, \gamma\} \) as follows.

\[
\begin{array}{c|ccc}
\alpha & 1 & 2 & 3 \\
\hline
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\beta & 1 & 2 & 3 \\
\hline
1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
3 & 2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|ccc}
\gamma & 1 & 2 & 3 \\
\hline
1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
3 & 2 & 2 & 3 \\
\end{array}
\]

Here \( S \) is a \( \Gamma \)-AG-groupoid because \((a\beta b)\gamma c = (c\beta b)\gamma a\) for all \( a, b, c \in S \) and \( S \) is non-associative, because \((1\alpha 2)\beta 3 \neq 1\alpha (2\beta 3)\). We define order \( \leq \) as:

\[\leq := \{(1, 1), (2, 2), (3, 3), (2, 1), (2, 3), (3, 1)\}.
\]

Clearly \((S, \leq)\) is a poset and for all \( a, b \) and \( x \in S \), \( a \leq b \) implies \( a\beta x \leq b\beta x \) and \( x\beta a \leq x\beta b \) for some \( \beta \in \Gamma \) so \( S \) is a ordered \( \Gamma \)-AG-groupoid.

Note that \( S \) is a \( \Gamma \)-ideal itself so by Lemma 1, \((S\Gamma S) \subseteq S \).

**Lemma 11.** If \( S \) is regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, \((2, 2)\)-regular and strongly regular ordered \( \Gamma \)-AG-groupoid, then \( S = (S\Gamma S) \).

**Proof.** Assume that \( S \) is a regular ordered \( \Gamma \)-AG-groupoid, then \((S\Gamma S) \subseteq S \) is obvious. Let for any \( a \in S \), \( a \in ((a\Gamma S)\Gamma a) \), then there exists \( x \in S \) and \( \beta, \gamma \in \Gamma \) such that \( a \leq (a\beta x)\gamma a \). Now \( a \leq (a\beta x)\gamma a \in S\Gamma S \), thus \( a \in (S\Gamma S) \).

Similarly if \( S \) is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, \((2, 2)\)-regular or strongly regular, then we can show that \( S = (S\Gamma S) \).}

The converse is not true in general, because in Example 2, \( S = (S\Gamma S) \) holds but \( S \) is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, \((2, 2)\)-regular and strongly regular, because \( 1 \in S \) is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, \((2, 2)\)-regular and strongly regular.
Example 2. Let us consider an ordered $\Gamma$-AG-groupoid $S = \{1, 2, 3, 4\}$ in the following Cayley’s table.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Let us define $\Gamma = \{\alpha, \beta, \gamma\}$ as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\beta$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Here $S$ is a $\Gamma$-AG-groupoid because $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in S$ and $S$ is non-associative, because $(1\alpha 2)\beta 3 \neq 1\alpha(2\beta 3)$. We define the order $\leq$ as:

$\leq := \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 4), (2, 4), (3, 4)\}$.

Clearly $(S, \leq)$ is a poset and for all $a, b$ and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for some $\beta \in \Gamma$ so $S$ is a ordered $\Gamma$-AG-groupoid. $A = \{1, 2, 4\}$ is an ideal of $S$ as $A\Gamma S \subseteq A$ and $STA \subseteq A$, also for every $1 \in A$ there exists $4 \in S$ such that $4 \leq 1 \in A$ implies that $4 \in A$, similarly for every $4 \in A$ there exists $2 \in S$ such that $2 \leq 4 \in A$ implies that $4 \in A$.

Theorem 12. If $S$ is an ordered $\Gamma$-AG**-groupoid, then $S$ is an intra-regular if and only if for all $a \in (S \setminus\{\gamma a\})$, $a \leq (x\beta a)\delta(\alpha \gamma z)$ holds for some $x, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Proof. Assume that $S$ is an intra-regular ordered $\Gamma$-AG**-groupoid. Let $a \in ((S \setminus\{\gamma a\})\Gamma S)$ for any $a \in S$ and $\gamma \in \Gamma$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta(a\gamma a))\delta y$. Now by using Lemma 11, $y \leq u\gamma v$ for some $u, v \in S$. Thus by using (2), (1) and (4), we have

$a \leq (x\beta(a\gamma a))\delta y = (a\beta(x\gamma a))\delta y = (y\beta(x\gamma a))\delta a = (y\beta(x\gamma a))\delta((x\beta(a\gamma a))\delta y)

\leq ((u\gamma v)\beta(x\gamma a))\delta((x\beta(a\gamma a))\delta y) = ((a\gamma x)\beta(v\gamma u))\delta((x\beta(a\gamma a))\delta y)

\leq ((a\gamma x)\beta t)\delta((x\beta(a\gamma a))\delta y) = (((x\beta(a\gamma a))\delta y)\beta t)\delta(a\gamma x)

= (((t\delta y)\beta(x\beta(a\gamma a)))\delta(a\gamma x) = (((a\gamma a)\delta x)\beta(y\beta t))\delta(a\gamma x)$.
≤ ((aγa)δx)βsxγx) = ((aδa)β(xγs))δ(aγx)
≤ ((aδa)βw)δ(aγx) = ((wδa)βa)δ(aγx) ≤ (zβa)δ(aγx)
= (xβa)δ(aγz),

where vγu ≤ t, yβt ≤ s, xγs ≤ w and wδa ≤ z for some t, s, w, z ∈ S.

Conversely, let for all a ∈ (S), a ≤ (xβa)δ(aγz) holds for some x, z ∈ S and β, γ, δ ∈ Γ. Now by using (3), (1), (2) and (4), we have

\[ a ≤ (xβa)δ(aγz) = aδ((xβa)γz) ≤ ((xβa)δ(aγz))δ((xβa)γz) = (aδ((xβa)γz))δ((xβa)γz) = (((xβa)γ(xβa))δ(zγz))δa = (((aβx)γ(aβx))δ(zγz))δa = (((aβx)δ((xβa)βx))δ(zγz))δa = (((zγz)δ((xβa)βx)γa)δa = (((zγz)δ((zγz)βx)βx)γa)δa = (((zγz)δ((zγz)βx)βx)γa)δa = (((aβx)γ((xβa)βx)(zγz)))δa = (aδ((xβa)βx)δ(zγz)))δ(aδa) ≤ (aγt)δ(aδa), \] where (xβa)δ(aγz) ≤ t for some t ∈ S.

Now by using (4) and (1), we have

\[ a ≤ (aγt)δ(aδa) = (aγt)δ(aδa) = (aγt)δ(tδa) ≤ (aγt)δ(tδa) = (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) ≤ (tδ(aδa))γ(aγa)δ(aδa) \]

Which implies that \( a ∈ ((SG(aγa))ΓS) \), thus S is intra-regular. □

**Theorem 13.** If S is an ordered Γ-AG**∗∗**-groupoid, then the following are equivalent.

(i) S is weakly regular.
(ii) S is intra-regular.

**Proof.** (i)⇒(ii) Assume that S is a weakly regular ordered Γ-AG**∗∗**-groupoid. Let \( a ∈ ((aΓS)Γ(aΓS)) \) for any \( a ∈ S \), then there exist \( x, y ∈ S \) and \( β, γ, δ ∈ Γ \)
such that $a \leq (a\beta x)\delta(a\gamma y)$. Now by Lemma 11, let $x \leq s\psi t$ for some $s, t \in S$, $\psi \in \Gamma$ and $\gamma s \leq u \in S$, then by using (4) and (1), we have

$$a \leq (a\beta x)\delta(a\gamma y) = (y\beta a)\delta(x\gamma a) = ((x\gamma a)\beta a)\delta y \leq (((s\psi t)\gamma a)\beta a)\delta y$$

$$= ((a\gamma a)\beta (s\psi t))\delta y = ((t\gamma s)\beta (a\psi a))\delta y = ((t\gamma s)\beta (a\psi a))\delta y \leq (u\beta (a\psi a))\delta y$$

$$\in (S\Gamma(a\psi a))\Gamma S.$$ 

Which implies that $a \in ((S\Gamma(a\psi a))\Gamma S]$, thus $S$ is intra-regular.

(ii)$\implies$(i) is the same as (i)$\implies$(ii).

**Theorem 14.** If $S$ is an ordered $\Gamma$-AG*-groupoid, then the following are equivalent.

(i) $S$ is weakly regular.

(ii) $S$ is right regular.

**Proof.** (i)$\implies$(ii) Assume that $S$ is a weakly regular ordered $\Gamma$-AG*-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$ and by using Lemma 11, let $x\gamma y \leq t$ for some $t \in S$. Now by using (2), we have

$$a \leq (a\beta x)\delta(a\gamma y) = (a\beta a)\delta(x\gamma y) \leq (a\beta a)\delta t \in (a\beta a)\Gamma S.$$ 

Which implies that $a \in ((a\beta a)\Gamma S]$, thus $S$ is right regular.

(ii)$\implies$(i) It follows from Lemma 11 and (2). 

**Theorem 15.** If $S$ is an ordered $\Gamma$-AG*-groupoid, then the following are equivalent.

(i) $S$ is weakly regular.

(ii) $S$ is left regular.

**Proof.** (i)$\implies$(ii) Assume that $S$ is a weakly regular ordered $\Gamma$-AG*-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$. Now let $y\beta x \leq t$ for some $t \in S$ then by (2) and (4), we have

$$a \leq (a\beta x)\delta(a\gamma y) = (a\beta a)\delta(x\gamma y) = (y\beta x)\delta(a\gamma a) = (y\beta x)\delta(a\gamma a)$$

$$\leq t\delta(a\gamma a) \in S\Gamma(a\gamma a).$$ 

Which implies that $a \in (S\Gamma(a\gamma a)]$, thus $S$ is left regular.

(ii)$\implies$(i) It follows from Lemma 11, (4) and (2).
Lemma 16. Every weakly regular ordered $\Gamma$-AG$^{**}$-groupoid is regular.

Proof. Assume that $S$ is a weakly regular ordered $\Gamma$-AG$^{**}$-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$. Let $x\gamma y \leq t \in S$ and by using (1), (2), (4) and (3), we have

\[
a \leq (a\beta x)\delta(a\gamma y) = ((a\gamma y)\beta x)\delta a = ((x\gamma y)\beta a)\delta a \leq (t\beta a)\delta a
\]

\[
\leq (t\beta((a\beta x)\delta(a\gamma y)))\delta a = (t\beta((a\beta a)\delta(x\gamma y)))\delta a
\]

\[
= (t\beta((y\beta x)\delta(a\gamma a)))\delta a = (t\beta(a\delta((y\beta x)\gamma a)))\delta a
\]

\[
= (a\beta(t\delta((y\beta x)\gamma a)))\delta a \leq (a\beta u)\delta a, \text{ where } t\delta((y\beta x)\gamma a) \leq u \in S
\]

\[
e \in (a\Gamma S)\Gamma a.
\]

Which implies that $a \in ((a\Gamma S)\Gamma a]$, thus $S$ is regular.

The converse of above Lemma is not true in general, as can be seen from the following example.

Example 3 [24]. Let us consider an AG-groupoid $S = \{1, 2, 3, 4\}$ in the following Cayley’s table.

\[
\begin{array}{cccc}
  & 1 & 2 & 3 & 4 \\
1 & 2 & 2 & 4 & 4 \\
2 & 2 & 2 & 2 & 2 \\
3 & 1 & 2 & 3 & 4 \\
4 & 1 & 1 & 1 & 2 \\
\end{array}
\]

Let us define $\Gamma = \{\alpha, \beta, \gamma\}$ as follows:

\[
\begin{array}{cccc|cccc}
\alpha & 1 & 2 & 3 & 4 & \beta & 1 & 2 & 3 & 4 & \gamma & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 1 & 1 & 1 & 1 & 3 & 2 & 2 & 4 & 4 & 3 & 2 & 2 & 2 \\
4 & 1 & 1 & 1 & 1 & 4 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 3 & 4 \\
\end{array}
\]

Here $S$ is a $\Gamma$-AG-groupoid because $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in S$. We define order $\leq$ as:

\[
\leq := \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (4, 2)\}.
\]

Clearly $(S, \leq)$ is a poset and for all $a, b$ and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for some $\beta \in \Gamma$ so $S$ is a ordered $\Gamma$-AG-groupoid. Also $S$ is regular, because $1 \leq (1\gamma 3)\alpha 1$, $2 \leq (2\beta 1)\gamma 2$, $3 \leq (3\beta 3)\gamma 3$ and $4 \leq (4\gamma 3)\beta 4$, but $S$ is not weakly regular, because $1 \notin ((1\Gamma S)\Gamma (1\Gamma S))]$. 

Theorem 17. If $S$ is an ordered $\Gamma$-$AG^{**}$-groupoid, then the following are equivalent.

(i) $S$ is weakly regular.
(ii) $S$ is completely regular.

**Proof.** (i)$\implies$(ii) It follows from Theorems 14, 15 and Lemma 16.
(ii)$\implies$(i) It follows from Theorem 15. $lacksquare$

Theorem 18. If $S$ is an ordered $\Gamma$-$AG^{**}$-groupoid, then the following are equivalent.

(i) $S$ is weakly regular.
(ii) $S$ is left quasi regular.

**Proof.** The proof of this Lemma is straight forward. $lacksquare$

Theorem 19. If $S$ is an ordered $\Gamma$-$AG^{**}$-groupoid, then the following are equivalent.

(i) $S$ is $(2,2)$-regular.
(ii) $S$ is completely regular.

**Proof.** (i)$\implies$(ii) Assume that $S$ is a $(2,2)$-regular ordered $\Gamma$-$AG^{**}$-groupoid. Let $a \in (((a\delta a)\Gamma S)\Gamma(a\delta a))$ for any $a \in S$ and $\delta \in \Gamma$, then there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (((a\delta a)\beta x)\gamma(a\delta a))$. Now let $(a\delta a)\beta x \leq y \in S$, then we have

\[ a \leq ((a\delta a)\beta x)\gamma(a\delta a) \leq y\gamma(a\delta a) \in S\Gamma(a\delta a). \]

Which implies that $a \in (S\Gamma(a\delta a)]$, thus $S$ is left regular. Now by using (4), we have

\[ a \leq ((a\delta a)\beta x)\gamma(a\delta a) = (a\beta a)\gamma(x\delta(a\delta a)) \leq (a\beta a)\gamma((uv)v)\delta(a\delta a)). \]

Which implies that $a \in ((a\delta a)\Gamma S]$, thus $S$ is right regular. Now let $x \leq u\psi v$ for some $u, v \in S$ and $\psi \in \Gamma$, then by using (4), (1) and (3), we have

\[ a \leq ((a\delta a)\beta x)\gamma(a\delta a) = (a\beta a)\gamma(x\delta(a\delta a)) \leq (a\beta a)\gamma((uv)v)\delta(a\delta a)) = (a\beta a)\gamma((uv)v)\delta(a\delta a)) \leq (a\beta a)\gamma((uv)v)\delta(a\delta a)) = (a\beta a)\gamma((uv)v)\delta(a\delta a)) \leq (a\beta a)\gamma((uv)v)\delta(a\delta a)) \leq (a\beta a)\gamma((uv)v)\delta(a\delta a)) \leq (a\beta a)\gamma((uv)v)\delta(a\delta a)) \leq (a\beta a)\gamma((uv)v)\delta(a\delta a)) \leq (a\beta a)\gamma((uv)v)\delta(a\delta a)). \]
\( \leq (a\beta y)\gamma a, \) where \((a\delta t)\psi a \leq y \in S \)

\( \in (a\Gamma S)\Gamma a. \)

Which implies that \(a \in ((a\Gamma S)\Gamma a],\) so \(S\) is regular. Thus \(S\) is completely regular.

(ii)\(\implies\)(i) Assume that \(S\) is a completely regular ordered \(\Gamma\)-AG**-groupoid. Let \(a \in ((a\Gamma S)\Gamma a], a \in ((a\delta a)\Gamma S]\) and \(a \in (S\Gamma(a\delta a)\] for any \(a \in S,\) then there exist \(x, y, z \in S\) and \(\beta, \gamma, \psi, \xi, \delta \in \Gamma\) such that \(a \leq (a\beta x)\gamma a, a \leq (a\delta a)\psi y\) and \(a \leq z\xi(a\delta a)\). Now by using (4), (1) and (3), we have

\[
a \leq (a\beta x)\gamma a \leq (a\beta x)\gamma(z\xi(a\delta a)) = ((a\delta a)\beta z)\gamma(x\xi a) = ((x\xi a)\beta z)\gamma(a\delta a)
\]

\[
\leq (((a\delta a)\xi t)\beta z)\gamma(a\delta a), \text{ where } x\psi y \leq t \in S
\]

\[
= ((z\xi t)\beta(a\delta a))\gamma(a\delta a) = ((a\xi a)\beta(t\delta z))\gamma(a\delta a)
\]

\[
\leq ((a\xi a)\beta w)\gamma(a\delta a), \text{ where } t\delta z \leq w \in S
\]

\[
= ((a\xi a)\beta w)\gamma(a\delta a) \in ((a\xi a)\Gamma S)\Gamma(a\delta a).
\]

Which implies that \(a \in (((a\xi a)\Gamma S)\Gamma(a\delta a)]\), this shows that \(S\) is \((2,2)\)-regular. ■

**Lemma 20.** Every strongly regular ordered \(\Gamma\)-AG**-groupoid is completely regular.

**Proof.** Assume that \(S\) is a strongly regular ordered \(\Gamma\)-AG**-groupoid, then for any \(a \in S\) there exists \(x \in S\) and \(\beta, \gamma \in \Gamma\) such that \(a \leq (a\beta x)\gamma a\) and \(a\beta x = x\beta a\). Now by using (1), we have

\[
a \leq (a\beta x)\gamma a = (x\beta a)\gamma a = (a\beta a)\gamma x \subseteq (a\beta a)\Gamma S.
\]

Which implies that \(a \in (a^2\Gamma S]\), this shows that \(S\) is right regular and by Theorems 14 and 17, it is clear to see that \(S\) is completely regular. ■

**Theorem 21.** In an ordered \(\Gamma\)-AG**-groupoid \(S\), the following are equivalent.

(i) \(S\) is weakly regular,

(ii) \(S\) is intra-regular,

(iii) \(S\) is right regular,

(iv) \(S\) is left regular,

(v) \(S\) is left quasi regular,
Characterizations of ordered $\Gamma$-Abel-Grassmann’s groupoids

(vi) $S$ is completely regular,

(vii) For all $a \in S$, there exist $x, y \in S$ such that $a \leq (x\beta a)\delta(a\gamma z)$ holds for some $x, z \in S$ and $\beta, \gamma, \delta \in \Gamma$,

(viii) $S$ is $(2,2)$-regular.

**Proof.** (i) $\iff$ (ii) It follows from Theorem 13.
   (ii) $\iff$ (iii) It follows from Theorems 13 and 14.
   (iii) $\iff$ (iv) It follows from Theorems 14 and 15.
   (iv) $\iff$ (v) It follows from Theorems 15 and 18.
   (v) $\iff$ (vi) It follows from Theorems 18 and 17.
   (vi) $\iff$ (i) It follows from Theorem 17.
   (ii) $\iff$ (vii) It follows from Theorem 12.
   (vi) $\iff$ (viii) It follows from Theorem 19.

**Remark 3.** Every intra-regular, right regular, left regular, left quasi regular $(2,2)$-regular and completely regular ordered $\Gamma$-$AG^{**}$-groupoids are regular.

The converse is not true in general, as can be seen from Example 3.

**Theorem 22.** Regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular $\Gamma$-$AG^{*}$-groupoids become a $\Gamma$-semigroups.

**Proof.** It follows from (6) and Lemma 11.

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**References**


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