

## CONGRUENCES AND BOOLEAN FILTERS OF QUASI-MODULAR $p$ -ALGEBRAS

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### Abstract

The concept of Boolean filters in  $p$ -algebras is introduced. Some properties of Boolean filters are studied. It is proved that the class of all Boolean filters  $BF(L)$  of a quasi-modular  $p$ -algebra  $L$  is a bounded distributive lattice. The Glivenko congruence  $\Phi$  on a  $p$ -algebra  $L$  is defined by  $(x, y) \in \Phi$  iff  $x^{**} = y^{**}$ . Boolean filters  $[F_a], a \in B(L)$ , generated by the Glivenko congruence classes  $F_a$  (where  $F_a$  is the congruence class  $[a]\Phi$ ) are described in a quasi-modular  $p$ -algebra  $L$ . We observe that the set  $F_B(L) = \{[F_a] : a \in B(L)\}$  is a Boolean algebra on its own. A one-one correspondence between the Boolean filters of a quasi-modular  $p$ -algebra  $L$  and the congruences in  $[\Phi, \nabla]$  is established. Also some properties of congruences induced by the Boolean filters  $[F_a], a \in B(L)$  are derived. Finally, we consider some properties of congruences with respect to the direct products of Boolean filters.

**Keywords:**  $p$ -algebras, quasi-modular  $p$ -algebras, Boolean filters, direct products, congruences.

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## 1. INTRODUCTION

The notion of pseudo-complements was introduced in semi-lattices and distributive lattices by O. Frink [7] and G. Birkhof [3]. The pseudo-complements in Stone algebras were studied and discussed by R. Balbes [1], O. Frink [7] and G. Grätzer [4] etc. Recently, the concept of Boolean filter of pseudo-complemented distributive lattices was introduced by M. Sambasiva Rao and K.P. Shum in [9].

In this paper, we further study the Boolean filters in a  $p$ -algebra  $L$  and many properties of Boolean filters are also given. We observe that every maximal filter of  $L$  is a Boolean filter, however the converse of this statement is not true. It is observed that a filter  $F$  of a  $p$ -algebra  $L$  is a prime Boolean filter if and only if it is a maximal filter. We will give a characterization theorem of Boolean filters of a quasi-modular  $p$ -algebra  $L$ . We also notice that the set of all Boolean filters of a quasi-modular  $p$ -algebra forms a bounded distributive lattice. Then, we introduce a Boolean filter  $[F_a]$  which is generated by the congruence class  $F_a$  (notice that  $[x]\Phi$  is denoted by  $F_a$ ) of the Glivenko congruence relation  $\Phi$  on  $L$ , where  $a$  is a closed element of a quasi-modular  $p$ -algebra  $L$ . It is proved that the set  $F_B(L) = \{[F_a] : a \in B(L)\}$  forms a Boolean algebra on its own. We also observe that  $F_B(L)$  is isomorphic to  $B(L)$ . The relationship between the Boolean filters and the congruences in  $[\Phi, \nabla]$  of a quasi-modular  $p$ -algebra  $L$  is introduced. It is proved that there is a one-one correspondence between the congruences in  $[\Phi, \nabla]$  and the Boolean filters of  $L$ . We also prove that the Boolean algebras  $F_B(L)$  and  $Con_B(L) = \{\theta_{[F_a]} : a \in B(L)\}$  are isomorphic, where  $\theta_{[F_a]}$  is the congruence on  $L$  induced by a Boolean filter  $F_a$  for a closed element  $a$  of  $L$ . Moreover, we show that the Boolean algebra  $Con_B(L)$  can be embedded into the interval  $[\Phi, \nabla]$  of  $Con(L)$ . It is proved that the lattice of all Boolean filters of a finite quasi-modular  $p$ -algebra  $L$  is isomorphic to the sublattice  $[\Phi, \nabla]$  of  $Con(L)$ . Finally, some properties of congruences with respect to the direct products of Boolean filters are explored and investigated.

Several results of [9] are still possible for  $p$ -algebras or quasi-modular  $p$ -algebras. Namely, Lemma 3.2, Lemma 3.3, Theorem 3.4, Theorem 3.6 and Theorem 6.1 correspond respectively to Proposition 2.3, Corollary 2.4, Theorem 2.6, Theorem 2.8 and Theorem 2.10 from [9].

## 2. PRELIMINARIES

In this section, we cite some known definitions and basic results which can be found in the papers [2, 5, 6, 7, 8] and [10].

A  $p$ -algebra is a universal algebra  $(L, \vee, \wedge, *, 0, 1)$ , where  $(L, \vee, \wedge, 0, 1)$  is a bounded lattice and the unary operation  $*$  is defined by  $x \wedge a = 0 \Leftrightarrow x \leq a^*$ .

It is well known that the class of all  $p$ -algebras is equational. We now call

a  $p$ -algebra  $L$  distributive (modular) if the lattice  $(L, \vee, \wedge, 0, 1)$  is distributive (modular). The variety of modular  $p$ -algebras contains the variety of distributive  $p$ -algebras. We call a  $p$ -algebra quasi-modular if  $((x \wedge y) \vee z^{**}) \wedge x = (x \wedge y) \vee (z^{**} \wedge x)$ . Clearly, the class of all modular  $p$ -algebras is a subclass of the class of quasi-modular  $p$ -algebras. If the Stone identity  $x^* \vee x^{**} = 1$  holds in a  $p$ -algebra, then we simply call this  $p$ -algebra an  $S$ -algebra. We usually call a distributive  $S$ -algebra a Stone algebra.

An element  $a$  of a  $p$ -algebra  $L$  is called closed if  $a^{**} = a$ . Then  $B(L) = \{a \in L : a = a^{**}\}$  is the set of all closed elements of  $L$ . It is known that  $(B(L), \nabla, \wedge, 0, 1)$ , where  $a \nabla b = (a^* \wedge b^*)^*$ , forms a Boolean algebra. The set  $D(L) = \{x \in L : x^* = 0\} = \{x \vee x^* : x \in L\}$  of all dense elements of  $L$  is a filter of  $L$ . If  $L$  is an  $S$ -algebra, then  $(x \wedge y)^* = x^* \vee y^*$  for all  $x, y \in L$ . It follows that  $a \nabla b = a \vee b$  for all  $a, b \in B(L)$ .

For an arbitrary lattice  $L$ , the set  $F(L)$  of all filters of  $L$  ordered by the set inclusion forms a lattice. It is known that  $F(L)$  is modular (distributive) if and only if  $L$  is a modular (distributive) lattice. Let  $a \in L$  and  $[a]$  be the principal filter of  $L$  generated by  $a : [a] = \{x \in L : x \geq a\}$ . A proper filter  $P$  of  $L$  is called prime if  $x \vee y \in P$  implies  $x \in P$  or  $y \in P$  for all  $x, y \in L$ . We call a proper filter  $M$  of  $L$  maximal if  $M \subseteq G$  for no proper filter  $G$ .

The following results on quasi-modular  $p$ -algebras may be found in [8].

Let  $L$  be a quasi-modular  $p$ -algebra. Then every element  $x \in L$  can be represented by  $x = x^{**} \wedge (x \vee x^*)$ , where  $x^{**} \in B(L)$  and  $x \vee x^* \in D(L)$ . The relation  $\Phi$  of a quasi-modular  $p$ -algebra  $L$  is defined by  $(x, y) \in \Phi \Leftrightarrow x^{**} = y^{**}$  and is called the Glivenko congruence relation. It is known that the Glivenko congruence is indeed a congruence on  $L$  such that  $L/\Phi \cong B(L)$  holds. Every congruence class of  $\Phi$  contains exactly one element of  $B(L)$  which is the greatest element in the congruence class, the greatest element of a congruence class  $[x]\Phi$  is  $x^{**}$ . Hence  $\Phi$  partitions  $L$  into  $\{F_a : a \in B(L)\}$ , where  $F_a = \{x \in L : x^{**} = a\} = [a]\Phi$ . It is clear that  $F_0 = \{0\}$  and  $F_1 = D(L)$ .

We frequently use the following rules in the computations of  $p$ -algebras (see [5, 10]):

- (1)  $0^{**} = 0$  and  $1^{**} = 1$ ;
- (2)  $a \wedge a^* = 0$ ;
- (3)  $a \leq b$  implies  $b^* \leq a^*$ ;
- (4)  $a \leq a^{**}$ ;
- (5)  $a^{***} = a^*$ ;
- (6)  $(a \vee b)^* = a^* \wedge b^*$ ;
- (7)  $(a \wedge b)^* \geq a^* \vee b^*$ ;
- (8)  $(a \wedge b)^{**} = a^{**} \wedge b^{**}$ ;
- (9)  $(a \vee b)^{**} = (a^* \wedge b^*)^* = (a^{**} \vee b^{**})^{**}$ .

3. BOOLEAN FILTERS OF  $p$ -ALGEBRAS

In this section, we introduce the concept of Boolean filter of a  $p$ -algebra. Some properties of Boolean filters in a  $p$ -algebra are derived. We show that the maximal filter and prime Boolean filter of a  $p$ -algebra are equivalent. A characterization theorem of Boolean filters of a quasi-modular  $p$ -algebra will be given. Also we will prove that the set of all Boolean filters of a quasi-modular  $p$ -algebra is a bounded distributive lattice.

**Definition 3.1.** Let  $L$  be a  $p$ -algebra. Then, we call a filter  $F$  of  $L$  a *Boolean filter* if  $x \vee x^* \in F$  for each  $x \in L$ .

We now give some examples of Boolean filters of a  $p$ -algebra  $L$ .

**Example 3.2.** (1) Let  $L$  be a  $p$ -algebra. Then the filter  $D(L)$  is a Boolean filter of  $L$  as  $x \vee x^* \in D(L)$  for all  $x \in L$ . Moreover  $D(L)$  is the smallest Boolean filter of  $L$  and  $L$  is the greatest Boolean filter of  $L$ ;

(2) Let  $B$  be a Boolean algebra. Then any filter  $F$  of  $B$  is a principal Boolean filter as  $x \vee x^* = 1 \in F$  for each  $x \in B$ ;

(3) Let  $C_4 = \{0, a, b, c : 0 < a, b < c\}$  be a four element Boolean lattice and a pentagon  $N_5 = \{u, x, y, z, 1 : u < x < y < 1, u < z < 1, x \wedge z = y \wedge z = u, x \vee z = y \vee z = 1\}$ . Clearly  $L = C_4 \oplus N_5$  is a quasi-modular  $p$ -algebra where  $\oplus$  stands for ordinal sum. Then the set of all Boolean filters of  $L$  is  $\{\{c, u, x, y, z, 1\}, \{a, c, u, x, y, z, 1\}, \{b, c, x, y, z, 1\}, L\}$ ;

We observe that the filters  $\{x, y, 1\}, \{y, 1\}, \{z, 1\}, \{u, x, y, z, 1\}$  and  $\{1\}$  are not Boolean filters.

The results in Corollary 2.4, Theorem 2.6 and Theorem 2.8 from [9] are already stated for the class of all bounded distributive pseudocomplemented lattices. Now we have the following Lemma

**Lemma 3.3.** *Every maximal filter of a  $p$ -algebra  $L$  is a Boolean filter.*

**Proof.** Let  $M$  be a maximal filter of  $L$ . Suppose that  $x \vee x^* \notin M$  for some  $x \in L$ . Then  $M \vee [x \vee x^*] = L$ . Hence  $a \wedge b = 0$  for some  $a \in M, b \in [x \vee x^*]$ . Now we have the following implications:

$$\begin{aligned} a \wedge b = 0 &\Rightarrow 0 = a \wedge b \geq a \wedge (x \vee x^*) \geq (a \wedge x) \vee (a \wedge x^*) \\ &\Rightarrow a \wedge x = 0 \text{ and } a \wedge x^* = 0 \\ &\Rightarrow a \leq x^* \text{ and } a \leq x^{**} \\ &\Rightarrow a \leq x^* \wedge x^{**} = 0 \end{aligned}$$

This result leads to  $0 = a \in M$  which is a contradiction. Hence  $x \vee x^* \in M$  for all  $x \in L$ . Therefore,  $M$  is a Boolean filter of  $L$ . ■

We note that it is not true that every Boolean filter of  $L$  is a maximal filter. For, in Example 3.1(3), the filter  $\{c, u, x, y, z, 1\}$  of  $L$  is a Boolean filter but not a maximal filter of  $L$ .

The proof of Corollary 2.4 of [9] is still appropriate for the following Lemma.

**Lemma 3.4.** *A proper filter of a  $p$ -algebra  $L$  which contains either  $x$  or  $x^*$  for all  $x \in L$  is a Boolean filter.*

Now, we characterize the maximal filters of a  $p$ -algebra.

**Theorem 3.5.** *Let  $F$  be a proper filter of a  $p$ -algebra  $L$ . Then the following conditions are equivalent*

- (1)  $F$  is a maximal of  $L$ ,
- (2)  $x \notin F$  implies  $x^* \in F$  for all  $x \in L$ ,
- (3)  $F$  is prime Boolean.

**Proof.** The most proof of Theorem 2.6 in [9] is still appropriate for this Theorem, we need only to prove that  $F$  is a prime filter of  $L$  without using distributivity in (2)  $\Rightarrow$  (3). Suppose that  $F$  is not prime. Let  $x \vee y \in F$  such that  $x \notin F$  and  $y \notin F$ . By condition (2), we immediately see that  $x^* \in F$  and  $y^* \in F$ . Hence  $(x \vee y)^* = x^* \wedge y^* \in F$ . Therefore  $0 = (x \vee y) \wedge (x \vee y)^* \in F$ , a contradiction (as  $F$  is a proper filter of  $L$ ). This shows that  $F$  is prime. ■

By Definition of Boolean filter, the following lemma is obvious.

**Lemma 3.6.** *Let  $L$  be a  $p$ -algebra. Then the following statements hold.*

- (1) Any filter of  $L$  containing a Boolean filter is a Boolean filter,
- (2) The class  $BF(L)$  of all Boolean filters of  $L$  is a  $\{1\}$ -sublattice of the lattice  $F(L)$ .

We now characterize the Boolean filters of a quasi-modular  $p$ -algebra  $L$ .

**Theorem 3.7.** *Let  $F$  be a proper filter of a quasi-modular  $p$ -algebra  $L$ . Then the following conditions are equivalent.*

- (1)  $F$  is a Boolean filter;
- (2)  $x^{**} \in F$  implies  $x \in F$ ;
- (3) For  $x, y \in L, x^* = y^*$  and  $x \in F$  imply  $y \in F$ .

**Proof.** We prove only that (1)  $\Rightarrow$  (2) without using distributivity. Assume that  $F$  is a Boolean filter of  $L$ . Suppose that  $x^{**} \in F$ . Since  $F$  is a Boolean filter, we have  $x \vee x^* \in F$  and so  $x^{**} \wedge (x \vee x^*) \in F$ . Since  $L$  is a quasi-modular  $p$ -algebra, it follows that  $x = x^{**} \wedge (x \vee x^*) \in F$  and condition (2) hold.

The proofs (2)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (1) are given in Theorem 2.8 of [9]. ■

**Theorem 3.8.** *The class of Boolean filters  $BF(L)$  of a quasi-modular  $p$ -algebra  $L$  forms a bounded distributive lattice on its own.*

**Proof.** Clearly  $(BF(L), \vee, \wedge, D(L), L)$  is a bounded lattice. For any  $F, H, G \in BF(L)$ , we have  $(F \cap G) \vee (H \cap G) \subseteq (F \vee H) \cap G$ . Then we have to prove that  $(F \vee H) \cap G \subseteq (F \cap G) \vee (H \cap G)$ . Let  $x \in (F \vee H) \cap G$ . Then by the distributivity of  $B(L)$  and the fact that  $(F \cap G) \vee (H \cap G)$  is a Boolean filter we deduce the following implications.

$$\begin{aligned}
x \in (F \vee H) \cap G &\Rightarrow x \in F \vee H \text{ and } x \in G \\
&\Rightarrow x \geq f \wedge h \text{ for some } f \in F, h \in H \\
&\Rightarrow x^{**} \geq (f \wedge h)^{**} = f^{**} \wedge h^{**} \\
&\Rightarrow x^{**} = x^{**} \nabla (f^{**} \wedge h^{**}) \\
&\Rightarrow x^{**} = (x^{**} \nabla f^{**}) \wedge (x^{**} \nabla h^{**}) \\
&\Rightarrow x^{**} = (x \vee f)^{**} \wedge (x \vee h)^{**} \in (F \cap G) \vee (H \cap G) \\
&\Rightarrow x \in (F \cap G) \vee (H \cap G) \text{ by Theorem 3.6 (2)}.
\end{aligned}$$

Notice that  $(x \vee f)^{**} \geq (x \vee f) \in F \cap G$  and  $(x \vee h)^{**} \geq (x \vee h) \in H \cap G$ . It follows that  $(BF(L), \vee, \wedge, D(L), L)$  is a bounded distributive lattice.  $\blacksquare$

#### 4. BOOLEAN FILTERS VIA GLIVENKO CONGRUENCE CLASSES

In this section, we will show that, for every closed element  $a$  of a quasi-modular  $p$ -algebra  $L$ , the congruence class  $F_a = [a]\Phi$  of the Glivenko congruence relation  $\Phi$  on  $L$  generates a Boolean filter  $[F_a]$ . Many properties of the Boolean filters  $[F_a]$  for all  $a \in B(L)$  are discovered. Also, we derive that the set  $F_B(L) = \{[F_a] : a \in B(L)\}$  forms a Boolean algebra. It is proved that  $F_B(L)$  is isomorphic to  $B(L)$ . Also we express a Boolean filter as a union of certain elements of  $F_B(L)$ .

**Theorem 4.1.** *Let  $L$  be a quasi-modular  $p$ -algebra. Then for any two closed elements  $a, b$  of  $L$ , the following statements hold.*

- (1)  $[F_a] = \{x \in L : x^{**} \geq a\} = [a] \vee D(L)$ ,
- (2)  $[F_a]$  is a Boolean filter of  $L$ ,
- (3)  $a \leq b$  in  $B(L)$  if and only if  $[F_b] \subseteq [F_a]$  in  $F_B(L)$ ,
- (4) The set  $F_B(L)$  forms a Boolean algebra on its own.  
Moreover,  $B(L) \cong F_B(L)$ ,
- (5)  $[F_{a \wedge b}] = [F_a] \vee [F_b]$ ,
- (6)  $[F_{a \nabla b}] = [F_a] \cap [F_b]$ ,
- (7)  $[F_{a \vee b}] = [F_a] \cap [F_b]$  whenever  $L$  is a quasi-modular  $S$ -algebra.

**Proof.** (1)  $x \in [F_a]$  if and only if there exists a positive integer  $n$  and  $f_1, f_2, \dots, f_n \in F_a$  such that  $x \geq f_1 \wedge f_2 \wedge \dots \wedge f_n$ . Then  $f_i^{**} = a, i = 1, \dots, n$ . Let  $H = \{x \in L : x^{**} \geq a\}$ . Clearly  $H$  is a filter of  $L$ . Firstly we verify that  $[F_a] = H$ . Let  $x \in [F_a]$ . Now we have the following implications.

$$\begin{aligned}
 x \in [F_a] &\Rightarrow x \geq f_1 \wedge f_2 \wedge \dots \wedge f_n \text{ for some } f_1, \dots, f_n \in F_a \\
 &\Rightarrow x^{**} \geq (f_1 \wedge f_2 \wedge \dots \wedge f_n)^{**} \\
 &\Rightarrow x^{**} \geq f_1^{**} \wedge f_2^{**} \wedge \dots \wedge f_n^{**} \\
 &\Rightarrow x^{**} \geq a \text{ as } f_i^{**} = a \\
 &\Rightarrow x \in H.
 \end{aligned}$$

Then  $[F_a] \subseteq H$ . Conversely, suppose that  $H \not\subseteq [F_a]$ . Then there exists  $y \in [F_a]$  with  $y \notin H$ . Hence  $y \geq f_1 \wedge f_2 \wedge \dots \wedge f_n$  for some  $f_1, f_2, \dots, f_n \in F_a$ . It follows that  $y^{**} \geq (f_1 \wedge f_2 \wedge \dots \wedge f_n)^{**} = f_1^{**} \wedge f_2^{**} \wedge \dots \wedge f_n^{**} = a$  as  $f_i^{**} = a$ . Then  $y^{**} \geq a$ , which is a contradiction. Consequently  $H \subseteq [F_a]$ . Therefore  $[F_a] = \{x \in L : x^{**} \geq a\}$ .

Now we prove that  $[F_a] = [a] \vee D(L)$ . Let  $x \in [F_a]$ . Then we have the following implications.

$$\begin{aligned}
 x \in [F_a] &\Rightarrow x^{**} \geq a \\
 &\Rightarrow x = x^{**} \wedge (x \vee x^*) \geq a \wedge (x \vee x^*) \\
 &\Rightarrow x \in [a] \vee D(L) \text{ as } x \vee x^* \in D(L).
 \end{aligned}$$

Then  $[F_a] \subseteq [a] \vee D(L)$ . Conversely, let  $y \in [a] \vee D(L)$ . Then  $y \geq a \wedge z$  for some  $z \in D(L)$ . It follows that  $y^{**} \geq (a \wedge z)^{**} = a^{**} \wedge z^{**} = a \wedge 1 = a$  as  $a^{**} = a$  and  $z$  is a dense element of  $L$ . Therefore  $y \in [F_a]$  and  $[a] \vee D(L) \subseteq [F_a]$ .

(2) By (1) above,  $D(L) \subseteq [F_a]$  for all  $a \in B(L)$ . Now, by Lemma 3.6(1),  $[F_a]$  is a Boolean filter of  $L$ .

(3) Let  $a \leq b$  in  $B(L)$ . Let  $x \in [F_b]$ . Then  $x^{**} \geq b \geq a$ . Hence  $x \in [F_a]$  and  $[F_b] \subseteq [F_a]$ . Conversely, suppose that  $[F_b] \subseteq [F_a]$ . Since  $b \in F_b \subseteq [F_b] \subseteq [F_a]$ , then  $b = b^{**} \geq a$ .

(4) Define the mapping  $g : B(L) \rightarrow F_B(L)$  by  $g(a) = [F_a]$ . It follows easily from (3) above that  $g$  is an order anti-isomorphism between  $B(L)$  and  $F_B(L)$ . Then  $F_B(L)$  is a Boolean algebra and  $g$  is a Boolean anti-isomorphism. It follows that the mapping  $f : B(L) \rightarrow F_B(L)$  defined by  $f(a) = [F_{a^*}]$  is a Boolean isomorphism. Therefore  $B(L) \cong F_B(L)$ .

(5), (6) Since  $g : B(L) \rightarrow F_B(L)$  defined by  $g(a) = [F_a]$  is an anti-isomorphism by (4) above between Boolean algebras  $B = (B, \nabla, \wedge, *, 0, 1)$  and  $F_B(L) = (F_B(L), \vee, \cap, \bar{\phantom{x}}, D(L), L)$ , where  $\overline{[F_a]} = [F_{a^*}]$ , we get

$$\begin{aligned}
[F_{a \wedge b}] &= g(a \wedge b) \\
&= g(a) \vee g(b) \\
&= [F_a] \vee [F_b]
\end{aligned}$$

and

$$\begin{aligned}
[F_{a \nabla b}] &= g(a \nabla b) \\
&= g(a) \cap g(b) \\
&= [F_a] \cap [F_b]
\end{aligned}$$

(7) If  $L$  is a quasi-modular  $S$ -algebra, then  $(x \wedge y)^* = x^* \vee y^*$  for all  $x, y \in L$ . Hence for all  $a, b \in B(L)$  we have  $a \nabla b = (a^* \wedge b^*)^* = a^{**} \vee b^{**} = a \vee b$ . Therefore  $[F_{a \vee b}] = [F_a] \cap [F_b]$  immediately follows from (6). ■

**Corollary 4.2.** *Let  $L$  be a finite quasi-modular  $p$ -algebra. Then we have.*

- (1) *Every Boolean filter can be expressed as  $[F_a]$  for some  $a \in B(L)$ ;*
- (2)  *$BF(L) \cong F_B(L)$ .*

Now, we are going to represent a Boolean filter of a quasi-modular  $p$ -algebra  $L$  as a union of certain elements of  $F_B(L)$ . We have the following theorem.

**Theorem 4.3.** *Let  $F$  be a Boolean filter of  $L$ . Then  $F = \bigcup_{x \in F} [F_{x^{**}}]$ .*

**Proof.** Let  $x \in F$ . Then  $x^{**} \in F$  and  $x \vee x^* \in D(L) \subseteq F$ . Thus  $x = x^{**} \wedge (x \vee x^*) \in [x^{**}] \vee D(L) = [F_{x^{**}}]$ . Then  $F \subseteq \bigcup_{x \in F} [F_{x^{**}}]$ . Conversely, let  $y \in \bigcup_{x \in F} [F_{x^{**}}]$ . Then  $y \in [F_{z^{**}}]$  for some  $z \in F$ . Hence  $y^{**} \geq z^{**} \in F$ . Then  $y^{**} \in F$ , which implies  $y \in F$  as  $F$  is Boolean. Therefore  $\bigcup_{x \in F} [F_{x^{**}}] \subseteq F$ . ■

## 5. BOOLEAN FILTERS AND CONGRUENCES

In this section we investigate the relationships between the set of Boolean filters of a quasi-modular  $p$ -algebra  $L$  and the set of congruences on the interval  $[\Phi, \nabla]$ , where  $\nabla$  is the universal congruence on  $L$ .

We first state the following lemma.

**Lemma 5.1.** *Let  $\theta$  be a congruence relation on a quasi-modular  $p$ -algebra  $L$  such that  $\theta \in [\Phi, \nabla]$ . Then  $Coker\theta$  is a Boolean filter of  $L$ .*

**Proof.** Obviously  $Coker\theta = \{x \in L : (x, 1) \in \theta\}$  is a filter of  $L$ . For every  $x \in L$ ,  $(x \vee x^*)^{**} = 1 = 1^{**}$ . Then  $(x \vee x^*, 1) \in \Phi \subseteq \theta$ . Hence  $x \vee x^* \in Coker\theta$ . Therefore,  $Coker\theta$  is a Boolean filter of  $L$ . ■

For a Boolean filter  $F$  of a quasi-modular  $p$ -algebra  $L$ , define a relation  $\theta_F$  on  $L$  as follows :

$$(x, y) \in \theta_F \Leftrightarrow x^{**} \wedge a = y^{**} \wedge a \text{ for some } a \in F \cap B(L)$$

We now establish the following theorem for a Boolean filter of  $L$ .

**Theorem 5.2.** *Let  $F$  be a Boolean filter of a quasi-modular  $p$ -algebra  $L$ . Then the following statements hold.*

- (1)  $\theta_F$  is a congruence on  $L$  such that  $\Phi \subseteq \theta_F$ ;
- (2)  $[x^{**}]_{\theta_F} = [x]_{\theta_F}$ , for all  $x \in L$ ;
- (3)  $\text{Coker}\theta_F = F$ ;
- (4)  $\theta_{D(L)} = \Phi$  and  $\theta_L = \nabla$  whenever  $F$  is identical with  $D(L)$ , respectively,  $L$ ;
- (5)  $L/\theta_F$  is a Boolean algebra.

**Proof.** (1) Clearly,  $\theta_F$  is an equivalence relation on  $L$ . Now we prove that  $\theta_F$  is a lattice congruence on  $L$ . Let  $(x, y), (c, d) \in \theta_F$ . Then  $x^{**} \wedge a = y^{**} \wedge a$  and  $c^{**} \wedge b = d^{**} \wedge b$  for some  $a, b \in F \cap B(L)$ . Now we have the following equalities.

$$\begin{aligned} (x \wedge c)^{**} \wedge (a \wedge b) &= x^{**} \wedge c^{**} \wedge a \wedge b \\ &= y^{**} \wedge d^{**} \wedge a \wedge b \\ &= (y \wedge d)^{**} \wedge (a \wedge b). \end{aligned}$$

Then  $(x \wedge c, y \wedge d) \in \theta_F$ . Now by distributivity of  $B(L)$  we have

$$\begin{aligned} (x \vee c)^{**} \wedge (a \wedge b) &= (x^* \wedge c^*)^* \wedge (a \wedge b) \\ &= (x^{***} \wedge c^{***})^* \wedge (a \wedge b) \\ &= (x^{**} \nabla c^{**}) \wedge (a \wedge b) \\ &= (x^{**} \wedge a \wedge b) \nabla (c^{**} \wedge a \wedge b) \\ &= (y^{**} \wedge a \wedge b) \nabla (d^{**} \wedge a \wedge b) \\ &= (y^{**} \nabla d^{**}) \wedge (a \wedge b) \\ &= (y \vee d)^{**} \wedge (a \wedge b). \end{aligned}$$

Then  $(x \vee c, y \vee d) \in \theta_{F_a}$  as  $a \wedge b \in F \cap B(L)$ . Now we show that  $\theta_F$  preserves the operation  $*$ . Let  $(x, y) \in \theta_F$ . Then  $x^{**} \wedge a = y^{**} \wedge a$  for some  $a \in F \cap B(L)$ . Now by the distributivity of  $B(L)$  we have the following set of implications.

$$\begin{aligned} x^{**} \wedge a = y^{**} \wedge a &\Rightarrow (x^{**} \wedge a) \nabla a^* = (y^{**} \wedge a) \nabla a^* \\ &\Rightarrow (x^{**} \nabla a^*) \wedge (a \nabla a^*) = (y^{**} \nabla a^*) \wedge (a \nabla a^*) \\ &\Rightarrow x^{**} \nabla a^* = y^{**} \nabla a^* \end{aligned}$$

$$\begin{aligned}
&\Rightarrow (x^{***} \wedge a^{**})^* = (y^{***} \wedge a^{**})^* \\
&\Rightarrow (x^{***} \wedge a)^{**} = (y^{***} \wedge a)^{**} \\
&\Rightarrow x^{***} \wedge a = y^{***} \wedge a \\
&\Rightarrow (x^*, y^*) \in \theta_F.
\end{aligned}$$

It is immediate that  $\theta_F$  is a congruence on  $L$ . Let  $(x, y) \in \Phi$ . Then  $x^{**} = y^{**}$ . Hence,  $x^{**} \wedge a = y^{**} \wedge a$ , for some  $a \in F \cap B(L)$ . Thus  $(x, y) \in \theta_F$  and  $\Phi \subseteq \theta_F$ .

(2) Since  $x^{****} \wedge a = x^{**} \wedge a$ ,  $(x^{**}, x) \in \theta_{F_a}$ , and thereby  $[x^{**}]_{\theta_F} = [x]_{\theta_F}$ ,  $\forall x \in L$ .

(3) It is known that  $Coker\theta_F = [1]_{\theta_{F_a}}$ . Let  $x \in Coker\theta_F$ . Then we get the following implications.

$$\begin{aligned}
x \in Coker\theta_F &\Rightarrow (x, 1) \in \theta_F \\
&\Rightarrow x^{**} \wedge a = 1^{**} \wedge a \text{ for some } a \in F \cap B(L) \\
&\Rightarrow x^{**} \wedge a = a \text{ as } 1^{**} = 1 \\
&\Rightarrow x^{**} \geq a \in F \\
&\Rightarrow x^{**} \in F \\
&\Rightarrow x \in F \text{ as } F \text{ is a Boolean filter of } L.
\end{aligned}$$

Then  $Coker\theta_F \subseteq F$ . Conversely, let  $y \in F$ . Then

$$\begin{aligned}
y \in F &\Rightarrow y^{**} \wedge y^{**} = y^{**} = 1^{**} \wedge y^{**} \\
&\Rightarrow (y, 1) \in \theta_F \text{ as } y^{**} \in F \cap B(L) \\
&\Rightarrow y \in Coker\theta_F.
\end{aligned}$$

Then  $F \subseteq Coker\theta_F$ .

(4) Since  $D(L) \cap B(L) = \{1\}$  and  $L \cap B(L) = B(L)$ , we deduce the following equalities:

$$\begin{aligned}
\theta_{D(L)} &= \{(x, y) \in L \times L : x^{**} \wedge 1 = y^{**} \wedge 1\} = \{(x, y) \in L \times L : x^{**} = y^{**}\} = \Phi, \\
\theta_L &= \{(x, y) \in L \times L : x^{**} \wedge 0 = y^{**} \wedge 0\} = \{(x, y) \in L \times L : x, y \in L\} \\
&= L \times L = \nabla.
\end{aligned}$$

(5) From (2) we have,  $L/\theta_F = \{[x]_{\theta_F} : x \in L\} = \{[x^{**}]_{\theta_F} : x \in L\}$ . Let  $[x]_{\theta_F}, [y]_{\theta_F}, [z]_{\theta_F} \in L/\theta_F$ . Then

$$\begin{aligned}
[x]_{\theta_F} \wedge ([y]_{\theta_F} \vee [z]_{\theta_F}) &= [x \wedge (y \vee z)]_{\theta_F} \\
&= [(x \wedge (y \vee z))^{**}]_{\theta_F} \\
&= [x^{**} \wedge (y \vee z)^{**}]_{\theta_F}
\end{aligned}$$

$$\begin{aligned}
 &= [x^{**} \wedge (y^{**} \nabla z^{**})]\theta_F \\
 &= [(x^{**} \wedge y^{**}) \nabla (x^{**} \wedge z^{**})]\theta_F \\
 &= [(x \wedge y)^{**} \nabla (x \wedge z)^{**}]\theta_F \\
 &= [((x \wedge y) \vee (x \wedge z))^{**}]\theta_F \\
 &= [(x \wedge y) \vee (x \wedge z)]\theta_F \\
 &= [x \wedge y]\theta_F \vee [x \wedge z]\theta_F \\
 &= ([x]\theta_F \wedge [y]\theta_F) \vee ([x]\theta_F \wedge [z]\theta_F).
 \end{aligned}$$

This shows that  $L/\theta_F$  is a distributive lattice. Clearly,  $[0]\theta_F$  and  $[1]\theta_F = F$  are the zero and the unit elements of  $L/\theta_F$ . This shows that  $L/\theta_F$  is a bounded distributive lattice. Now we proceed to show that every  $[x]\theta_F$  of  $L/\theta_F$  has a complement. Since  $x \wedge x^* = 0$ ,  $[x]\theta_F \wedge [x^*]\theta_F = [x \wedge x^*]\theta_F = [0]\theta_F$ . Since  $F$  is a Boolean filter,  $x \vee x^* \in F$ . Hence, we have  $[x]\theta_F \vee [x^*]\theta_F = [x \vee x^*]\theta_F = F$ . Thus we have proved that  $L/\theta_F$  is a Boolean algebra.  $\blacksquare$

Now, let  $F = [F_a]$  be a Boolean filter of  $L$  for some  $a \in B(L)$ . Then  $a \in F \cap B(L)$ . For brevity, we write  $\theta_{F_a}$  instead of  $\theta_{[F_a]}$ .

In the following Corollary, we state some congruence properties of a quasi-modular  $p$ -algebra.

**Corollary 5.3.** *Let  $L$  be a quasi-modular  $p$ -algebra. Then the following statements hold.*

- (1)  $(x, y) \in \theta_{F_a} \Leftrightarrow x^{**} \wedge a = y^{**} \wedge a$ ,
- (2)  $Coker\theta_{F_a} = [F_a]$  and  $Ker\theta_{F_a} = [a^*]$ ,
- (3)  $\theta_{F_1} = \Phi$  and  $\theta_{F_0} = \nabla$ .

**Proof.** (1) Let  $(x, y) \in \theta_{F_a}$ . Then

$$\begin{aligned}
 (x, y) \in \theta_{F_a} &\Rightarrow x^{**} \wedge b = y^{**} \wedge b \text{ for some } b \in [F_a] \cap B(L) \\
 &\Rightarrow x^{**} \wedge b \wedge a = y^{**} \wedge b \wedge a \\
 &\Rightarrow x^{**} \wedge a = y^{**} \wedge a \text{ as } b = b^{**} \geq a
 \end{aligned}$$

Conversely, let  $x^{**} \wedge a = y^{**} \wedge a$ . Then  $(x, y) \in \theta_{F_a}$  as  $a \in [F_a] \cap B(L)$ .

(2) By Theorem 5.2(3), we have  $Coker\theta_{F_a} = [F_a]$ . Now we prove the second equality in (2) as follows:

$$\begin{aligned}
 Ker\theta_{F_a} &= \{x \in L : (x, 0) \in \theta_{F_a}\} \\
 &= \{x \in L : x^{**} \wedge a = 0^{**} \wedge a\} \\
 &= \{x \in L : x^{**} \wedge a = 0\} \text{ as } 0^{**} = 0 \\
 &= \{x \in L : x \leq x^{**} \leq a^*\} \\
 &= [a^*].
 \end{aligned}$$

(3) Using Theorem 5.2(4), we get  $\theta_{F_1} = \theta_{D(L)} = \Phi$  and  $\theta_{F_0} = \theta_{[F_0]} = \theta_L = \nabla$  ■

By combining Lemma 5.1 and Theorem 5.2(1), (3) we establish the following characterization theorem of a Boolean filter of  $L$ .

**Theorem 5.4.** *A filter  $F$  of a quasi-modular  $p$ -algebra  $L$  is a cokernel of a congruence  $\theta \in [\Phi, \nabla]$  if and only if  $F$  is a Boolean filter.*

Consider  $Con_B(L) = \{\theta_{F_a} : a \in B(L)\}$ , we observe that  $Con_B(L)$  is a partially ordered set under set inclusion. We now study properties of the elements in the set  $Con_B(L)$ .

**Theorem 5.5.** *Let  $L$  be a quasi-modular  $p$ -algebra. Then for every  $a, b \in B(L)$ , the following statement hold in  $Con_B(L)$ .*

- (1)  $a \leq b$  if and only if  $\theta_{F_b} \subseteq \theta_{F_a}$ ,
- (2) The set  $Con_B(L)$  is a Boolean algebra on its own. Moreover,  $F_B(L) \cong Con_B(L)$ ,
- (3)  $\theta_{F_a} \sqcup \theta_{F_b} = \theta_{F_{a \wedge b}}$  and  $\theta_{F_a} \sqcap \theta_{F_b} = \theta_{F_{a \nabla b}}$ ,
- (4)  $\theta_{F_a} \sqcap \theta_{F_{a^*}} = \Phi$  and  $\theta_{F_a} \sqcup \theta_{F_{a^*}} = \nabla$ .

**Proof.** (1) Let  $a \leq b$  and  $(x, y) \in \theta_{F_b}$ . Then  $x^{**} \wedge b = y^{**} \wedge b$ . Hence  $x^{**} \wedge b \wedge a = y^{**} \wedge b \wedge a$ . This leads to  $x^{**} \wedge a = y^{**} \wedge a$ . Thus  $(x, y) \in \theta_{F_a}$  and  $\theta_{F_b} \subseteq \theta_{F_a}$ . Conversely, let  $\theta_{F_b} \subseteq \theta_{F_a}$ . Then we have  $(b, 1) \in \theta_{F_b} \subseteq \theta_{F_a}$ . This implies that  $b \wedge a = 1 \wedge a = a$ . Thus  $a \leq b$ .

(2) Define the mapping  $\Psi : B(L) \rightarrow Con_B(L)$  as follows :

$$\Psi(a) = \theta_{F_a} \text{ for all } a \in B(L).$$

By (1) above,  $\Psi$  is an order anti-isomorphism between  $B(L)$  and  $Con_B(L)$ . This immediately implies that  $Con_B(L)$  is a Boolean algebra. Now if we define the mapping  $f : B(L) \rightarrow Con_B(L)$  by  $f(a) = \theta_{F_{a^*}}$ , then  $f$  is an isomorphism between Boolean algebras  $B(L)$  and  $Con_B(L)$ . Then  $B(L) \cong Con_B(L)$  and  $B(L) \cong F_B(L)$  imply  $F_B(L) \cong Con_B(L)$ .

(3) Since by (2) above  $\Psi$  is a anti-isomorphism, we have  $\Psi(a \wedge b) = \Psi(a) \sqcup \Psi(b)$  and  $\Psi(a \nabla b) = \Psi(a) \sqcap \Psi(b)$ , where  $\sqcup$  and  $\sqcap$  are the join and meet operations on  $Con_B(L)$ . Now

$$\theta_{F_a} \sqcup \theta_{F_b} = \Psi(a) \sqcup \Psi(b) = \Psi(a \wedge b) = \theta_{F_{a \wedge b}}$$

and

$$\theta_{F_a} \sqcap \theta_{F_b} = \Psi(a) \sqcap \Psi(b) = \Psi(a \nabla b) = \theta_{F_{a \nabla b}}.$$

(4) From (3) above we have

$$\theta_{F_a} \sqcap \theta_{F_{a^*}} = \theta_{F_{a \nabla a^*}} = \theta_{F_1} = \Phi$$

and

$$\theta_{F_a} \sqcup \theta_{F_{a^*}} = \theta_{F_{a \wedge a^*}} = \theta_{F_0} = \nabla.$$

Therefore  $Con_B(L) = (Con_B(L), \sqcup, \sqcap, \bar{\theta}, \Phi, \nabla)$ , where  $\bar{\theta}_{F_a} = \theta_{F_{a^*}}$  is the complement of  $\theta_{F_a}$  on  $Con_B(L)$  and  $\Phi, \nabla$  are the smallest and greatest elements of  $Con_B(L)$  respectively. ■

In the following Corollary an isomorphism between the sublattice  $[\Phi, \nabla]$  of  $Con(L)$  and the lattice  $BF(L)$  of all Boolean filters of  $L$  is obtained.

**Corollary 5.6.** *Let  $L$  be a finite quasi-modular  $p$ -algebra. Then  $[\Phi, \nabla] \cong BF(L)$ .*

**Proof.** Since  $L$  is finite,  $BF(L) = F_B(L)$  and hence  $Con_B(L) = [\Phi, \nabla]$ . By the above Theorem 5.5 (2), we deduce that  $BF(L) \cong [\Phi, \nabla]$ . ■

## 6. CONGRUENCES AND DIRECT PRODUCT OF BOOLEAN FILTERS

Let  $L_1$  and  $L_2$  be two  $p$ -algebras. Then the direct product  $L_1 \times L_2$  is also a  $p$ -algebra, where  $*$  is defined on  $L_1 \times L_2$  by  $(a, b)^* = (a^*, b^*)$ . Now we study the direct product of Boolean filters of  $p$ -algebras. Some properties of congruences with respect to direct product are given.

We first consider the Boolean filters of the  $p$ -algebras in the following theorem.

**Theorem 6.1.** *If  $F_1$  and  $F_2$  are Boolean filters of  $p$ -algebras  $L_1$  and  $L_2$  respectively, then  $F_1 \times F_2$  is a Boolean filter of  $L_1 \times L_2$ . Conversely, every Boolean filter  $F$  of  $L_1 \times L_2$  can be expressed as  $F = F_1 \times F_2$  where  $F_1$  and  $F_2$  are Boolean filters of  $L_1$  and  $L_2$  respectively.*

**Proof.** Let  $F_1$  and  $F_2$  be Boolean filters of  $L_1$  and  $L_2$  respectively. Then, it is clear that  $F_1 \times F_2$  is a filter of  $L_1 \times L_2$ . Since  $F_1$  and  $F_2$  are Boolean filters of  $L_1$  and  $L_2$  respectively, we get  $a \vee a^* \in F_1$  for each  $a \in L_1$  and  $b \vee b^* \in F_2$  for each  $b \in L_2$ . Hence we have  $(a, b) \vee (a, b)^* = (a, b) \vee (a^*, b^*) = (a \vee a^*, b \vee b^*) \in F_1 \times F_2$ . This shows that  $F_1 \times F_2$  is a Boolean filter of  $L_1 \times L_2$ . Conversely, if  $F$  is a Boolean filter of  $L_1 \times L_2$ , then we consider  $F_1$  and  $F_2$  as follows:

$$F_1 = \{x \in L_1 : (x, 1) \in F\} \text{ and } F_2 = \{y \in L_2 : (1, y) \in F\}$$

Clearly  $F_1$  and  $F_2$  are filters of  $L_1$  and  $L_2$  respectively. We now prove that  $F_1$  and  $F_2$  are Boolean filters of  $L_1$  and  $L_2$  respectively. For all  $x \in F_1$ , we have  $(x, 1) \in F$ . Since  $F$  is Boolean,  $(x \vee x^*, 1) = (x, 1) \vee (x, 1)^* \in F$ . Hence, we have  $x \vee x^* \in F_1$ . Therefore,  $F_1$  is a Boolean filter of  $L_1$ . Similarly,  $F_2$  is a Boolean filter of  $L_2$ . Now we prove that  $F = F_1 \times F_2$ . For this purpose, we let  $(x, y) \in F$ . Then we have the following implications.

$$\begin{aligned} (x, y) \in F &\Rightarrow (x, 1) \in F \text{ and } (1, y) \in F \\ &\Rightarrow x \in F_1 \text{ and } y \in F_2 \\ &\Rightarrow (x, y) \in F_1 \times F_2. \end{aligned}$$

Hence,  $F \subseteq F_1 \times F_2$ . Conversely, if  $(x, y) \in F_1 \times F_2$ , then the following implications hold.

$$\begin{aligned} (x, y) \in F_1 \times F_2 &\Rightarrow x \in F_1 \text{ and } y \in F_2 \\ &\Rightarrow (x, 1) \in F \text{ and } (1, y) \in F \\ &\Rightarrow (x, y) = (x, 1) \wedge (1, y) \in F. \end{aligned}$$

Consequently, we have  $F_1 \times F_2 \subseteq F$ . This shows that  $F_1 \times F_2 = F$ . ■

In closing this paper, we state two equalities concerning Boolean filters of quasi-modular  $p$ -algebras.

**Theorem 6.2.** *Let  $[F_a]$  and  $[F_b]$  be two Boolean filters of the quasi-modular  $p$ -algebras  $L_1$  and  $L_2$ , respectively. Then*

- (1)  $[F_a] \times [F_b] = [F_{(a,b)}]$
- (2)  $\theta_{F_a \times F_b} = \theta_{F_{(a,b)}}$ .

**Proof.** (1) From the above Theorem 6.1, we see immediately that  $[F_a] \times [F_b]$  is a Boolean filter of  $L_1 \times L_2$ . Now, we have

$$\begin{aligned} (x, y) \in [F_a] \times [F_b] &\Leftrightarrow x \in [F_a] \text{ and } y \in [F_b] \\ &\Leftrightarrow x^{**} \geq a \text{ and } y^{**} \geq b \\ &\Leftrightarrow (x, y)^{**} = (x^{**}, y^{**}) \geq (a, b) \\ &\Leftrightarrow (x, y) \in [F_{(a,b)}]. \end{aligned}$$

Therefore,  $[F_a] \times [F_b] = [F_{(a,b)}]$ .

- (2) By (1), we obtain  $\theta_{F_a \times F_b} = \theta_{[F_a] \times [F_b]} = \theta_{[F_{(a,b)}]} = \theta_{F_{(a,b)}}$ . ■

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## REFERENCES

- [1] R. Balbes and A. Horn, *Stone lattices*, Duke Math. J. **37** (1970) 537–543.  
doi:10.1215/S0012-7094-70-03768-3
- [2] R. Balbes and Ph. Dwinger, *Distributive Lattices* (Univ. Miss. Press, 1975).
- [3] G. Birkhoff, *Lattice theory*, Amer. Math. Soc., Colloquium Publications, **25**, New York, 1967.
- [4] G. Grätzer, *A generalization on Stone's representations theorem for Boolean algebras*, Duke Math. J. **30** (1963) 469–474. doi:10.1215/S0012-7094-63-03051-5
- [5] G. Grätzer, *Lattice Theory, First Concepts and Distributive Lattice* (W.H. Freeman and Co., San-Francisco, 1971).
- [6] G. Grätzer, *General Lattice Theory* (Birkhäuser Verlag, Basel and Stuttgart, 1978).
- [7] O. Frink, *Pseudo-complements in semi-lattices*, Duke Math. J. **29** (1962) 505–514.  
doi:10.1215/S0012-7094-62-02951-4
- [8] T. Katriňák and P. Mederly, *Construction of  $p$ -algebras*, Algebra Universalis **4** (1983) 288–316.
- [9] M. Sambasiva Rao and K.P. Shum, *Boolean filters of distributive lattices*, Int. J. Math. and Soft Comp. **3** (2013) 41–48.
- [10] P.V. Venkatanarasimhan, *Ideals in semi-lattices*, J. Indian. Soc. (N.S.) **30** (1966) 47–53.

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