Discussiones Mathematicae
General Algebra and Applications 34 (2014) 45–53
doi:10.7151/dmgaa.1211

AN IDEAL-BASED ZERO-DIVISOR GRAPH OF DIRECT PRODUCTS OF COMMUTATIVE RINGS

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Abstract

In this paper, specifically, we look at the preservation of the diameter and girth of the zero-divisor graph with respect to an ideal of a commutative ring when extending to a finite direct product of commutative rings.

Keywords: zero-divisor graph, ideal-based, diameter, girth, finite direct product.

2010 Mathematics Subject Classification: 05C40, 05C45, 13A99.

1. Introduction

Finding the relationship between the algebraic structure of rings using properties of graphs associated to them has become an interesting topic in recent years. Indeed, it is worthwhile to relate algebraic properties of rings to combinatorial properties of their assigned graphs. One of the associated graphs to a ring R is the zero-divisor graph, denoted by $\Gamma(R)$. It is a simple graph with vertex set $Z(R) \setminus \{0\}$, and two vertices x and y are adjacent if and only if $xy = 0$. It is due to Anderson and Livingston [1]. This graph was first introduced by Beck, in [5], where all the elements of R were considered as the vertices. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions [1–8]. In [8], Redmond introduced and investigated the zero-divisor graph with respect to an ideal. Let I be an ideal of a commutative ring R. The zero-divisor graph of R with respect to I, denoted by $\Gamma_I(R)$, is the graph whose vertices are the set $Z_I(R)^* = Z_I(R) \setminus I = \{x \in R : xy \in I \text{ for some } y \in R \setminus I\} \setminus I$ with distinct vertices x and y adjacent if and only if $xy \in I$. In [8], Redmond showed that for an ideal I of R, $\mathrm{diam}(\Gamma_I(R)) \leq 3$ and $\mathrm{gr}(\Gamma_I(R)) \leq 4$ (if it contains cycle). In [2], Axtell, Stickles, and Warfel studied zero-divisor graphs of direct products of commutative rings. In this paper, we completely characterize the diameter and girth of the zero-divisor graph with respect to an ideal of a finite direct product of rings.

In order to make this paper easier to follow, we recall in this section various notions from graph theory which will be used in the sequel. For a graph Γ, we denote the set of all edges and vertices by $E(\Gamma)$ and $V(\Gamma)$, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by $d(a, b)$, is the length of a shortest path connecting them ($d(a, a) = 0$ and $d(a, b) = \infty$ if there is no such path). The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$.

2. Diameter and direct products

In this section, we will investigate the relation between the diameter of an ideal-based zero-divisor graph of a finite direct product $R_1 \times R_2 \times \cdots \times R_n$ with the diameters of the zero-divisor graphs with respect to ideals of $R_1, R_2, \cdots, R_{n-1}$, and R_n.

Proposition 1. *Let I be an ideal of a commutative ring S. Then the following hold.*

(1) *If $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{S})) = 1$ and $S = Z_I(S)$, then $S^2 \subseteq I$, where $S^2 = \{rs : r, s \in S\}$.*

(2) *If $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{S})) = 2$ and $Z_I(S)$ is a (not necessarily proper) subring of S, then for all $x, y \in Z_I(S)$, there exists $z \in Z_I(S)^*$ such that $zx, zy \in I$.*

Proof. (1) Suppose that $x^2 \notin I$ for some $x \in S$. If $S = \{0, x\}$, then we have $S \neq Z_I(S)$, which is a contradiction. Hence, there is an element $y \in S \setminus I$ such that $x \neq y$. Observe that $x + y \neq x$. By assumption, $xy, x(x + y) \in I$; hence $x^2 \in I$ since I is an ideal of S, a contradiction.

(2) Let $x, y \in Z_I(S)$. We split the proof into two cases.

Case 1. $x = y$. If $xy \in I$, we choose $z = x$. If $xy \notin I$, then there exists $z \in Z_I(S)^*$ such that $zx, zy \in I$ since $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{S})) = 2$.

Case 2. $x \neq y$. If $xy \notin I$, we are done. So we may assume that $xy \in I$. If $x^2 \in I$ (resp., $y^2 \in I$), then $z = x$ (resp., $z = y$) yields the desired element. So, suppose $x^2, y^2 \notin I$. Also, if $x + y \in I$, then $x(x + y) \in I$ gives $x^2 \in I$, which is a contradiction. Hence $x + y \notin I$. Let $X' = \{x' \in Z_I(S)^* : xx' \in I\}$ and $Y' = \{y' \in Z_I(S)^* : yy' \in I\}$. Observe that $x \in Y'$ and $y \in X'$; hence X' and Y' are nonempty. If $X' \cap Y' \neq \emptyset$, choose $z \in X' \cap Y'$. Suppose $X' \cap Y' = \emptyset$ and consider $x + y$. By assumption, $x + y \neq x$, $x + y \neq y$, $x + y \notin X'$, and $x + y \notin Y'$. Since $\mathrm{diam}(\Gamma_I(S)) = 2$ and $Z_I(S)$ is a subring (so $x + y \in Z_I(S)$), there exists $w \in Z_I(S)^*$ such that the following path exists: $x - w - x + y$. Then $w(x + y) - wx = wy \in I$, and so $w \in X' \cap Y'$, which is a contradiction. ∎

Remark 2. Assume that $R_1, R_2, \ldots, R_n$ ($n \geq 2$) are commutative rings. If I is an ideal of $R = R_1 \times R_2 \times \cdots \times R_n$, then for each i ($1 \leq i \leq n$), $I_i = \{a_i : (0, 0, \ldots, 0, a_i, 0, \ldots, 0) \in I\}$ is an ideal of R_i.

Remark 3. Throughout this paper, we shall assume, unless otherwise stated, that R, I_i, and I are as described in Remark 2.

Compare the next theorem with [2, Theorem 3.3].

Theorem 4. *Let R, I_i, and I be as in Remark* 3 *such that $R_n = Z_{I_n}(R_n)$ and $R_1, R_2, \ldots, R_{n-1}$ are domains.*

(1) *If* $\mathrm{diam}(\Gamma_{I_n}(R_n)) \leq 2$, *then* $\mathrm{diam}(\Gamma_I(R)) = 2$.

(2) *If* $\mathrm{diam}(\Gamma_{I_n}(R_n)) = 3$, *then* $\mathrm{diam}(\Gamma_I(R)) = 3$.

Proof. (1) Let $x = (x_1, \ldots, x_n) \in R$. By assumption, there exists an element $y_n \in Z_{I_n}(R_n)^*$ such that $x_n y_n \in I_n$. Then $(0, 0, \ldots, 0, y_n) \notin I$ and $x(0, 0, \ldots, y_n) \in I$ since $R_n = Z_{I_n}(R_n)$; hence $Z_I(R) = R$. If $z_n \in Z_{I_n}(R_n)^*$, then $(1, 1, 0, \ldots, 0)(1, 1, \ldots, z_n) \notin I$; so

$$d((1, 1, 0, \ldots, 0), (1, 1, \ldots, z_n)) \geq 2.$$

Now if $\mathrm{diam}(\Gamma_{I_n}(R_n)) \leq 2$, then for $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in R$, we either have $ab \in I$ or for some $c_n \in Z_{I_n}(R_n)^*$, we have

$$a(0, 0, \ldots, c_n), b(0, \ldots, c_n) \in I$$

using Proposition 2 (2) in the diameter two case. So we have $\mathrm{diam}(\Gamma_I(R)) = 2$. If $\mathrm{diam}(\Gamma_{I_n}(R_n)) = 3$, then there exist $x_n, y_n \in Z_{I_n}(R_n)^*$ such that $d(x_n, y_n) = 3$. Then for $b_i \in Z_{I_i}(R_i)^*$ ($1 \leq i \leq n - 1$), we have $d(e, f) = 3$, where $e = (b_1, \ldots, b_{n-1}, x_n)$ and $f = (b_1, \ldots, b_{n-1}, y_n)$, as required. ∎

For the remainder of this section, we assume that $R_1, R_2, \ldots, R_{n-1}$, and R_n are commutative rings, not necessarily with identity, such that $Z_{I_1}(R_1), \ldots, Z_{I_{n-1}}(R_{n-1})$, and $Z_{I_n}(R_n)$ are nonempty.

Theorem 5. *Let R, I_i, and I be as in Remark* 3 *such that* $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 1$ *for all $i = 1, \ldots, n$.*

(1) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 1$ *if and only if $R_i^2 \subseteq I_i$ for every $i \in \{1, 2, \ldots, n\}$.*

(2) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$ *if and only if $R_i^2 \subseteq I_i$ and $R_j^2 \nsubseteq I_j$ for some $i, j \in \{1, 2, \ldots, n\}$.*

(3) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 3$ *if and only if $R_i^2 \nsubseteq I_i$ for every $i \in \{1, 2, \ldots, n\}$.*

Proof. (1) Assume that $R_i^2 \subseteq I_i$ for all i, and let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in Z_I(R)^*$. Then $xy \in I_1 \times I_2 \times \cdots \times I_n \subseteq I$; hence $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 1$. Conversely, assume that $R_j^2 \nsubseteq I_j$ for some $j \in \{1, 2, \ldots, n\}$. Then $x_j y_j \notin I_j$ for some $x_j, y_j \in R_j$. Let $z_i \in Z_{I_i}(R_i)$ for $i \neq j$. Set $X = (0, \ldots, x_j, \ldots, 0), Y = (0, \ldots, y_j, \ldots, 0)$ and $Z = (0, \ldots, z_i, \ldots, 0)$. Then $XZ, YZ \in I$; hence $X - Y - Z$ is a path of length 2 from X to Y in $Z_I(R)^*$, which is a contradiction.

(2) Let $R_i^2 \subseteq I_i$ and $R_j^2 \nsubseteq I_j$ for some $i, j \in \{1, 2, \ldots, n\}$. Then $\mathrm{diam}(\Gamma_I(R)) \neq 1$ by (1). Let $c_i \in Z_{I_i}(R_i)^*$, and set $c = (0, \ldots, c_i, \ldots, 0)$. For every $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Z_I(R)^*$, at worst we have $x - c - y$ is a path from x to y in $Z_I(R)^*$. So, $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) \leq 2$. The result then follows from (1). Conversely, assume that $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$. If $R_i^2 \subseteq I_i$, then $R_i = Z_{I_i}(R_i)$ for all $i = 1, \ldots, n$ (see Proposition 1); so $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 1$ by (1), a contradiction. If for each i, $Z_{I_i}(R_i) \neq R_i$, then there must exists $x_i \in R_i$ with $x_i \notin Z_{I_i}(R_i)$ for all $i = 1, \ldots, n$. For each i, let $z_i \in Z_{I_i}(R_i)^*$. So for all i, there is an element $w_i \in Z_{I_i}(R_i)^*$ such that $z_i w_i \in I_i$. If $a = (z_1, x_2, \ldots, x_n)$ and $b = (x_1, z_2, x_3, \ldots, x_n)$, then $a(w_1, 0, \ldots, 0), b(0, w_2, \ldots, 0) \in I$; hence $a, b \in Z_I(R)^*$. Since $ab \notin I$, we get $d(a, b) > 1$. As $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$, there exists $c = (c_1, \ldots, c_n) \in Z_I(R)^*$ such that $ac, bc \in I$. It follows that there exists i $(1 \leq i \leq n)$ such that $x_i \in Z_{I_i}(R_i)^*$, a contradiction. Thus the proof is complete. (3) follows from (1) and (2). ■

Compare the next theorem with [2, Theorem 3.5].

Theorem 6. *Let R, I_i, and I be as in Remark* 3 *such that* $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 2$ *for all $i = 1, \ldots, n$.*

(1) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) \neq 1$.

(2) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$ *if and only if $R_i = Z_{I_i}(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.*

(3) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 3$ *if and only if $R_i \neq Z_{I_i}(R_i)$ for every $i \in \{1, 2, \ldots, n\}$.*

Proof. (1) Since $\operatorname{diam}(\Gamma_{I_n}(R_n)) = 2$, there exist distinct $y_n, w_n \in Z_{I_n}(R_n)^*$ with $y_n w_n \notin I$. Set $a = (0, 0, \ldots, y_n)$ and $b = (0, 0, \ldots, w_n)$. Then $ab \notin I$. Therefore $\operatorname{diam}(\Gamma_I(R)) > 1$.

(2) Assume that $R_i = Z_{I_i}(R_i)$ for some $i \in \{1, 2, \ldots, n\}$. So for $x_i, y_i \in Z_{I_i}(R_i)$, there exists $z_i \in Z_{I_i}(R_i)^*$ such that $x_i z_i, y_i z_i \in I$ by Proposition 2 (2). So, for any $x = (x_1, \cdots, x_n), y = (y_1, \ldots, y_n) \in Z_I(R)^*$, there exists $z = (0, 0, \ldots, z_i, 0, \ldots, 0) \in Z_I(R)^*$ such that $xz, yz \in I$. If, without loss of generality, $y = z$, we have $xy \in I$. Therefore, $\operatorname{diam}(\Gamma_I(R)) \leq 2$. By (1), it must be that $\operatorname{diam}(\Gamma_I(R)) = 2$. Conversely, suppose that $\operatorname{diam}(\Gamma_I(R)) = 2$ and $R_i \neq Z_{I_i}(R_i)$ for all $i \in \{1, 2, \ldots, n\}$. Let $e_i \in Z_{I_i}(R_i)$ and $m_i \in R_i \setminus Z_{I_i}(R_i)$ for all i. Set $a = (e_1, m_2, \ldots, m_n)$ and $b = (m_1, e_2, m_3, \ldots, m_n)$. Then $ab \notin I$. Since $\operatorname{diam}(\Gamma_I(R)) = 2$, there exists $z = (z_1, \ldots, z_n) \in Z_I(R)^*$ such that $az, bz \in I$. Then $e_1 z_1 \in I_1$, $m_i z_i \in I_i$ $(2 \leq i \leq n)$, $m_1 z_1 \in I_1$, $e_2 z_2 \in I_2$, and $m_i z_i \in I_i$ $(3 \leq i \leq n)$, which is a contradiction. (3) follows from (1) and (2). ■

Compare the next theorem with [2, Theorem 3.9].

Theorem 7. *Let R, I_i, and I be as in Remark* 3 *such that* $\operatorname{diam}(\Gamma_{I_i}(R_i)) = 3$ *for all* $i = 1, \ldots, n$. *Then* $\operatorname{diam}(\Gamma_I(R)) = 3$.

Proof. Since for each i, $\operatorname{diam}(\Gamma_{I_i}(R_i)) = 3$, there exist distinct $x_i, y_i \in Z_{I_i}(R_i)^*$ with $x_i y_i \notin I_i$ and there is no $z_i \in Z_{I_i}(R_i)^*$ such that $x_i y_i, y_i z_i \in I_i$. Consider $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Now for every $i \in \{1, 2, \ldots, n\}$, there are elements $x_i', y_i' \in Z_{I_i}(R_i)^*$ such that $x_i x_i', y_i y_i' \in I_i$; hence $x, y \in Z_I(R)^*$. Since $xy \notin I$, we have $\operatorname{diam}(\Gamma_I(R)) > 1$. If $\operatorname{diam}(\Gamma_I(R)) = 2$, there exists $a = (a_1, \cdots, a_n) \in Z_I(R)^*$ such that $ax, ay \in I$. Since $a \notin I$, $a_i \notin I_i$ for some i; hence $x_i a_i, y_i a_i \in I_i$, which is a contradiction. Thus $\operatorname{diam}(\Gamma_I(R)) = 3$. ■

Compare the next theorem with [2, Theorem 3.5].

Theorem 8. *Let R, I_i, and I be as in Remark* 3 *such that* $\operatorname{diam}(\Gamma_{I_i}(R_i)) = 1$, $\operatorname{diam}(\Gamma_{I_j}(R_j)) = 2$ *for some* $i, j \in \{1, 2, \ldots, n\}$, *and there is no* $k \in \{1, 2, \ldots, n\}$ *with* $\operatorname{diam}(\Gamma_{I_k}(R_k)) = 3$.

(1) $\operatorname{diam}(\Gamma_I(R)) \neq 1$.

(2) $\operatorname{diam}(\Gamma_I(R)) = 2$ *if and only if* $R_i = Z_{I_i}(R_i)$ *for some* $i \in \{1, 2, \ldots, n\}$.

(3) $\operatorname{diam}(\Gamma_I(R)) = 3$ *if and only if* $R_i \neq Z_{I_i}(R_i)$ *for every* $i \in \{1, 2, \ldots, n\}$.

Proof. (1) Same as Theorem 6 (1).

(2) Let $R_i = Z_{I_i}(R_i)$ and $\operatorname{diam}(\Gamma_{I_i}(R_i)) = 1$. Thus we have $R_i^2 \subseteq I_i$ by Proposition 1 (1). Let $x_i \in R_i \setminus \{0\}$. Since $(0, \ldots, 0, x_i, 0, \ldots, 0)(y_1, \ldots, y_n) \in I$ for all $(y_1, \ldots, y_n) \in Z_I(R)^*$, we have $\operatorname{diam}(\Gamma_I(R)) \leq 2$. It follows from (1) that $\operatorname{diam}(\Gamma_I(R)) = 2$. Conversely, assume that $\operatorname{diam}(\Gamma_I(R)) = 2$. Suppose

$R_i \neq Z_{I_i}(R_i)$ for every $i \in \{1, 2, \ldots, n\}$. Without loss of generality, let $z_1 \in Z_{I_1}(R_1)^*$. Then there exists $w_1 \in Z_{I_1}(R_1)^*$ such that $z_1 w_1 \in I_1$. For each i, let $r_i \in R_i \setminus Z_{I_i}(R_i)$, and set $a = (r_1, 0, \ldots, 0), b = (0, r_2, 0, \ldots, 0), c = (z_1, 0, \ldots, 0)$, and $d = (w_1, r_2, r_3, \ldots, r_n)$. Then $a - b - c - d$ is a path of length 3. Now we show that $d(a,d) \neq 2$. Assume contrary $d(a,d) = 2$. Then there exists $x = (x_1, \ldots, x_n) \in Z_I(R)^*$ such that $ax, dx \in I$. Since $ax \in I$, $r_1 x_1 \in I_1$ with $r_1 \in R_1 \setminus Z_{I_1}(R_1)$; thus $x_1 \in I_1$. As $dx \in I$, $r_i x_i \in I_i$ with $r_i \in R_i \setminus Z_{I_i}(R_i)$; so $x_i \in I_i$ $(2 \leq i \leq n)$. Thus $x \in I$, a contradiction. Therefore $d(a,d) = 3$, and hence $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 3$, which is a contradiction. (3) follows from (1) and (2). ■

Theorem 9. *Let R, I_i, and I be as in Remark* 3 *such that* $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 1$, $\mathrm{diam}(\Gamma_{\mathrm{I_j}}(\mathrm{R_j})) = 3$ *for some $i, j \in \{1, 2, \ldots, n\}$, and there is no $k \in \{1, 2, \ldots, n\}$ with* $\mathrm{diam}(\Gamma_{\mathrm{I_k}}(\mathrm{R_k})) = 2$.

(1) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) \neq 1$.

(2) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$ *if and only if $R_i = Z_{I_i}(R_i)$ and* $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 1$ *for some $i \in \{1, 2, \ldots, n\}$.*

(3) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 3$ *if and only if there is no $k \in \{1, 2, \ldots, n\}$ with $R_k \neq Z_{I_k}(R_k)$ and* $\mathrm{diam}(\Gamma_{\mathrm{I_k}}(\mathrm{R_k})) = 1$.

Proof. (1) Same as Theorem 6 (1).

(2) ($\Longleftarrow$) Same as Theorem 8 (2). Conversely, assume that $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R}) = 2$; we show that $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 1$ and $R_i = Z_{I_i}(R_i)$ for some $i \in \{1, 2, \ldots, n\}$. Suppose either $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) \neq 1$ or $R_i \neq Z_{I_i}(R_i)$ for every $i \in \{1, 2, \ldots, n\}$. Let $i_1, \ldots, i_k$ be such that $\mathrm{diam}(\Gamma_{\mathrm{I_{i_r}}}(\mathrm{R_{i_r}})) = 1$ $(1 \leq r \leq k)$, and let $j_1, \ldots, j_t$ be such that $\mathrm{diam}(\Gamma_{\mathrm{I_{j_s}}}(\mathrm{R_{j_s}})) = 3$ $(1 \leq s \leq t)$. Since for each s $(1 \leq s \leq t)$, $\mathrm{diam}(\Gamma_{\mathrm{I_{j_s}}}(\mathrm{R_{j_s}})) = 3$, there exist distinct $x_{j_s}, y_{j_s} \in Z_{I_{j_s}}(R_{j_s})^*$ with $x_{j_s} y_{j_s} \notin I_{j_s}$ such that there is no $z_{j_s} \in Z_{I_{j_s}}(R_{j_s})^*$ with $x_{j_s}, z_{j_s} \in I_{j_s}$. Moreover for each s $(1 \leq s \leq t)$, there must exist $x'_{j_s}, y'_{j_s} \in Z_{I_{j_s}}(R_{j_s})^*$ with $x_{j_s} x'_{j_s}, y_{j_s} y'_{j_s} \in I_{j_s}$. Now for each r $(1 \leq r \leq k)$, let $m_{i_r} \in R_{i_r} \setminus Z_{I_{i_r}}(R_{i_r})$. Set $c = (m_{i_1}, \ldots, x_{j_1}, \ldots, x_{j_t}, \ldots, 0)$ and $d = (m_{i_1}, \ldots, y_{j_1}, \ldots, y_{j_t}, \ldots, 0)$. Then $c(0, \ldots, x'_{j_1}, 0, \ldots, 0) \in I$; so $c \in Z_I(R)^*$. As $cd \notin I$ and $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$, there must be some $e = (e_1, \ldots, e_n) \in Z_I(R)^*$ such that $ce, de \in I$. But this is a contradiction, as needed.

(3) Since $\Gamma_I(R)$ is connected and $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) \leq 3$, the diameter of $\Gamma_I(R)$ is either 2 or 3 by (1). If $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$, then by (2), $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 1$ and $R_i = Z_{I_i}(R_i)$ for some $i \in \{1, 2, \ldots, n\}$, which is a contradiction. Thus $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 3$. The proof of other implication is clear. ■

Compare the next theorem with [2, Theorem 3.7].

Theorem 10. *Let R, I_i, and I be as in Remark* 3 *such that* $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 2$, $\mathrm{diam}(\Gamma_{\mathrm{I_j}}(\mathrm{R_j})) = 3$ *for some $i, j \in \{1, 2, \dots, n\}$, and there is no $k \in \{1, 2, \dots, n\}$ with* $\mathrm{diam}(\Gamma_{\mathrm{I_k}}(\mathrm{R_k})) = 1$.

(1) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) \neq 1$.

(2) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$ *if and only if $R_i = Z_{I_i}(R_i)$ and* $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 2$ *for some $i \in \{1, 2, \dots, n\}$.*

(3) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R}))) = 3$ *if and only if there is no $k \in \{1, 2, \dots, n\}$ with $R_k \neq Z_{I_k}(R_k)$ and* $\mathrm{diam}(\Gamma_{\mathrm{I_k}}(\mathrm{R_k})) = 2$.

Proof. (1) Same as Theorem 6 (1).

(2) ($\Longleftarrow$) Same as Theorem 6 (2). Conversely, assume that $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$; we show that $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 2$ and $R_i = Z_{I_i}(R_i)$ for some i. Suppose not. Let $i_1, \dots, i_k$ be such that $\mathrm{diam}(\Gamma_{\mathrm{I_{i_r}}}(\mathrm{R_{i_r}})) = 2$ $(1 \leq r \leq k)$, and let $j_1, \dots, j_t$ be such that $\mathrm{diam}(\Gamma_{\mathrm{I_{j_s}}}(\mathrm{R_{j_s}})) = 3$ $(1 \leq s \leq t)$. Since for each s $(1 \leq s \leq t)$, $\mathrm{diam}(\Gamma_{\mathrm{I_{j_s}}}(\mathrm{R_{j_s}})) = 3$, there exist distinct $x_{j_s}, y_{j_s} \in Z_{I_{j_s}}(R_{j_s})$ with $x_{j_s} y_{j_s} \notin I_{j_s}$. Moreover for each s $(1 \leq s \leq t)$, there must exist $x'_{j_s}, y'_{j_s} \in Z_{I_{j_s}}(R_{j_s})^*$ with $x_{j_s} x'_{j_s}, y_{j_s} y'_{j_s} \in I_{j_s}$. Now for each r $(1 \leq r \leq k)$, let $m_{i_r} \in R_i \setminus Z_{I_{i_r}}(R_{i_r})$. Set $c = (m_{i_1}, \dots, x_{j_1}, \dots, x_{j_t}, \dots, 0)$ and $d = (m_{i_1}, \dots, y_{j_1}, \dots, y_{j_t}, \dots, 0)$. Then $c(0, \dots, x'_{j_1}, 0, \dots, 0) \in I$; so $c \in Z_I(R)^*$. Similarly, $d \in Z_I(R)^*$. As $cd \in I$ and $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$, there must be some $e = (e_1, \dots, e_n) \in Z_I(R)^*$ such that $ce, de \in I$. But this is a contradiction, as required. (3) follows from (1) and (2). ∎

Theorem 11. *Let R, I_i, and I be as in Remark* 3 *such that* $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 1$, $\mathrm{diam}(\Gamma_{\mathrm{I_j}}(\mathrm{R_j})) = 2$, *and* $\mathrm{diam}(\Gamma_{\mathrm{I_k}}(\mathrm{R_k})) = 3$ *for some $i, j, k \in \{1, 2, \dots, n\}$.*

(1) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) \neq 1$.

(2) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$ *if and only if $R_i = Z_{I_i}(R_i)$ and* $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) \leq 2$ *for some $i \in \{1, 2, \dots, n\}$.*

(3) $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 3$ *if and only if there is no $k \in \{1, 2, \dots, n\}$ with $R_k \neq Z_{I_k}(R_k)$ and* $\mathrm{diam}(\Gamma_{\mathrm{I_k}}(\mathrm{R_k})) \leq 2$.

Proof. (1) Is clear.

(2) Let $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) \leq 2$ and $R_i = Z_{I_i}(R_i)$ for some $i \in \{1, 2, \cdots, n\}$. If $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 1$ and $R_i = Z_{I_i}(R_i)$ for some i, then by a similar argument as in Theorem 8 (2), we get $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R}))) = 2$. If $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i})) = 2$ and $R_i = Z_{I_i}(R_i)$ for some i, then by a similar argument as in Theorem 9 (2), we obtain $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$. Conversely, assume that $\mathrm{diam}(\Gamma_{\mathrm{I}}(\mathrm{R})) = 2$. It is easy to see from Theorem 10 (2) that $\mathrm{diam}(\Gamma_{\mathrm{I_i}}(\mathrm{R_i}) \leq 2$ and $R_i = Z_{I_i}(R_i)$ for some $i \in \{1, 2, \dots, n\}$. (3) follows from (1) and (2). ∎

3. Girth and direct products

We continue to use the notation already established; so R, I_i, and I are as in Remark 3. We are now ready to turn our attention toward describing the girth of the zero-divisor graph with respect to an ideal of a direct product of commutative rings, not necessarily with identity. Compare the next theorem with [2, Theorem 4.1].

Theorem 12. *Let R, I_i, and I be as in Remark 3. Then $\mathrm{gr}(\Gamma_I(R)) = 3$ if and only if one (or both) of the following hold.*

(1) $|Z_{I_i}(R_i)^*| \geq 2$ *for some* $i \in \{1, 2, \dots, n\}$.

(2) $|\sqrt{I_i}| \geq 2$ *and* $|\sqrt{I_j}| \geq 2$ *for some* $i, j \in \{1, 2, \dots, n\}$ *with* $i \neq j$.

Proof. If (1) holds, there exists $i \in \{1, 2, \dots, n\}$ such that $|Z_{I_i}(R_i)| \geq 2$. Since $\Gamma_{I_i}(R_i)$ is connected, there must exist $a_i, b_i \in Z_{I_i}(R_i)^*$ with $a_i \neq b_i$ such that $a_i b_i \in I_i$. Then

$$(0, \dots, 0, a_i, \dots, 0) - (0, \dots, b_i, \dots, 0) - (0, \dots, c_j, \dots, 0) - (0, \dots, 0, a_i, \dots, 0)$$

is a cycle of length 3, where $c_j \in Z_{I_j}(R_j)$ and $i \neq j$. If (2) holds, let $a_i \in R_i^*$ and $b_j \in R_j^*$ with $a_i^2 \in I_i$ and $b_j^2 \in I_j$. We may assume that $j > i$. Then $(0, \dots, a_i, \dots, 0) - (0, \dots, a_i, \dots, b_j, \dots, 0) - (0, \dots, b_j, \dots, 0) - (0, \dots, a_i, \dots, 0)$ is a cycle of length 3. Conversely, suppose, without loss of generality, $\sqrt{I_i}$ has no nonzero elements for $i \in \{2, 3, \dots, n\}$. If $|Z_{I_i}(R_i)| < 2$, then $|Z_{I_i}(R_i)| = 0$ $(2 \leq i \leq n)$. Let $(a_1, \dots, a_n) - (b_1, \dots, b_n) - (c_1, \dots, c_n) - (d_1, \dots, d_n) - (a_1, \cdots, a_n)$ be a cycle in $\Gamma_I(R)$. Since $|Z_{I_i}(R_i)| = 0$ for each i $(2 \leq i \leq n)$, there must exist $b_1, c_1 \in R_1$ such that $b_1, c_1 \notin I_1$ and $b_1 c_1 \in I_1$; hence $b_1, c_1 \in Z_{I_1}(R_1)$. Thus, $|Z_{I_1}(R_1)| \geq 2$. ■

Compare the next theorem with [2, Theorem 4.2].

Theorem 13. *Let R, I_i, and I be as in Remark 3 (for $n = 2$). Then $\mathrm{gr}(\Gamma_I(R)) = 4$ if and only if both of the following hold.*

(1) $|R_1| \geq 3$ *and* $|R_2| \geq 3$.

(2) *Without loss of generality, R_1 is a domain and $|Z_{I_2}(R_2)| \leq 1$.*

Proof. ($\Longleftarrow$) Clearly, $\mathrm{gr}(\Gamma_I(R)) \neq 3$ by Theorem 12. Now, let $x_1, x_2 \in R_1 \setminus \{0\}$ be distinct and $y_1, y_2 \in R_2 \setminus \{0\}$ be distinct. Then $(x_1, 0) - (0, y_1) - (x_2, 0) - (0, y_2) - (x_1, 0)$ is a cycle. Thus $\mathrm{gr}(\Gamma_I(R)) = 4$. Conversely, assume that $\mathrm{gr}(\Gamma_I(R)) = 4$. Then Theorem 12 gives $|Z_{I_1}(R_1)| \leq 1$ and $|Z_{I_2}(R_2)| \leq 1$. Without loss of generality, assume R_2 is not a domain; so there exists $x \in Z_{I_2}(R_2)$

such that $x \notin I_2$. It follows that $|Z_{I_2}(R_2)| = |\sqrt{I_2}| = 1$. If R_1 is not a domain, then $|Z_{I_1}(R_1)| = |\sqrt{I_1}| = 1$. Thus $\mathrm{gr}(\Gamma_I(R)) = 3$, a contradiction. Therefore R_1 is a domain; so $Z_{I_1}(R_1) = \emptyset$. Now a cycle must have the form $(x_1, y_1) - (0, y_2) - (x_2, y_3) - (0, y_4) - (x_1, y_1)$. In this cycle, y_2 and y_4 must be nonzero and distinct. Thus $|R_2| \geq 3$. If either x_1 or x_2 is zero, then $|Z_{I_2}(R_2)| \geq 2$; whence $\mathrm{gr}(\Gamma_I(R)) = 3$ by Theorem 3.1, a contradiction. If $x_1 = x_2$, then y_1 and y_3 are distinct. If $y_3 = 0$, then $y_1, y_2, y_4 \in Z_{I_2}(R_2)$, implying $y_1 = y_2 = y_4$, a contradiction. If $y_3 \neq 0$, then $y_2, y_3, y_4 \in Z_{I_2}(R_2)$, implying $y_2 = y_3 = y_4$, another contradiction. Therefore we must have $x_1 \neq x_2$ and $|R_1| \geq 3$. ■

Acknowledgments

The authors are grateful to the referee for his comments and valuable suggestions.

REFERENCES

[1] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra. **217** (1999) 434–447. doi:10.1006/jabr.1998.7840

[2] M. Axtell, J. Stickles and J. Warfel, *Zero-divisor graphs of direct products of commutative rings*, Houston J. Math. **32** (2006) 985–994.

[3] D.F. Anderson, M.C. Axtell and J.A. Stickles Jr., Zero-divisor graphs in commutative rings in commutative Algebra-Noetherian and Non-Noetherian Perspectives (M. Fontana, S.E. Kabbaj, B. Olberding, I. Swanson, Eds), (Springer-Verlag, New York, 2011) 23–45. doi:10.1007/978-1-4419-6990-3_2

[4] D.F. Anderson and A. Badawi, *On the zero-divisor graph of a ring*, Comm. Algebra **36** (2008) 3073–3092. doi:10.1080/00927870802110888

[5] I. Beck, *Coloring of commutative rings*, J. Algebra. **116** (1998) 208–226. doi:10.1016/0021-8693(88)90202-5

[6] S. Ebrahimi Atani and M. Shajari Kohan, *On L-ideal-based L-zero-divisor graphs*, Discuss. Math. Gen. Algebra Appl. **31** (2011) 127–145. doi:10.7151/dmgaa.1178

[7] S. Ebrahimi Atani and M. Shajari Kohan, *L-zero-divisor graphs of direct products of L-commutative rings*, Discuss. Math. Gen. Algebra Appl. **31** (2011) 159–174. doi:10.7151/dmgaa.1180

[8] S.P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003) 4425–4443. doi:10.1081/AGB-120022801

Received 9 August 2013
First Revision 8 November 2013
Second Revision 15 January 2014