

CLIFFORD CONGRUENCES ON GENERALIZED QUASI-ORTHODOX GV-SEMIGROUPS

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Abstract

A semigroup S is said to be completely π -regular if for any $a \in S$ there exists a positive integer n such that a^n is completely regular. A completely π -regular semigroup S is said to be a GV-semigroup if all the regular elements of S are completely regular. The present paper is devoted to the study of generalized quasi-orthodox GV-semigroups and least Clifford congruences on them.

Keywords: Clifford semigroup, Clifford congruence, generalized quasi-orthodox semigroup.

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1. INTRODUCTION

The study of the structure of semigroups is essentially influenced by the study of the congruences defined on them. We know that the set of all congruences defined on a semigroup S is a partially ordered set with respect to inclusion and relative to this partial order it forms a lattice, the lattice of congruences on S . The study of lattice of congruences on different types of semigroups such as regular semigroups and eventually regular semigroups led to breakthrough innovations made by T.E. Hall [3], LaTorre [5], S.H. Rao and P. Lakshmi [10]. The congruences that they looked into were mostly group congruences. In paper [10], S.H. Rao and P. Lakshmi characterized group congruences on eventually regular semigroups in which they used self-conjugate subsemigroups. Further studies were continued by S. Sattayaporn [11] with weakly self-conjugate subsets.

Over the years, congruence structures have been an integral part of discussion in mathematics.

In this paper, we study various types of congruences on GV-semigroups. To be more precise, we characterize least Clifford congruences on generalized quasi-orthodox GV-semigroups.

2. PRELIMINARIES

An element a in a semigroup (S, \cdot) is said to be regular if there exists an element $x \in S$ such that $axa = a$. A semigroup (S, \cdot) is said to be regular if every element of S is regular. In this case there also exists $y \in S$ such that $aya = a$ and $yay = y$. Such an element y is called an inverse of a . An element a in a semigroup (S, \cdot) is said to be π -regular (or power regular) if there exists a positive integer n such that a^n is regular. Naturally, a semigroup (S, \cdot) is said to be π -regular (or power regular) if every element of S is π -regular. An element a in a semigroup (S, \cdot) is said to be completely regular if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. We know that an element a in a semigroup S is completely regular if and only if it belongs to a subgroup of G . We call a semigroup S , a completely regular semigroup if every element of S is completely regular.

An element a in a semigroup (S, \cdot) is said to be completely π -regular if there exists a positive integer n such that a^n is completely regular. Naturally, a semigroup S is said to be completely π -regular if every element of S is completely π -regular.

Lemma 2.1 [7]. *Let S be a semigroup and let x be an element of S such that x^n belongs to a subgroup G of S for some positive integer n . Then, if e is the identity of G , we have*

- (a) $ex = xe \in G$,
- (b) $x^m \in G$ for any integer $m > n$.

Let a be a completely π -regular element in a semigroup S . Then a^n lies in a subgroup G of S for some positive integer n . The inverse of a^n in G is denoted by $(a^n)^{-1}$. From the above lemma, it follows that for a completely π -regular element a in a semigroup S , all its completely regular powers lie in the same subgroup of S . Let a^0 be the identity of this group and $\bar{a} = (aa^0)^{-1}$. Then clearly, $a^0 = a\bar{a} = \bar{a}a$ and $aa^0 = a^0a$.

Throughout this paper, we always let $E(S)$ be the set of all idempotents of the semigroup S . Also we denote the set of all inverses of a regular element a in a semigroup S by $V(a)$. For $a \in S$, by " a^n is a -regular" we mean that n is the smallest positive integer for which a^n is regular.

As usual, we denote the Green's relations [4] on the semigroup (S, \cdot) by \mathcal{L} , \mathcal{R} , \mathcal{D} , \mathcal{J} and \mathcal{H} . For any $a \in S$, we let H_a be the \mathcal{H} -class in S containing a . If (S, \cdot) be a π -regular semigroup, we consider the relations \mathcal{L}^* , \mathcal{R}^* , \mathcal{J}^* , \mathcal{H}^* and \mathcal{D}^* defined by

$$\begin{aligned} a \mathcal{L}^* b &\text{ if and only if } a^p \mathcal{L} b^q, \\ a \mathcal{R}^* b &\text{ if and only if } a^p \mathcal{R} b^q, \\ a \mathcal{J}^* b &\text{ if and only if } a^p \mathcal{J} b^q, \\ \mathcal{H}^* &= \mathcal{L}^* \cap \mathcal{R}^* \quad \text{and} \quad \mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^* \end{aligned}$$

where a^p is a -regular and b^q is b -regular.

A semigroup (S, \cdot) is said to be a band if each element of S is idempotent, i.e., $a^2 = a$ for all $a \in S$. A commutative band is called a semilattice. A band S is said to be a rectangular band if it satisfies the identity $axa = a$ for all $a, x \in S$. A congruence ρ on a semigroup S is called a semilattice congruence if S/ρ is a semilattice. A semigroup S is called a semilattice Y of semigroups $S_\alpha (\alpha \in Y)$ if S admits a semilattice congruence ρ on S such that $Y = S/\rho$ and each S_α is a ρ -class mapped onto α by the natural epimorphism $\rho^\# : S \rightarrow Y$. We write $S = (Y; S_\alpha)$. For other notations and terminologies not given in this paper, the reader is referred to the texts of Bogdanovic [1] and Howie [4].

3. COMPLETELY ARCHIMEDEAN SEMIGROUPS AND GV-SEMIGROUPS

In this section we recall some definitions and state some of their important properties.

Definition 3.1. A semigroup S is said to be an Archimedean semigroup if for any two elements $a, b \in S$ there exists a positive integer n such that $a^n \in SbS$.

Definition 3.2. An Archimedean semigroup S is said to be completely Archimedean if it is completely π -regular.

Definition 3.3. Let I be an ideal of a semigroup S . We define a relation ρ_I on S by $a\rho_I b$ if and only if either $a, b \in I$ or $a = b$ where $a, b \in S$. It is easy to verify that ρ_I is a congruence on S . This congruence is said to be Rees congruence on S and the quotient semigroup S/ρ_I contains a zero, namely I . This quotient semigroup S/ρ_I is said to be the Rees quotient semigroup and is denoted by S/I . In this case the semigroup S is said to be an ideal extension or simply an extension of I by the semigroup S/I . An ideal extension S of a semigroup I is said to be a nil-extension of I if S/I is nil semigroup, i.e., for any $a \in S$ there exists a positive integer n such that $a^n \in I$.

Theorem 3.4 [1]. *The following conditions on a semigroup are equivalent:*

- (i) *S is a completely Archimedean semigroup;*
- (ii) *S is a nil-extension of a completely simple semigroup.*

In a completely π -regular semigroup, regular elements may not be completely regular. A completely π -regular semigroup in which every regular element is completely regular is said to be a GV-semigroup. A completely π -regular semigroup containing a single idempotent is called a π -group. It is well known that a semigroup S is a π -group if and only if S is nil-extension of a group. The following theorem gives the complete characterization of GV-semigroups.

Theorem 3.5 [1]. *The following conditions on a semigroup S are equivalent:*

- (i) *S is a GV-semigroup;*
- (ii) *S is π -regular and every \mathcal{H}^* -class of S is a π -group;*
- (iii) *S is semilattice of completely Archimedean semigroups.*

In this connection, it is interesting to mention that \mathcal{J}^* is a semilattice congruence on a GV-semigroup S and each \mathcal{J}^* -class is a completely Archimedean subsemigroup of S .

4. GENERALIZED QUASI-ORTHODOX GV-SEMIGROUPS

In this section we define generalized quasi-orthodox semigroups and study some important properties of generalized quasi-orthodox GV-semigroups.

Definition 4.1. A semigroup (S, \cdot) is said to be an orthodox semigroup if $E(S)$ forms a subsemigroup of S .

Definition 4.2. A semigroup (S, \cdot) is said to be a generalized quasi-orthodox semigroup if for any two elements $e, f \in E(S)$, there exists a positive integer n such that $(ef)^n = (ef)^{n+1}$.

Clearly, every orthodox semigroup is generalized quasi-orthodox. But the converse is not true in general. We cite some examples to ensure that generalized quasi-orthodox semigroup may not be an orthodox semigroup.

Example 4.3 [6]. Let $S = \{e, f, a, 0\}$. On S we define a multiplication $'\cdot'$ with the following Cayley table:

\cdot	e	f	a	0
e	e	a	a	0
f	0	f	0	0
a	0	a	0	0
0	0	0	0	0

Then (S, \cdot) is a generalized quasi-orthodox semigroup but not an orthodox.

Example 4.4. Let $S = \{0, 1, 2, 3, 4, 5\}$. On S we define a multiplication \cdot with the following Cayley table:

\cdot	0	1	2	3	4	5
0	0	2	2	3	0	0
1	3	1	3	3	1	5
2	3	2	3	3	2	0
3	3	3	3	3	3	3
4	0	1	2	3	4	5
5	5	1	1	3	5	5

Then (S, \cdot) is a semigroup with $E(S) = \{0, 1, 3, 4, 5\}$. Here, $0 \cdot 1 = 2 \notin E(S)$. Hence S is not an orthodox semigroup. But, one can easily verify that (S, \cdot) is a generalized quasi-orthodox semigroup.

Remark 4.5. A completely π -regular semigroup S is generalized quasi-orthodox if and only if for any $e, f \in E(S)$, $(ef)(ef)^0 = (ef)^0$.

Lemma 4.6. Let S be a GV-semigroup; $e, f \in E(S)$, $g = (ef)^0e$ and $h = f(ef)^0$. Then $ef(ef)^0 = gh$, $g^2 = g$, $h^2 = h$ and $(ef) \mathcal{J}^*g \mathcal{J}^*h$.

Proof. Firstly, $gh = (ef)^0(ef)(ef)^0 = (ef)(ef)^0$. Again, $g^2 = (ef)^0e(ef)^0e = (ef)^0e = g$. Similarly, $h^2 = h$. Since S is a semilattice of its \mathcal{J}^* -classes [1], it follows that $(ef) \mathcal{J}^*g \mathcal{J}^*h$. ■

Definition 4.7. A GV-semigroup S is said to be generalized quasi-orthodox GV-semigroup if S is a generalized quasi-orthodox semigroup.

Theorem 4.8. Let $S = (Y; S_\alpha)$ be a GV-semigroup, where Y is a semilattice and S_α ($\alpha \in Y$) is a completely Archimedean semigroup. Then the following conditions are equivalent:

- (i) S is generalized quasi-orthodox;
- (ii) For all $\alpha \in Y$, S_α is orthodox;
- (iii) For all $e \in E(S)$ and for all $x \in S$, there exist $m, n \in \mathbb{N}$ such that

$$\left(x^{m-1}(x^m)^{-1}ex\right)^n = \left(x^{m-1}(x^m)^{-1}ex\right)^{n+1};$$

- (iv) For all $a, b \in S$ there exists a positive integer n such that $(a^0b^0)^n = (a^0b^0)^{n+1}$.

Proof. (i) \iff (ii) This follows from Theorem X:2.1 [1].

(i) \implies (iii) For $x \in S$ there exists a positive integer m such that x^m is x -regular. Let $e \in E(S)$. As S is quasi-orthodox and $x^0 \in E(S)$, then there exists a positive integer n such that $(ex^0)^{n-1} = (ex^0)^n$. Therefore,

$$\begin{aligned} \left(x^{m-1}(x^m)^{-1}ex\right)^n &= x^{m-1}(x^m)^{-1}(ex^0)^{n-1}ex \\ &= x^{m-1}(x^m)^{-1}(ex^0)^n ex \\ &= \left(x^{m-1}(x^m)^{-1}ex\right)^{n+1}. \end{aligned}$$

Hence, for all $e \in E(S)$ and for all $x \in S$, there exist positive integers $m, n \in \mathbb{N}$ such that $(x^{m-1}(x^m)^{-1}ex)^n = (x^{m-1}(x^m)^{-1}ex)^{n+1}$.

(iii) \implies (i) Let $e, f \in E(S)$. Then by the given condition we have, $(fef)^n = (fef)^{n+1}$, for some positive integer n .

Now, $(ef)^{n+1} = e(fef)^n = e(fef)^{n+1} = (ef)^{n+2}$. Hence, S is generalized quasi-orthodox.

(i) \implies (iv) Since S is generalized quasi-orthodox and for all $a, b \in S$, $a^0, b^0 \in E(S)$, hence there exists a positive integer n such that $(a^0b^0)^n = (a^0b^0)^{n+1}$.

(iv) \implies (i) This part is obvious. ■

Lemma 4.9. Let $S = (Y; S_\alpha)$ be a generalized quasi-orthodox GV-semigroup, where Y is a semilattice and S_α ($\alpha \in Y$) is a completely Archimedean semigroup. Then for all $\alpha \in Y$, $E(S_\alpha)$ is a rectangular band and for any two elements $a, b \in S_\alpha$, $e \in E(S_\beta)$, $a^0b^0 = a^0eb^0$, where $\alpha, \beta \in Y$ with $\alpha \leq \beta$.

Proof. Since each S_α is a completely Archimedean semigroup, then S_α has a completely simple kernel K_α . As every element $e \in E(S_\alpha)$ is completely regular, it follows that $e \in E(K_\alpha)$ and thus $E(S_\alpha) = E(K_\alpha)$. Since K_α is completely simple and orthodox, so $E(K_\alpha)$ is a rectangular band (Corollary III.5.3 [9]). Hence $E(S_\alpha)$ is a rectangular band.

Now $a^0, a^0e, b^0 \in K_\alpha$. Since S is generalized quasi-orthodox, so there exists a positive integer n such that $(a^0e)^n = (a^0e)^{n+1}$. Here $(a^0e)^n \in E(K_\alpha)$.

Therefore,

$$\begin{aligned} a^0b^0 &= a^0(a^0e)^nb^0 \\ &= a^0(a^0e)^{n+1}b^0 \\ &= a^0(a^0e)^na^0eb^0 \\ &= a^0eb^0. \end{aligned} \quad \blacksquare$$

Next we prove a very important result on generalized quasi-orthodox GV-semigroup.

Theorem 4.10. *Let $S = (Y; S_\alpha)$ be a generalized quasi-orthodox GV-semigroup, where Y is a semilattice and S_α ($\alpha \in Y$) is a completely Archimedean semigroup. Let $a \in S_\alpha$ be a completely regular element and $b \in S_\beta$ where $\alpha \leq \beta$. Then ab is completely regular.*

Proof. For all $\alpha \in Y$, S_α is the nil-extension of its kernel K_α , that is completely simple.

Clearly, $(ab) \in S_\alpha$. We show that $(ab) \in K_\alpha$. As S is a GV-semigroup, then there exists a positive integer n such that $(ab)^n$ is completely regular and $(ab)^n$ is in a subgroup $G \subseteq K_\alpha$. Let g be the identity of G . Then $abg = gab \in G$.

Now,

$$\begin{aligned} ab &= a^0 ab, \\ &= (a^0 ga^0) ab, \end{aligned}$$

since $E(S_\alpha)$ is a rectangular band and $a^0, g \in E(S_\alpha)$.

Therefore,

$$\begin{aligned} ab &= (a^0 ga^0) ab \\ &= a^0 (ga^0 ab) \\ &= a^0 (gab) \in K_\alpha, \end{aligned}$$

since $a^0 \in K_\alpha$, $gab \in G \subseteq K_\alpha$. Consequently, ab is completely regular. ■

5. LEAST CLIFFORD CONGRUENCES

In this section, we characterize least Clifford congruences on generalized quasi-orthodox GV-semigroups. For this purpose we define a relation ν on a GV-semigroup and finally we establish a necessary and sufficient condition for ν to be a least Clifford congruence on a GV-semigroup.

Recall that a regular semigroup in which idempotents are central is said to be a Clifford semigroup. It is interesting to mention that a semigroup S is a Clifford semigroup if and only if S is semilattice of groups.

In order to characterize further the least Clifford congruence on a GV-semigroup, we define the following relation ν .

Definition 5.1. Let $S = (Y; S_\alpha)$ be GV-semigroup, where Y is a semilattice and S_α ($\alpha \in Y$) is a completely Archimedean semigroup.

On S we define a relation ν as follows. For $a, b \in S$,

$$a \nu b \text{ if and only if } aa^0 = a^0 bb^0 a^0 \text{ and } bb^0 = b^0 aa^0 b^0.$$

Theorem 5.2. *Let $S = (Y; S_\alpha)$ be GV-semigroup, where Y is a semilattice and S_α ($\alpha \in Y$) is a completely Archimedean semigroup. Then the relation ν , as*

defined in Definition 5.1 is the least Clifford congruence on S if and only if S is generalized quasi-orthodox.

Proof. Let $S = (Y; S_\alpha)$ be a generalized quasi-orthodox GV-semigroup. We prove that ν is the least Clifford congruence on S .

We first show that ν is an equivalence relation on S . Clearly, ν is reflexive and symmetric.

Let $a \nu b$ and $b \nu c$ holds for $a, b, c \in S$. Then $a, b, c \in S_\alpha$ for some $\alpha \in Y$.

Now, $a \nu b$ implies $aa^0 = a^0bb^0a^0$ and $bb^0 = b^0aa^0b^0$. Also, $b \nu c$ implies $bb^0 = b^0cc^0b^0$ and $cc^0 = c^0bb^0c^0$.

Therefore,

$$\begin{aligned} aa^0 &= a^0b^0cc^0b^0a^0, \\ \text{i.e., } c^0aa^0c^0 &= c^0a^0b^0cc^0b^0a^0c^0 \\ &= c^0a^0b^0cc^0, \quad [\text{since } E(S_\alpha) \text{ is a rectangular band}] \\ &= c^0a^0b^0c^0c, \\ &= c^0c, \\ &= cc^0. \end{aligned}$$

Similarly, $aa^0 = a^0cc^0a^0$. Thus, $a \nu c$. Hence ν is an equivalence relation.

Let $a \nu b$ and $c \in S$. Let $a, b \in S_\alpha$ and $c \in S_\beta$. Therefore, $aa^0 = a^0bb^0a^0$ and $bb^0 = b^0aa^0b^0$. By using Lemma 4.9., we have

$$\begin{aligned} (ca)^0(cb)(cb)^0(ca)^0 &= (ca)^0(cb)b^0(cb)^0(ca)^0 \\ &= (ca)^0cb^0aa^0b^0(cb)^0(ca)^0 \\ &= (ca)^0caa^0b^0(ca)^0 \\ &= (ca)^0caa^0(ca)^0 \\ &= (ca)^0ca(ca)^0 \\ &= (ca)(ca)^0. \end{aligned}$$

Similarly, $(cb)^0(ca)(ca)^0(cb)^0 = (cb)(cb)^0$. Therefore, $(ca) \nu (cb)$. Dually, we can obtain, $(ac) \nu (bc)$. Consequently, ν is a congruence on S .

Now, for any $a \in S$, $e \in E(S)$ we have $e(ae)^0 \in E(S)$.

Also,

$$\begin{aligned} (ea)^0(ae)(ae)^0(ea)^0 &= (ea)^0(ae)(ea)^0 \\ &= (ea)^0e(ae)(ea)^0 \\ &= (ea)^0(ea)(ea)^0 \\ &= (ea)(ea)^0. \end{aligned}$$

Similarly, $(ae)^0(ea)(ea)^0(ae)^0 = (ae)(ae)^0$. Therefore, $(ae) \nu (ea)$. Thus, we conclude that ν is a Clifford congruence on S .

Now, we verify that ν is the least Clifford congruence on S .

Let ρ be any other Clifford congruence on S and $a \nu b$, for $a, b \in S$. Then, $(aa^0) \nu (bb^0)$. Now, $aa^0 = a^0bb^0a^0 = (a^0b^0aa^0b^0a^0) \rho (b^0a^0aa^0a^0b^0) = b^0aa^0b^0 = bb^0$. Hence, $(aa^0) \rho (bb^0)$. Now, $a \rho (aa^0) \rho (bb^0) \rho b$ implies, $\nu \subseteq \rho$. Hence, ν is the least Clifford congruence on S .

To prove the converse, let ν be the least Clifford congruence on a GV-semigroup S . We show that S is generalized quasi-orthodox. Now, let $e, f \in E(S)$. Then, by definition, $(ef) \nu (fe)$. This implies, $(ef)\nu = (fe)\nu$. Let $(ef)^n$ be (ef) -regular. Then, $(ef)^{n+1}\nu = (fe)\nu(ef)^n\nu$, i.e., $(ef)^{n+1}\nu = (ef)^n\nu$. This implies $((ef)(ef)^0)\nu = (ef)^0\nu$, i.e., $(ef)(ef)^0 = (ef)^0$, i.e., $(ef)^{n+1} = (ef)^n$. This proves that S is generalized quasi-orthodox. ■

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