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CONGRUENCES ON BANDS OF π-GROUPS

Sunil K. Maity

Department of Mathematics, University of Burdwan,
Golapbag, Burdwan – 713104, West Bengal, India

e-mail: skmaity@math.buruniv.ac.in

Abstract

A semigroup S is said to be completely π-regular if for any $a \in S$ there exists a positive integer n such that a^n is completely regular. The present paper is devoted to the study of completely regular semigroup congruences on bands of π-groups.

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1. Introduction

The study of the structure of semigroups are is essentially influenced by the study of the congruences defined on them. We know that the set of all congruences defined on a semigroup S is a partially ordered set with respect to inclusion and relative to this partial order it forms a lattice, the lattice of congruences on S. The study of the lattice of congruences on different types of semigroups such as regular semigroups and eventually regular semigroups led to breakthrough innovations made by T.E. Hall [3], LaTorre [5], S.H. Rao and P. Lakshmi [10]. The congruences that they looked into were group congruences on regular and eventually regular semigroups. In paper [10], S.H. Rao and P. Lakshmi characterized group congruences on eventually regular semigroups in which they used self-conjugate subsemigroups. Further studies were continued by S. Sattayaporn [11] with weakly self-conjugate subsets. Over the years, congruence structures have been an integral part of discussion in mathematics.

In this paper, we study various types of congruences on bands of π-groups. To be more precise, we characterize least completely regular semigroup congruences on bands of π-groups.

2. Preliminaries

A semigroup $(S, \cdot)$ is called regular if for every element $a \in S$ there exists an element $x \in S$ such that $axa = a$. In this case there also exists $y \in S$ such that $aya = a$ and $yay = y$. Such an element y is called an inverse of a. A semigroup $(S, \cdot)$ is said to be π-regular (or power regular) if for every element $a \in S$ there exists a positive integer n such that a^n is regular. An element a in a semigroup $(S, \cdot)$ is said to be completely regular if there exists $x \in S$ such that $a = axa$ and $ax = xa$. We call a semigroup S, a completely regular semigroup if every element of S is completely regular.

An element a in a semigroup $(S, \cdot)$ is said to be completely π-regular if there exists a positive integer n such that a^n is completely regular. Naturally, a semigroup S is said to be completely π-regular if every element of S is completely π-regular.

Lemma 1 [7]**.** *Let S be a semigroup and let x be an element of S such that x^n belongs to a subgroup G of S for some positive integer n. Then, if e is the identity of G, we have*

(a) $ex = xe \in G$,

(b) $x^m \in G$ *for any integer* $m > n$.

Let a be a completely π-regular element in a semigroup S. Then a^n lies in a subgroup G of S for some positive integer n. The inverse of a^n in G is denoted by $(a^n)^{-1}$. From the above lemma, it follows that for a completely π-regular element a of a semigroup S, all its completely regular powers lie in the same subgroup of S, and let a^0 be the identity of this group and $\overline{a} = (aa^0)^{-1}$. Then clearly, $a^0 = a\overline{a} = \overline{a}a$ and $aa^0 = a^0a$. By a nil-extension of a semigroup we mean any of its ideal extension by a nil-semigroup.

Throughout this paper, we always let $E(S)$ be the set of all idempotents of the semigroup S. Also we denote the set of all inverses of a regular element a in a semigroup S by $V(a)$. For $a \in S$, by "a^n is a-regular" we mean that n is the smallest positive integer for which a^n is regular.

A semigroup $(S, \cdot)$ is said to be a band if each element of S is an idempotent, i.e., $a^2 = a$ for all $a \in S$. A congruence ρ on a semigroup S is called a band congruence if S/ρ is a band. A semigroup S is called a band B of semigroups $S_\alpha(\alpha \in B)$ if S admits a band congruence ρ on S such that $B = S/\rho$ and each S_α is a ρ-class mapped onto α by the natural epimorphism $\rho^{\#} : S \longrightarrow B$. We write $S = (B; S_\alpha)$. For other notations and terminologies not given in this paper, the reader is referred to the texts of Bogdanovic [1] and Howie [4].

3. Least completely regular semigroup congruences

In this section we characterize the least completely regular semigroup congruences on bands of π-groups. We introduce a relation on π-groups and then extend this relation on bands of π-groups.

Definition 1 [1]**.** Let S be a semigroup and G be a subgroup of S. If for every $a \in S$ there exists a positive integer n such that $a^n \in G$, then S is said to be a π-group.

Theorem 2 [1]**.** *Let S be a π-regular semigroup. Then S is a π-group if and only if S has exactly one idempotent.*

Theorem 3 [1]**.** *A semigroup S is a π-group if and only if S is a nil-extension of a group.*

In order to characterize further the least completely regular semigroup congruence on a band of π-groups, we define the following relation σ.

Definition 2. Let S be a π-group. Then by Theorem 3, S is nil-extension of a group G. We define a relation σ on S as follows. For $a, b \in S$,

$$a \,\sigma\, b \text{ if and only if } ab^{m-1}(b^m)^{-1} = e,$$

where e is the identity of G and b^m is b-regular.

Lemma 4. *Let S be a π-group which is nil-extension of a group G. Then the relation σ as defined in Definition 2 is the least group congruence on S such that $S/\sigma \cong G$.*

Proof. Clearly, σ is reflexive. Let $a \,\sigma\, b$. Then $ab^{m-1}(b^m)^{-1} = e$, where e is the identity of G and b^m is b-regular.

Let a^n be a-regular. Now, $a^{n-1}(a^n)^{-1}a \in E(S)$. Since S contains exactly one idempotent, it follows that $a^{n-1}(a^n)^{-1}a = e$. Now, $ba^{n-1}(a^n)^{-1} = ba^{n-1}(a^n)^{-1}e = ba^{n-1}(a^n)^{-1}ab^{m-1}(b^m)^{-1} = beb^{m-1}(b^m)^{-1} = ebb^{m-1}(b^m)^{-1} = eb^m(b^m)^{-1} = e$, i.e., $b \,\sigma\, a$. Thus, σ is symmetric.

Let $a \,\sigma\, b$ and $b \,\sigma\, c$ hold. Then, $ab^{m-1}(b^m)^{-1} = e$ and $bc^{k-1}(c^k)^{-1} = e$, where b^m is b-regular and c^k is c-regular.

Now, $ab^{m-1}(b^m)^{-1}bc^{k-1}(c^k)^{-1} = e$ implies $aec^{k-1}(c^k)^{-1} = e$, i.e., $ac^{k-1}(c^k)^{-1} = e$. This implies $a \,\sigma\, c$, and hence σ is transitive. Thus, σ is an equivalence relation.

Let $a \,\sigma\, b$ and $c \in S$. Then $ab^{m-1}(b^m)^{-1} = e$ and b^m is b-regular.

Let $a^n, (bc)^l$ and c^k be a-regular, (bc)-regular and c-regular, respectively.

Now $c(bc)^{l-1}((bc)^l)^{-1}b = e$ implies $ac(bc)^{l-1}((bc)^l)^{-1}ba^{n-1}(a^n)^{-1} = e$, i.e., $(ac)(bc)^{l-1}((bc)^l)^{-1} = e$, i.e., $(ac)\,\sigma\,(bc)$. Similarly, we can prove $(ca)\,\sigma\,(cb)$. Consequently, σ is a congruence on S.

Clearly, $a\,\sigma\,(ae)$ and $(ae)\sigma$ is regular. Hence $a\sigma$ is regular. Again, $(ae) \in G$ and let x be the inverse of (ae) in G. Then, $(a\sigma)(x\sigma)(a\sigma) = a\sigma$ and $(a\sigma)(x\sigma) = (x\sigma)(a\sigma) = e\sigma$.

Thus, σ is a group congruence. To show σ is the least group congruence on S, let γ be any group congruence on S and let $a\,\sigma\,b$. Then $ab^{m-1}(b^m)^{-1} = e$, where b^m is b-regular.
Therefore, $b\,\gamma\,(eb) = ab^{m-1}(b^m)^{-1}b = (ae)\,\gamma\,a$, i.e., $a\,\gamma\,b$. Hence $\sigma \subseteq \gamma$. Thus, σ is the least group congruence on S.

One can easily prove that the mapping $\psi : S/\sigma \to G$ defined by $\psi(a\sigma) = ae$ is a group isomorphism. ∎

Remark. It follows from Theorem 1 [10] that the relation σ on a π-group S defined in Definition 2 is a group congruence if $\{a \in S : ae = e\}$ is substituted for H in Theorem 1 [10].

Using the above lemma, we now characterize the least completely regular semigroup congruence on a band of π-groups.

Definition 3. Let $S = (B; T_\alpha)$ be a band of π-groups, where B is a band and $T_\alpha\,(\alpha \in B)$ is a π-group. Let T_α be the nil-extension of the group G_α and e_α be the identity of G_α for all $\alpha \in B$. For $a \in T_\alpha\,(\alpha \in B)$, where a^n is a-regular, let $(a^n)^{-1}$ denote the inverse of a^n in G_α.

On S we define a relation ρ as follows. For $a, b \in S$, $a\,\rho\,b$ if and only if $a, b \in T_\alpha$ for some $\alpha \in B$ and $ab^{m-1}(b^m)^{-1} = e_\alpha$, where b^m is b-regular; i.e., $\rho = \bigcup_{\alpha \in B} \sigma_\alpha$, where σ_α is the least group congruence on T_α for all $\alpha \in B$.

Theorem 5. *Let $S = (B; T_\alpha)$ be a band of π-groups. Then the relation ρ as defined in Definition 3 is the least completely regular semigroup congruence on S.*

Proof. Clearly, ρ is an equivalence relation on S.

To show ρ is a congruence on S, let $a\,\rho\,b$ and $c \in S$. Therefore, $a, b \in T_\alpha$ and $c \in T_\gamma$ for some $\alpha, \gamma \in B$. Now, $a\,\rho\,b$ implies $ab^{m-1}(b^m)^{-1} = e_\alpha$, where e_α is the identity of G_α and b^m is b-regular. This implies $ba^{n-1}(a^n)^{-1} = e_\alpha$, where a^n is a-regular.

Let $(bc)^l$ be (bc)-regular. Now, $c(bc)^{l-1}((bc)^l)^{-1}b = e_{\gamma\alpha}$ implies

$$(ac)(bc)^{l-1}\Big((bc)^l\Big)^{-1}ba^{n-1}(a^n)^{-1} = ae_{\gamma\alpha}a^{n-1}(a^n)^{-1} = e_{\alpha\gamma\alpha},$$

i.e., $$(ac)(bc)^{l-1}\Big((bc)^l\Big)^{-1}e_\alpha = e_{\alpha\gamma\alpha},$$

i.e., $$(ac)(bc)^{l-1}\Big((bc)^l\Big)^{-1}e_\alpha e_{\alpha\gamma} = e_{\alpha\gamma\alpha}e_{\alpha\gamma} = e_{\alpha\gamma},$$

i.e., $$(ac)(bc)^{l-1}\Big((bc)^l\Big)^{-1}e_{\alpha\gamma} = e_{\alpha\gamma},$$

i.e., $$(ac)(bc)^{l-1}\Big((bc)^l\Big)^{-1} = e_{\alpha\gamma},$$

i.e., $$ac\,\rho\,bc.$$

Similarly, we can prove that $ca\,\rho\,cb$. Hence, ρ is a congruence on S.

Also, for any $a \in S$, $a\,\rho\,(ae_\alpha)$ (where $a \in T_\alpha$) and $ae_\alpha \in G_\alpha$ is completely regular. This implies $a\rho$ is completely regular. Moreover, it is easy to verify that ρ is the least completely regular semigroup congruence on S. ∎

In a semigroup S with nonempty set of idempotents, $E(S)$ is a subsemigroup of S if and only if for all idempotents e, f in S, $(ef)^2 = ef$. However, a semigroup S with nonempty set of idempotents and the property that for any two elements $e, f \in E(S)$, there exists a positive integer n such that $(ef)^n = (ef)^{n+1}$ does not necessarily have $E(S)$ as its subsemigroup. We provide an example of such a semigroup.

Example [6]**.** Let $S = \{e, f, a, 0\}$. On S we define a multiplication $'\cdot'$ with the following Cayley table:

$\cdot$	e	f	a	0
e	e	a	a	0
f	0	f	0	0
a	0	a	0	0
0	0	0	0	0

Then $(S, \cdot)$ is a semigroup with $E(S) = \{e, f, 0\}$. Here, $ef = a \notin E(S)$. Hence $E(S)$ is not a subsemigroup of S. But, for any two elements $x, y \in E(S)$, there exists a positive integer n such that $(xy)^n = (xy)^{n+1}$.

Theorem 6. *Let $S = (B; T_\alpha)$ be a band of π-groups. Then the following two statements are equivalent.*

(i) *For any two elements* $e, f \in E(S)$*, there exists a positive integer* n *such that* $(ef)^n = (ef)^{n+1}$.

(ii) S/ρ *is an orthogroup, where* ρ *is the least completely regular semigroup congruence on* S *as defined in Definition* 3.

Proof. Let $S = (B; T_\alpha)$ be a band of π-groups, where B is a band and $T_\alpha (\alpha \in B)$ is a π-group. Furthermore, let T_α be the nil-extension of the group $G_\alpha\,(\alpha \in B)$.

Suppose S satisfies statement (i) of Theorem 6. Let $e\rho, f\rho \in E(S/\rho)$, where $e, f \in E(S)$. Then there exists a positive integer n such that $(ef)^n = (ef)^{n+1}$, i.e., $(ef)^2(ef)^{n-1}((ef)^n)^{-1} = e_\alpha$, where e_α is the identity of the group G_α containing $(ef)^n$. Therefore, $(ef)^2\,\rho\,(ef)$, i.e., $(e\rho)(f\rho) \in E(S/\rho)$. Hence S/ρ is an orthogroup.

Conversely, let us assume that S/ρ is an orthogroup. Let $e, f \in E(S)$ and $ef \in T_\alpha$, where $\alpha \in B$. Let $(ef)^n$ be (ef)-regular.

Clearly, $e\rho, f\rho \in E(S/\rho)$. Since S/ρ is orthodox, $(ef)\rho \in E(S/\rho)$. Thus, we have, $(ef)\rho(ef)\rho = (ef)\rho$, i.e., $(ef)^2\,\rho\,(ef)$, i.e., $(ef)^2(ef)^{n-1}((ef)^n)^{-1} = e_\alpha$, i.e., $(ef)^{n+1} = (ef)^n$. Thus, S satisfies statement (i) of Theorem 6. ∎

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